University of the Peloponnese
Electrical and Computer
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DIGITAL SIGNAL PROCESSING

Solved Examples

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SET #7 – Series and Fourier Transform of Discrete-Time Signals

- Fourier series Discrete Time Signals
- Fourier Transform of Discrete Time Signals
- DTFT properties
- Inverse DTFT

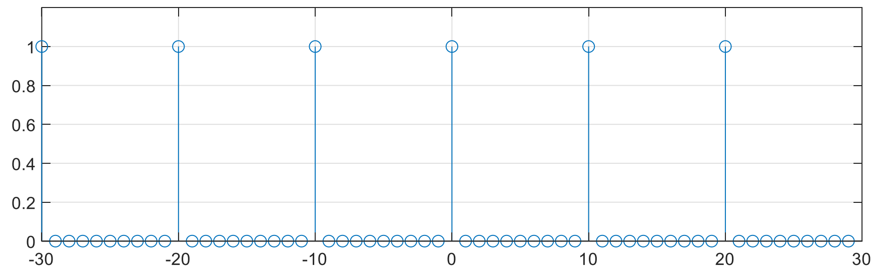
1. Fourier Series of Discrete Time Signals

Example 1

To find the exponential Fourier series expansion of the series of Delta functions with period N:

$$\delta_N[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$$

Answer: The sequence $\delta_N[n]$ has the following graph (for $N = 10$):



String of impulse functions $x[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$

We will calculate the coefficients $\Delta[k]$ of the exponential Fourier series from equation (10.2) choosing for period $[-N/2 + 1, N/2]$:

$$\Delta[k] = \frac{1}{N} \sum_{k=-N/2+1}^{N/2} \delta_N[n]\, e^{-jk(2\pi/N)n} = \frac{1}{N}\delta[0] = \frac{1}{N}$$

The exponential series expansion of the sequence $\delta_N[n]$is:

$$\delta_N[n] = \sum_{k=0}^{N-1} \Delta[k]\, e^{jk(2\pi/N)n} = \frac{1}{N}\sum_{k=0}^{N-1} e^{jk(2\pi/N)n}$$

Therefore, the Magnitude spectrum of the sequence $\delta_N[n]$ consists of $\delta[n]$ Magnitude functions $1/N$, placed in sequence with each other.

Example 2

Find the exponential growth of the sequence $x[n] = 1 + \cos(\pi n) + \sin(\pi n/2)$, $-\infty < n < \infty$.

Answer: Since the exponential growth is calculated only for periodic sequences, we will first examine whether the given sequence is periodic. This will only happen if the ratio of the periods of the periodic sequences $\cos(\pi n)$ and $\sin(\pi n/2)$ can be written as a quotient of integers. Term 1 gives the dc component (dc component). It $\cos(\pi n)$has a cyclic frequency $\omega_1 = \pi$and period $N_1 = 2\pi/\omega_1 = 2$, while it $\sin(\pi n/2)$ has a cyclic frequency $\omega_2 = \pi/2$ and period $N_2 = 2\pi/(\pi/2) = 4$. The signal $x[n]$ is periodic because it has infinite duration and satisfies the equation:

$$\frac{N_1}{N_2} = \frac{2}{4} = \frac{1}{2}$$

Its period $x[n]$is:

$$N = \frac{N_1 N_2}{MK\Delta(N_1, N_2)} = \frac{2x4}{2} = 4$$

Based on the Euler equation it is $x[n]$ written:

$$\begin{aligned} x[n] &= 1 + \frac{1}{2}\left[e^{j\pi n} + e^{-j\pi n}\right] + \frac{1}{2}\left[e^{j\pi n/2} - e^{-j\pi n/2}\right] \\ &= 1 + 0.5e^{j\pi n/2} - 0.5e^{-j\pi n/2} + 0.5e^{j\pi n} + 0.5e^{-j\pi n} \\ &= X[0] + X[1]e^{j\omega_0 n} + X[-1]e^{-j\omega_0 n} + X[2]e^{j2\omega_0 n} \\ &\quad + X[-2]e^{-j2\omega_0 n} \end{aligned}$$

where $\omega_0 = \pi/2$. Therefore, the coefficients of the exponential Fourier series are:

$$X[0] = 1, X[1] = X[-1]^* = -0.5, X[2] = X[-2]^* = 0.5$$

2. Fourier Transform of Discrete Time Signals

Example 3

Calculate the DTFT of the sequence$x[n] = \begin{cases} 0.5^n & n = 0,2,4,\dots \\ 0 & \text{elsewhere} \end{cases}$

Answer: Using the definition of DTFT we have:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} \\ &= \sum_{n=0,2,4,\dots}^{\infty} 0.5^n e^{-jn\omega} = \sum_{n=0}^{\infty} 0.5^{2n} e^{-2jn\omega} \\ &= \sum_{n=0}^{\infty} \left(0.25\, e^{-2j\omega}\right)^n = \frac{1}{1 - 0.25\, e^{-2j\omega}} \end{aligned}$$

Example 4

Calculate the DTFT of the sequence $x[n] = 0.5^n\, u[n+3]$

Answer: Using the definition of DTFT we have:

$$X(e^{j\omega}) = \sum_{n=-3}^{\infty} 0.5^n\, e^{-jn\omega} = \sum_{n=-3}^{\infty} \left(0.5\, e^{-j\omega}\right)^n$$
$$= \left(0.5\, e^{-j\omega}\right)^{-3} \sum_{n=0}^{\infty} \left(0.5\, e^{-j\omega}\right)^n = \frac{8\, e^{j3\omega}}{1-0.5\, e^{-j\omega}}$$

Example 5

Calculate the DTFT of the sequence $x[n] = A(u[n]-u[n-N])$.

Answer: Using the definition of DTFT we have:

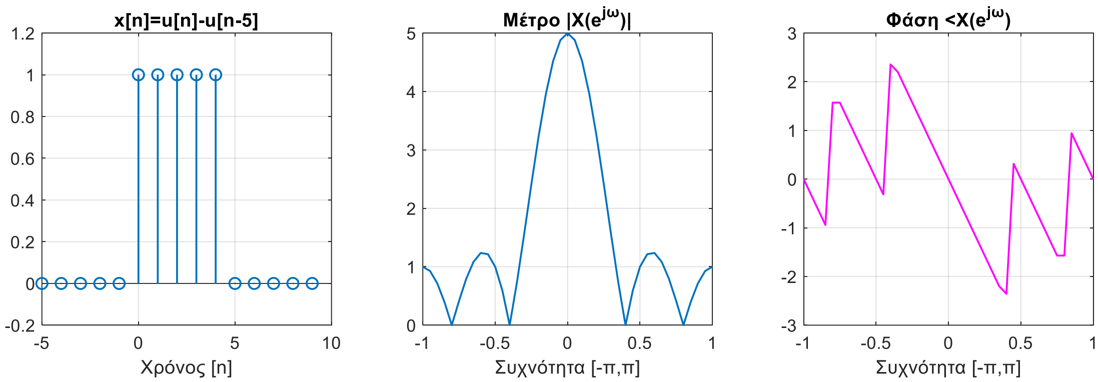
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} = \sum_{n=0}^{N-1} A\, e^{-jn\omega} = A\sum_{n=0}^{N-1} e^{-jn\omega} = A\sum_{n=0}^{N-1} \left(e^{-j\omega}\right)^n$$
$$= \frac{A\left(1-e^{-j\omega N}\right)}{1-e^{-j\omega}} = \frac{Ae^{-j\omega N/2}\left(e^{j\omega N/2}-e^{-j\omega N/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)}$$
$$= \frac{Ae^{-j\omega N/2}\, 2j\, sin(\omega N/2)}{e^{-j\omega/2}\, 2j\, sin(\omega/2)} = Ae^{-j\omega(N-1)/2}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

The measure of DTFT is:

$$\left|X(e^{j\omega})\right| = |A|\left|\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right| \quad (1)$$

and the phase is:

$$\varphi_X(\omega) = -\frac{\omega(N-1)}{2} \quad (2)$$



(a) Signal $x[n] = u[n]-u[n-5]$, (b) Magnitude spectrum,
(c) Phase spectrum, in one period$[-\pi,\pi)$

Comments: The following comments apply to the DTFT measure:

- Since the numerator and denominator of equation (1) are odd functions, it follows that the measure of the DTFT is an even function, as expected.

- By rule de l ' Hospital we find that for the frequency $\omega = 0$ the measure takes the maximum value which is $\left|X(e^{j0})\right| = A$.
- The zeroing points of the measure are those that satisfy the equation $\sin(\omega N/2) = 0$, so the measure is zeroed at the frequencies $\omega = 2k\pi/N$.
- The measure of the DTFT is a function of:
    - Period magazine 2π, when it N is unnecessary.
    - Non-periodic when it N is even.

3. DTFT of Periodic Discrete-Time Signals

Example 6

Prove that the DTFT of the signal $x[n] = e^{j\omega_0 n}$, όπου $\omega \in (-\pi, \pi]$ is given by the equation:

$$x[n] = e^{j\omega_0 n} \leftrightarrow X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 + 2\pi m), \qquad m \in Z$$

Answer: Because the signal is not absolutely summable the DTFT cannot be calculated from its definition. For this reason, we will work in reverse, i.e. we will calculate the inverse DTFT. We notice that the function

$$X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 + 2\pi m)$$

is an infinite sum of impulse functions spaced apart $2\pi m$ on the frequency axis. In other words, its $e^{j\omega_0 n}$ DTFT contains impulse functions at frequencies $\omega_0 \pm 2\pi m$. The **inverse** DTFT is calculated in the frequency domain $(-\pi, \pi]$ from the equation:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 + 2\pi m) \right\} e^{j\omega n}\, d\omega$$

But in the region $(-\pi, \pi]$ there is only the function $\delta(\omega - \omega_0)$, so the integral is:

$$x[n] = \int_{-\pi}^{\pi} \delta(\omega - \omega_0)\, e^{j\omega n}\, d\omega = e^{j\omega n}\Big|_{\omega=\omega_0} = e^{j\omega_0 n}$$

Example 7

Calculate the DTFT of the signal $x[n] = \sum_{k=-\infty}^{\infty} A_k\, e^{j\omega_0 k n}$

Answer: From the previous example we know:

$$e^{j\omega_0 n} \leftrightarrow \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 + 2\pi m)$$

Taking into account the linearity of the DTFT we have:

$$X(e^{j\omega}) = F\left\{\sum_{k=-\infty}^{\infty} A_k\, e^{j\omega_0 kn}\right\} = \sum_{k=-\infty}^{\infty} A_k \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 + 2\pi m)$$

$$= \sum_{k=-\infty}^{\infty} 2\pi\, A_k\, \delta(\omega - \omega_0 + 2\pi m)$$

4. DTFT properties

📖Example 8

Calculate the DTFT of the signal $x[n] = a^n\, sin(n\omega_0)\, u[n]$ and plot the spectra.

Answer: Using Euler's formula we express the sine as a sum of complex functions and then calculate the DTFT of each term. We have:

$$x[n] = \frac{1}{2j}\left[a^n e^{jn\omega_0} - a^n e^{-jn\omega_0}\right] u[n]$$

The DTFT of the first term is:

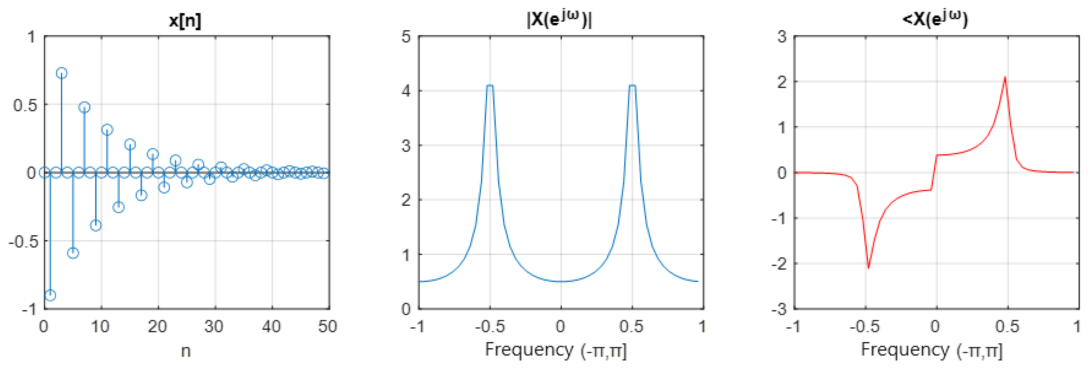
$$\frac{1}{2j}\sum_{n=0}^{\infty} a^n e^{jn\omega_0} e^{-jn\omega} = \frac{1}{2j}\sum_{n=0}^{\infty}\left(ae^{-j(\omega-\omega_0)}\right)^n = \frac{1}{2j}\frac{1}{1 - ae^{-j(\omega-\omega_0)}}$$

The DTFT of the second term is:

$$-\frac{1}{2j}\sum_{n=0}^{\infty} a^n e^{-jn\omega_0} e^{-jn\omega} = -\frac{1}{2j}\frac{1}{1 - ae^{-j(\omega+\omega_0)}}$$

Therefore, it is:

$$X(e^{j\omega}) = \frac{1}{2j}\left[\frac{1}{1 - ae^{-j(\omega-\omega_0)}} \frac{1}{1 - ae^{-j(\omega+\omega_0)}}\right] = \frac{a\, e^{-j\omega} \sin\omega_0}{1 - 2a\, e^{-j\omega}\cos\omega_0 + a^2 e^{-2j\omega}}$$



(a) Sequence $x[n] = a^n\, sin(n\omega_0)\, u[n]$, (b) Magnitude spectrum, (c) Phase spectrum in the frequency domain$[-\pi, \pi]$

📖Example 9

Calculate the DTFT of the signal $y[n] = x[n]\, c[n]$, where $x[n] = u[n + n_0] - u[n - n_0]$ where $n_0 = 5$ and $c[n] = \cos(\omega_0 n)$ where $\omega_0 = 0.7\ (rad)$.

Answer: We know that DTFT of $c[n] = \cos(\omega_0 n)$ is:

$$C(e^{j\omega}) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

By the property of multiplication in time, the DTFT of the product $y[n] = x[n]c[n]$ is:

$$Y(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) * C(e^{j\omega})$$
$$= \frac{1}{2\pi} X(e^{j\omega}) * [\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]] \qquad (1)$$

Using the convolution property $f(x) * \delta(x - x_0) = f(x - x_0)$ we have:

$$X(e^{j\omega}) * \delta(\omega - \omega_0) = X(e^{j(\omega-\omega_0)})$$
$$X(e^{j\omega}) * \delta(\omega + \omega_0) = X(e^{j(\omega+\omega_0)})$$

So, equation (1) is written:

$$Y(e^{j\omega}) = \frac{1}{2}X(e^{j(\omega-\omega_0)}) + \frac{1}{2}X(e^{j(\omega+\omega_0)}) \qquad (2)$$

We find that we arrived at the same result as in Example 10.16 but in a different way. From Example 10.8 we know that the DTFT of the pulse $z[n] = u[n] - u[n - N]$for $N = 10$is:

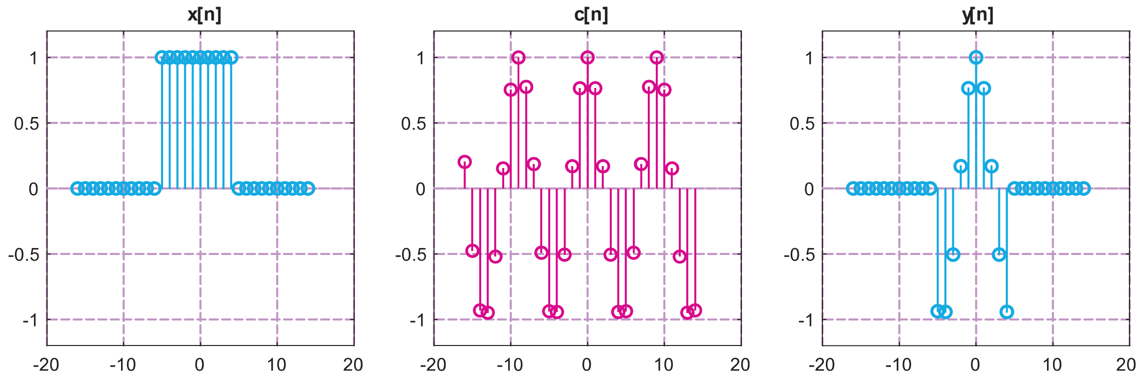
$$Z(e^{j\omega}) = e^{-j\omega(N-1)/2}\frac{\sin(\omega N/2)}{\sin(\omega/2)} = e^{-j9\omega/2}\frac{\sin(5\omega)}{\sin(\omega/2)} \qquad (3)$$

Since the signal $x[n] = u[n + 5] - u[n - 5]$ is its displacement $z[n] = u[n] - u[n - 10]$ by -5 time units, i.e. holds $x[n] = z[n + 5]$, by the time-shift property DTFT of $x[n]$ is:

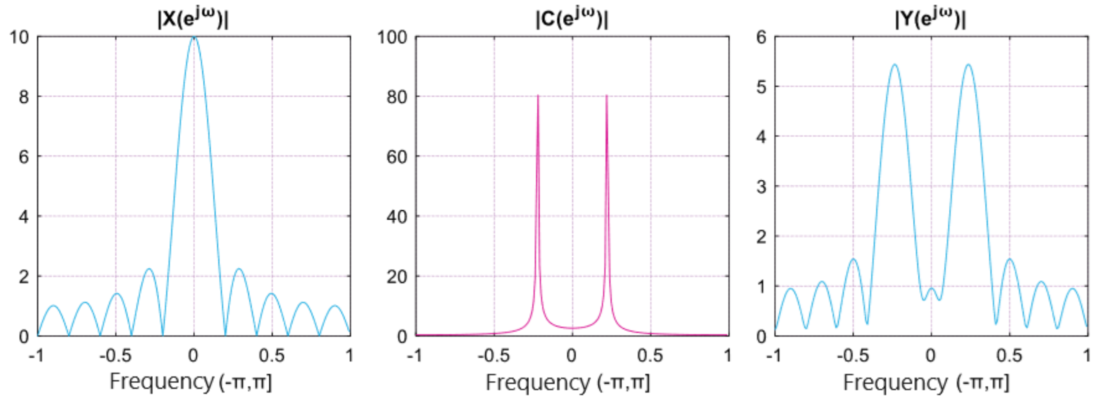
$$X(e^{j\omega}) = e^{-j(-5)\omega} Z(e^{j\omega}) = e^{j5\omega}e^{-j9\omega/2}\frac{\sin(5\omega)}{\sin(\omega/2)} = e^{j\omega/2}\frac{\sin(5\omega)}{\sin(\omega/2)} \qquad (4)$$

From relations (1) and (4) we obtain:

$$Y(e^{j\omega}) = \frac{1}{2}\left[e^{j(\omega-0.7)/2}\frac{\sin(5(\omega - 0.7))}{\sin((\omega - 0.7)/2)} + e^{j(\omega+0.7)/2}\frac{\sin(5(\omega + 0.7))}{\sin((\omega + 0.7)/2)}\right]$$



Signals (a) $x[n] = u[n + 5] - u[n - 5]$, (b) $c[n] = \cos(\omega_0 n)$, (c) $y[n] = x[n]\, c[n]$.



Magnitude spectra (a) $|X(e^{j\omega})|$, (b) $|C(e^{j\omega})|$, (c) $|Y(e^{j\omega})|$.

Comments:

- The Example describes both the **formation** of an information signal $x[n]$ from a carrier signal $c[n] = \cos(\omega_0 n)$ and the **windowing** of a cosine $\cos(\omega_0 n)$ from a square window $x[n] = u[n + n_0] - u[n - n_0]$.
- Regarding the modulation, we notice that the spectrum of the square pulse was transferred to the frequencies $\pm\omega_0$ with half the width of the original one.
- By the term windowing we describe the multiplication of a signal $c[n]$ by a window (square in our example), so we extract a **part** from the signal $c[n]$. In this case DTFT of the part (of the cosine in our example) no longer consists of the two impulse functions at the frequencies $-\omega_0$ and ω_0(spectrum of the cosine), but of two sampling functions (spectrum of the square pulse) placed at the frequencies $-\omega_0$ and ω_0.
- Windowing causes **diffusion** of the signal $\pm\omega_0$ **spectrum at frequencies on either side of the** cosine frequency. Spreading of the spectrum is an undesirable distortion, especially in the case where we seek to distinguish cosines with adjacent frequencies because the lobes of the spectra are entangled with each other.
- Minimizing the effect of the square window on the signal spectrum is achieved by increasing the duration $[-n_0, n_0]$ of the window, because this leads to a reduction in the width of the lobes of the window spectrum. A longer window causes less distortion in the signal spectrum, while an infinite window causes no effect on the signal spectrum, however this no longer causes signal windowing, as we have discussed for continuous-time signals.

📖 Example 10

We sample the analog signal $x_\alpha(t) = 1 + \cos(15\pi t)$with a sampling period $T_s = 0{,}1\ sec$ and pass it through a low-pass filter with a cutoff frequency $f_c = 2{,}5\ Hz$. What is the signal produced at the output of the filter?

Answer: The analog signal contains a DC component with zero frequency and a cosine component with frequency $2\pi F = 15\pi \Rightarrow F = 7{,}5\ Hz$, which is also the maximum frequency (F_{max}) of the analog signal. The sampling frequency is $f_s = 1/T_s = 1/0{,}1\ sec\ = 10\ Hz$. Since, it follows that the $f_s = 7{,}5 < 10 = 2F_{max}$ Nyquist criterion is not satisfied, so frequency folding will appear for those frequencies that are outside the spectral range defined based on the sampling frequency, i.e. the spectral range $[-f_s/2, f_s/2] =$

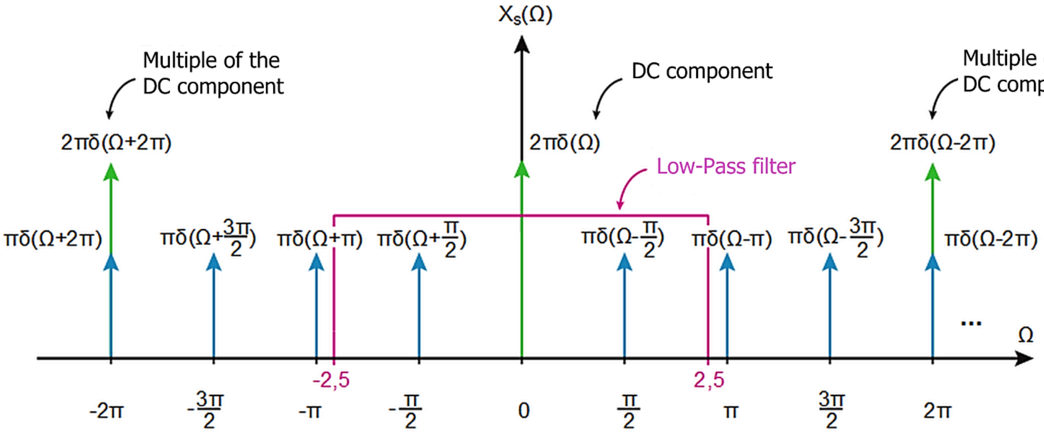
$[-5Hz, 5Hz]$. Therefore, the frequency $F = F_{max} = 7{,}5\ Hz$ of the signal will be folded and show the aliased frequency $F' = F - kf_s = 7{,}5 - k10 = 7{,}5 - 1x10 = -2{,}5\ Hz$. So, the $\cos(15\pi t)$7.5 Hz cosine component of the analog signal, when sampled will look like a 2.5 Hz cosine. The DC component (DC component) is unaffected by sampling. Based on the above, the discrete-time signal resulting from the sampling is:

$$x_s[n] = x_a(t)|_{t=nTs=n/10} = 1 + \cos\left(\frac{15\pi}{10}n\right)$$
$$= 1 + \cos\left(\frac{3\pi}{2}n\right) = 1 + \cos\left(2\pi - \frac{\pi}{2}\right)n = 1$$
$$+ \cos\left(\frac{\pi n}{2}\right)$$

To calculate the signal at the output of the low-pass filter, we need to obtain the spectral form of the discrete-time signal. The DTFT of the sampled signal $x_s[n]$ is given by equation (6.6) and is:

$$X_s(e^{j\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\,(\Omega - k\Omega_s)$$
$$= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - k\Omega_s)$$
$$+ \pi\left[\delta\left(\Omega - k\Omega_s - \frac{\pi}{2}\right) + \delta\left(\Omega - k\Omega_s + \frac{\pi}{2}\right)\right]$$

where $\Omega_s = 2\pi\ T_s$. The spectrum of the discrete-time signal and the spectrum of the low-pass filter are represented in the figure.



Spectrum of a discrete-time signal and spectrum of the low-pass filter.

We notice that the only components of the signal spectrum that come out of the low-pass filter are:

$$\hat{X}_s(\Omega) = 2\pi\delta(\Omega) + \pi\left[\delta\left(\Omega - \frac{\pi}{2}\right) + \delta\left(\Omega + \frac{\pi}{2}\right)\right]$$

Fourier transform we find that the analog signal produced at the output of the low-pass filter is:

$$\hat{x}_s(t) = 1 + \cos\left(\frac{\pi t}{2}\right)$$

5. Inverse DTFT

Example 11

To find the discrete-time signal $x[n]$with DTFT the function:

$$X(e^{j\omega}) = \frac{1}{1 + 0.2\, e^{-j\omega} - 0.35\, e^{-2j\omega}}$$

Answer: We factor the denominator of the function $X(e^{j\omega})$ and we have:

$$X(e^{j\omega}) = \frac{1}{1 + 0.2\, e^{-j\omega} - 0.35\, e^{-2j\omega}} = \frac{1}{(1 - 0.5e^{-j\omega})} \frac{1}{(1 + 0.7e^{-j\omega})}$$
$$= \frac{A}{(1 - 0.5e^{-j\omega})} + \frac{B}{(1 + 0.7e^{-j\omega})}$$

We calculate the coefficients A and B from the relations:

$$A = \frac{1}{(1 - 0.5e^{-j\omega})} \frac{1}{(1 + 0.7e^{-j\omega})} (1 - 0.5e^{-j\omega}) \Big|_{e^{-j\omega}=2} = \frac{5}{12}$$
$$B = \frac{1}{(1 - 0.5e^{-j\omega})} \frac{1}{(1 + 0.7e^{-j\omega})} (1 + 0.7e^{-j\omega}) \Big|_{e^{-j\omega}=-10/7} = \frac{12}{7}$$

So, the function is written:

$$X(e^{j\omega}) = \left(\frac{5}{12}\right) \frac{1}{1 - 0.5e^{-j\omega}} + \left(\frac{12}{7}\right) \frac{1}{1 + 0.7e^{-j\omega}}$$

Using Table 10.1 we find the inverse DTFT:

$$x[n] = \left[\frac{5}{12} (0.5)^{-n} + \frac{12}{7} (-0.7)^{-n}\right] u[n]$$