University of the Peloponnese
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DIGITAL SIGNAL PROCESSING

Solved Examples

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SET #5 – Z Transformation

- Bilateral Z-Transform and Region of Convergence
- Equation of Z Transformation to other Transformations
- Transformation Properties G
- Poles and Zeros

1. Bilateral Z-Transform and Region of Convergence

Example 1

Calculate the Z-transform of discrete-time signals of infinite duration:

(a) $x_1[n] = a^n u[n],\ 0 < |a| < \infty$,

(b) $x_2[n] = -b^n u[-n-1],\ 0 < |b| < \infty$ and

(c) $x[n] = x_1[n] + x_2[n]$

Answer: (a) The signal $x_1[n]$ is causal (right-sided) sequence and has values only for positive - time. The bilateral Z-transform is:

$$X_1(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} a^n\, z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

The function $X_1(z)$ converges when $|az^{-1}| < 1 \Rightarrow |z| > |a|$. So the region of convergence is the outer surface of a circle defined by the set of points for which R_{x1}: $|z| > |a|$. That is:

$$R_{x1}: |\alpha| < |z| < \infty$$

Also, there is a pole for $z = a$ and a zero for $z = 0$.

(b) The signal $x_2[n]$ is anti-causal (left-sided) sequence and has values only for negative - time. The bilateral Z-transform is:

$$X_2(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} -b^n\, u[-n-1]\, z^{-n} = -\sum_{n=-\infty}^{-1} (bz^{-1})^n$$

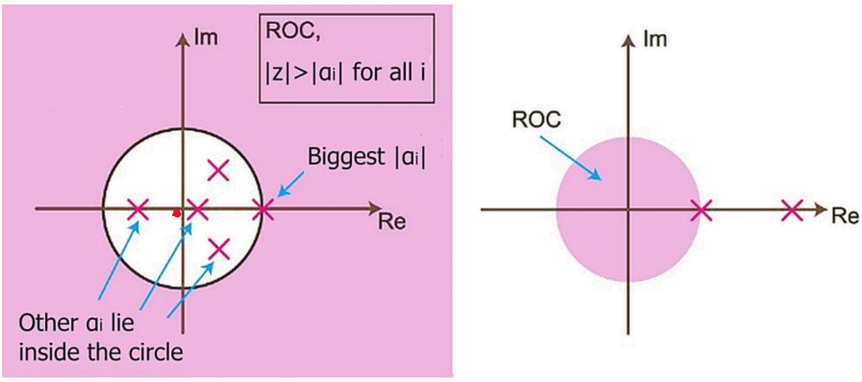
We set $m = -n$ and have:

$$X_2(z) = -\sum_{m=1}^{\infty}(b^{-1}z)^m = 1 - \sum_{m=1}^{\infty}(b^{-1}z)^m = 1 - \frac{1}{1 - bz^{-1}} = \frac{z}{z - b}$$

The function $X_2(z)$ converges when $|b^{-1}z| < 1 \Rightarrow |z| < |b|$. So the region of convergence is the interior surface of a circle defined by the set of points for which $R_{x2}: |z| < |b|$. That is:

$$R_{x2}: 0 < |z| < |b|$$

Also, there is a pole for $z = b$ and a zero for $z = 0$.



Regions of convergence (ROC) of sequences $x_1[n]$ and $x_2[n]$.

We notice that if in the above sequences we put $a = b$, then while the sequences will be different $x_1[n] \neq x_2[n]$, the functions of the Z-transforms will be the same, that is $X_1(z) = X_2(z)$, but with different regions of convergence ($R_{x1} \neq R_{x2}$). Therefore, the complete calculation of the Z-transform requires, not only the calculation of the function $X(z)$, but also the determination of the region of convergence (ROC).

(c) The signal $x[n]$ is the sum $x_1[n] + x_2[n] = a^n u[n] - b^n u[-n-1]$ and is called a two - side sequence or non-causal. The Z-transform is:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=-\infty}^{-1} b^n z^{-n}$$
$$= \left\{\frac{z}{z-a}, R_{x1}: |z| > |a|\right\} + \left\{\frac{z}{z-b}, R_{x2}: |z| < |b|\right\}$$
$$= \frac{z}{z-a} + \frac{z}{z-b}, R_x = R_{x1} \cap R_{x2}$$

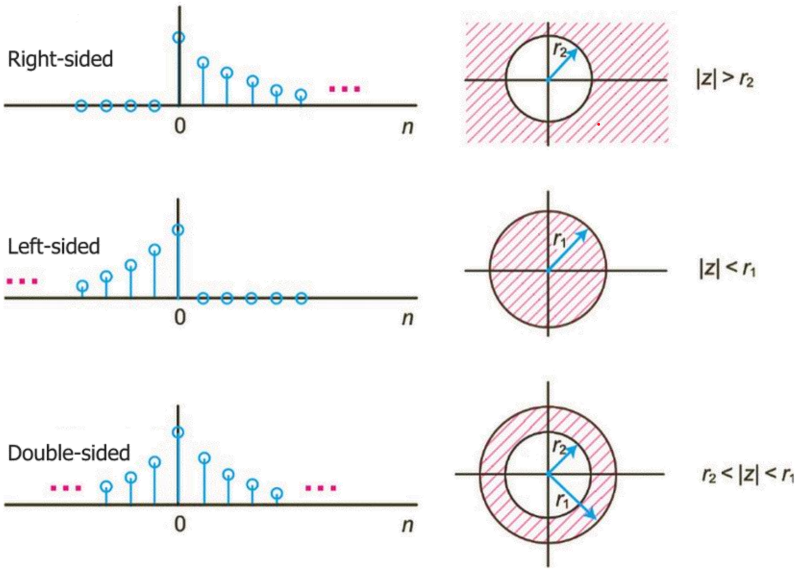
- If $|b| < |a|$, then the region of convergence R_x does not exist, because the intersection of the regions of convergence R_{x1} and R_{x2} is the empty set.
- If $|a| < |b|$, then the region of convergence is $R_x: |a| < |z| < |b|$.

Comments: From the solution of the above example it follows that for signals **of infinite duration** the convergence region is distinguished in the following cases:

- **Right side signals (causal)**: the region of convergence (ROC) is the exterior of a circle with radius R_{x-} the maximum radius of the poles of $X(z)$ or $|z| > R_{x-}$.

- **Right side signs (anti-causal):** The region of convergence (ROC) is the inner circle with radius R_{x+} the minimum radius of the poles of $X(z)$ or $|z| < R_{x+}$.
- **Double-sided signals (non-causal):** The region of convergence (ROC) is the interior of a ring with an inner radius R_{x-} and an outer radius R_{x+}, which correspond to the maximum and minimum radius of its poles $X(z)$, i.e. holds $R_{x-} < |z| < R_{x+}$.

The region of convergence of infinite sequences is shown in the next figure.



Regions of convergence of sequences of infinite duration

Example 2

Find the region of convergence (ROC) of the Z transform. without directly calculating it, $X(z)$, for the following signals of infinite duration:

(a) $x_1[n] = [(0.5)^n + (0.25)^n]u[n]$ (b) $x_2[n] = 3^n u[-n]$

Answer: (a) Since $x_1[n]$ is a right-sided sequence, the region of convergence is the outer surface of a circle with radius the maximum radius of the poles. The poles are: $z_1 = 0.5$ derived from the term $(0.5)^n$, and $z_2 = 0.25$ derived from the term $(0.25)^n$. Therefore, the region of convergence is R_x: $|z| > 0.5$.

(b) Since $x_2[n]$ is a left-sided sequence, the region of convergence covers the interior surface of a circle with radius the minimum radius of the poles. The function has a pole at $z = 3$. Therefore, the region of convergence is R_x: $|z| < 3$.

2. Relationship between Z-Transform and other Transforms

Example 3

The Z-transform of a sequence $x[n]$ is:

$$X(z) = \frac{z + 2z^{-2} + z^{-3}}{1 - 3z^{-4} + z^{-5}}$$

If the region of convergence includes the unit circle, find the DTFT of $x[n]$, για $\omega = \pi$.

Answer: If $X(z)$is the Z -transform of the sequence $x[n]$ and the unit circle is inside the region of convergence, then its $x[n]$ DTFT can be found by computing it $X(z)$ on the unit circle, that is:

$$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$$

We remind you that $e^{j\omega} = cos\omega + jsin\omega$: Therefore for $\omega = \pi$ we have $e^{j\pi} = cos\pi + jsin\pi = -1$ and the DTFT at the point $\omega = \pi$, is:

$$X(e^{j\omega})\big|_{\omega=\pi} = X(z)|_{z=e^{j\pi}} = X(z)|_{z=-1}$$

Therefore we have:

$$X(e^{j\omega})\big|_{\omega=\pi} = \left.\frac{z + 2z^{-2} + z^{-3}}{1 - 3z^{-4} + z^{-5}}\right|_{z=-1} = \frac{-1 + 2 - 1}{1 - 3 - 1} = 0$$

3. Properties of Z-Transform

Example 4

Assuming known the Z- transform of the sequence $x[n]$, find the Z- transform of the sequence:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Answer: The given equation $y[n] = \sum_{k=-\infty}^{n} x[k]$ can be written $y[n] = y[n-1] + x[n]$. Therefore:

$$x[n] = y[n] - y[n-1]$$

If we transform both members of the equation and use the time-shift property of the Z transform, we find:

$$X(z) = Y(z) - z^{-1}Y(z) \Rightarrow X(z) = Y(z)[1 - z^{-1}]$$

We solve in terms of $Y(z)$:

$$Y(z) = \frac{1}{1 - z^{-1}} X(z)$$

Therefore:

$$y[n] = \sum_{k=-\infty}^{n} x[k] \overset{Z}{\leftrightarrow} \frac{1}{1 - z^{-1}} X(z)$$

This equation is also referred to as **the property of overlap.**

Example 5

Calculate the Z-transform of the discrete-time signal:

$$x[n] = (0.4)^{-n}\, u[-n]$$

Answer: We know that:

$$(0.4)^n\, u[n] \overset{Z}{\leftrightarrow} \frac{1}{1-0.4z^{-1}}, \qquad R_x: |z| > 0.4$$

From the time inverse property it follows:

$$(0.4)^{-n}\, u[-n] \overset{Z}{\leftrightarrow} \frac{1}{1-0.4(\mathrm{z}^{-1})^{-1}} = \frac{1}{1-0.4\, z}$$

and region of convergence (ROC):

$$R'_x = 1/R_x \ = |z| < 1/0.4$$

Example 6

Calculate the Z-transform of the discrete-time signal:

$$x[n] = (0.4)^{n/2}\, u[n/2]$$

Answer: We know that:

$$(0.4)^n\, u[n] \overset{Z}{\leftrightarrow} \frac{1}{1-0.4z^{-1}}, \qquad R_x: |z| > 0.4$$

From the time scaling property it follows:

$$(0.4)^{n/2}\, u[n/2] \overset{Z}{\leftrightarrow} \frac{1}{1-0.4(\mathrm{z}^{-1})^2} = \frac{1}{1-0.4\, z^{-2}} = \frac{1}{1-0.2\, z^{-1}} \cdot \frac{1}{1+0.2\, z^{-1}}$$

Example 7

Calculate the Z-transform of the discrete-time signal:

$$x[n] = \left(3e^{j\pi}\right)^n (0.4)^n\, u[n]$$

Answer: We know that:

$$(0.4)^n\, u[n] \overset{Z}{\leftrightarrow} \frac{1}{1-0.4z^{-1}}, \qquad R_x: |z| > 0.4$$

From the complex frequency scaling property it follows:

$$\left(e^{j\pi}\right)^n (0.4)^n\, u[n] \overset{Z}{\leftrightarrow} \frac{1}{1-0.4(\mathrm{z}/3e^{j\pi})^{-1}} = \frac{1}{1-1.2\, e^{j\pi}\, z^{-1}}$$

The region of convergence (ROC) is:

$$R'_x: \left|3e^{j\pi}\right||z| > 0.4 \Rightarrow\ |z| > 1.2$$

and region of convergence (ROC):

$$R'_x = R_x^{1/2}: |z| > \sqrt{0.4} = 0.2$$

📖Example 8

Using the Z-transform calculate the convolution between the sequences $x[n] = \{\hat{1}, -2, 0, 3, -1\}$ and $h[n] = \{2, \hat{3}, 0, 1\}$.

Answer: We calculate the Z-transform of each sequence using the time shift property and we have:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n} = \sum_{n=0}^{4} x[n]\, z^{-n} = 1 - z^{-1} + 3z^{-3} - z^{-4}$$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n} = \sum_{n=-1}^{2} h[n]\, z^{-n} = 2z + 3 + z^{-2}$$

Based on the convolution property we have:

$$\begin{aligned} Y(z) = X(z)\, H(z) &= (1 - 2z^{-1} + 3z^{-3} - z^{-4})(2z + 3 + z^{-2}) \\ &= 2z + 3 + z^{-2} - 4 - 6z^{-1} - 2z^{-3} + 6z^{-2} + 9z^{-3} + 3z^{-5} - 2z^{-3} \\ &- 3z^{-4} - z^{-6} \\ &= 2z - 1 - 6z^{-1} + 7z^{-2} + 5z^{-3} - 3z^{-4} + 3z^{-5} - z^{-6} \\ &= \sum_{n=-1}^{6} y[n]\, z^{-n} \end{aligned}$$

Using the time shift property once more, we get the result:

$$y[n] = \{2, -\hat{1}, -6, 7, 5, -3, 3, -1\}$$

📖 Example 9

Calculate the Z-transform of the discrete-time signal:

$$x[n] = -\, n\, (0.4)^n\, u[n]$$

Answer: We know that:

$$(0.4)^n\, u[n] \overset{z}{\leftrightarrow} \frac{1}{1 - 0.4z^{-1}}, \qquad R_x: |z| > 0.4$$

From the derivation property at frequency z it follows:

$$-n\, (0.4)^n\, u[n] \overset{z}{\leftrightarrow} z \left(\frac{1}{1 - 0.4\, z^{-1}} \right)' = \frac{-0.4}{(1 - 0.4z^{-1})^2}$$

and region of convergence (ROC):

$$R'_x = R_x: |z| > 0.4$$

📖 Example 10

A discrete-time causal signal has a Z-transform given by the equation:

$$X(z) = \frac{1}{1 - az^{-1}}$$

Calculate the value of the signal $x[n]$ at position $n = 0$ and for$n \to \infty$.

Answer: For $x[0]$ from the initial value theorem it follows:

$$x[0] = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{1}{1 - az^{-1}} = 1$$

📖 Example 11

A discrete-time causal signal has a Z-transform given by the equation:

$$X(z) = \frac{4z^2 + 3z + 1}{(z-1)(z+2)^2}$$

Calculate the value of the signal $x[n]$ at position $n = 0$ and for $n \to \infty$.

Answer: For $x[\infty]$ from the final value theorem it follows:

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1}(z-1)\frac{4z^2 + 3z + 1}{(z-1)(z+2)^2}$$
$$= \lim_{z\to 1}\frac{4z^2 + 3z + 2}{(z+2)^2} = \frac{4+3+1}{3^2} = \frac{8}{9}$$

4. Poles and Zeros

📖 Example 12

Draw the pole and zero diagram of the function:

$$X(z) = \frac{2z^2 + 3z}{z^2 + 0.4z + 1}$$

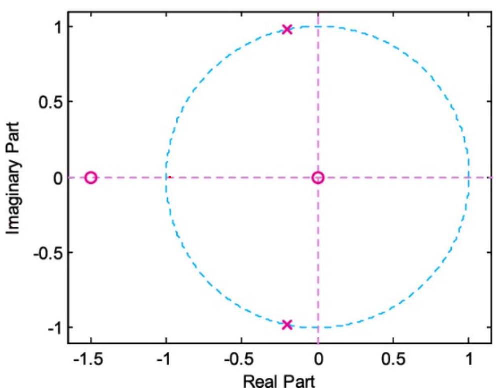
Answer: To find the zeros we calculate the roots of the numerator. Is:

$$2z^2 + 3z = 0 \Rightarrow z(2z+3) = 0 \Rightarrow z_1 = 0 \text{ και } z_2 = -3/2$$

To find the poles we calculate the roots of the denominator. Is:

$$z^2 + 0.4z + 1 = 0 \Rightarrow p_{1,2} = -0.20 \pm j0.98$$

We observe that there are two complex conjugate poles $p_{1,2} = -0.20 \pm j0.98$, placed on the unit circle and two zeros placed at the positions $z_1 = 0$ and $z_2 = -1.5$.



Pole-zero diagram