University of the Peloponnese
Electrical and Computer Engineering Department

DIGITAL SIGNAL PROCESSING

Solved Examples

Teacher's Examples: M. Paraskevas

SET #2 – Discrete Time Signals

- Periodic Discrete-Time Signals
- Even and Odd discrete-time signals
- Energy Signals and Power Signals
- Complex Exponential Sequence
- Sine Sequence

1. Periodic Discrete-Time Signals

Example 1

The sampling of a continuous-time signal $x(t) = sin(t + \pi/6), -\infty < t < \infty$ with a sampling period $T_s = 1\ sec$.

(a) To check if the signal $x[n]$ is periodic and if positive to calculate its fundamental period.

(b) In the negative case, determine the values of the sampling period that satisfy the Nyquist criterion and make the signal periodic $x[n]$.

Answer: The discrete-time signal $x[n]$ is derived by sampling it $x(t)$ according to the equation:

$$x[n] = x(t)|_{t=nT_s} = sin(n + \pi/6)$$

and has frequency $\omega = 1\ rad$. This value cannot be expressed in the form $2\pi m/N$ with positive integers m that $N$are not divisible by each other, due to the existence of the implicit π. Therefore, the signal $x[n]$ is not periodic.

The frequency of the continuous-time signal $x(t) = sin(t + \pi/6)$is $\Omega_0 = 1\ rad/sec$, so the sampling period according to the Nyquist criterion is:

$$T_s \leq \frac{\pi}{\Omega_0} = \pi$$

If we sample the continuous-time signal with a sampling period T_s we get the discrete-time signal:

$$x[n] = x(t)|_{t=nT_s} = sin(nT_s + \pi/6)$$

For this signal to be periodic, the equation must be satisfied:

$$sin((n + N)T_s + \pi/6) = sin(nT_s + \pi/6)$$

that is, it is necessary for it to be valid $NT_s = 2k\pi$ for an integer k. Thus, the sampling period $T_s = 2k\pi/N \leq \pi$ must satisfy the Nyquist criterion and at the same time ensure the periodicity of the signal. For example, if we want a sine wave with a fundamental period $N = 10$, then $T_s = k\pi/5$ for a value of k that satisfies the Nyquist criterion

$$0 < T_s = \frac{k\pi}{5} \leq \pi \Rightarrow 0 < k \leq 5$$

From all possible values of k we choose the values 1 and 3 that are not divisible by $N = 10$ (we exclude 2 and 4 because they are divisible by 10). For $k = 1$, which satisfies the Nyquist criterion and produces the discrete-time signal, it follows: $T_s = \pi/5 < \pi$

$$x[n] = sin(n\pi/5 + \pi/6) = sin\left(\frac{2n\pi}{10} + \frac{\pi}{6}\right)$$

Accordingly $T_s = 3\pi/5 < \pi$, for $k = 3$, which satisfies the Nyquist criterion and produces the discrete-time signal, we obtain:

$$x[n] = sin(3n\pi/5 + \pi/6) = sin\left(\frac{2\pi\ x\ 3}{10} + \frac{\pi}{6}\right)$$

Comments: When we sample a continuous time sinusoidal signal:

$$x(t) = A\,cos(\Omega_0 t + \theta),\ -\infty < n < \infty$$

we get a **periodic discrete-time sinusoidal signal:**

$$x[n] = A\,cos(\Omega_0 n T_s + \theta) = Acos\left(\frac{2\pi T_s}{T_0}n + \theta\right)$$

only if:

$$\frac{T_s}{T_0} = \frac{m}{N}$$

where m and $N$are positive integers that are not divisible by each other. In order for the phenomenon of frequency folding not to appear, the sampling period must also satisfy the Nyquist criterion:

$$T_s \leq \frac{\pi}{\Omega_0} = \frac{T_0}{2}$$

2. Even and Odd Discrete Time Signals

🕮 Example 2

To calculate the even and the odd part of the discrete-time signal $x[n] = u[n]$

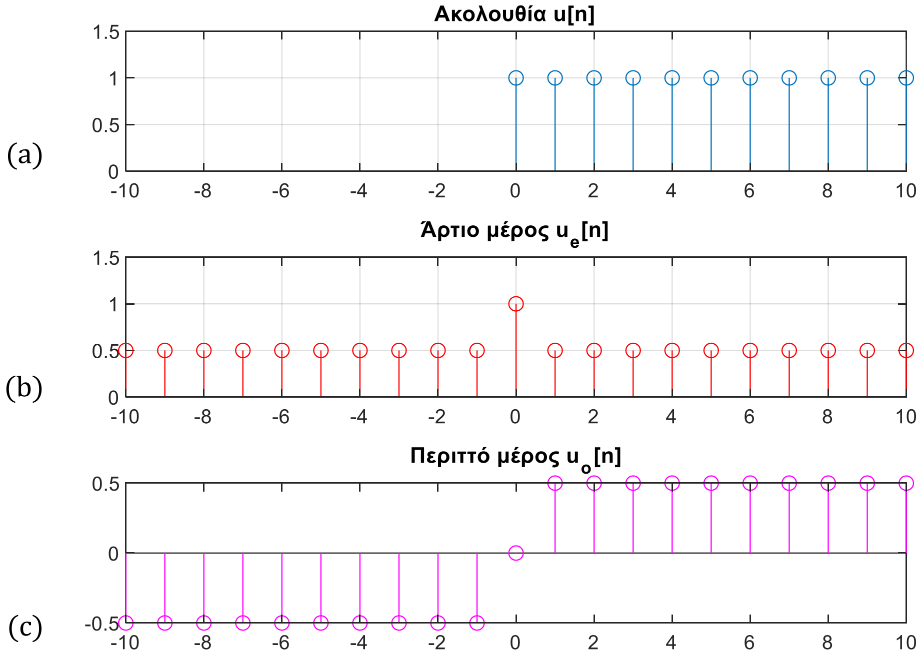
Answer: The even part is given by the equation:

$$x_e[n] = \frac{1}{2}\big[x[n] + x[-n]\big] = \frac{1}{2}[u[n] + u[-n]] = \begin{cases} 1, n = 0 \\ \frac{1}{2}, n \neq 0 \end{cases} = \frac{1}{2} + \delta[n]$$

The odd part is given by the equation:

$$x_o[n] = \frac{1}{2}\big[x[n] - x[-n]\big] = \frac{1}{2}[u[n] - u[-n]] = \begin{cases} \frac{1}{2}, n > 0 \\ 0, n = 0 \\ -\frac{1}{2}, n < 0 \end{cases} = \frac{1}{2}\ \mathrm{sgn}(n)$$

where $\text{sgn}(n)$ is the sign function, which returns: $+1$ when $n > 0$, 0 when $n = 0$ and 1 when $n < 0$.



(a) Unit step sequence $u[n]$, (b) Even part $u_e[n]$, (c) Odd part$u_o[n]$

Example 3

Find the conjugate symmetric (even) and the conjugate antisymmetric (odd) part of the complex signal $x[n] = je^{\,j\pi/4}$.

Answer: Its conjugate symmetric part $x[n]$ is:

$$x_e[n] = \frac{1}{2}[x[n] + x^*[-n]] = \frac{1}{2}\left[je^{j\pi/4} - je^{j\pi/4}\right] = 0$$

The antisymmetric conjugate is:

$$x_e[n] = \frac{1}{2}[x[n] - x^*[-n]] = \frac{1}{2}\left[je^{j\pi/4} + je^{j\pi/4}\right] = je^{j\pi/4}$$

So, the signal is conjugate antisymmetric (odd).

3. Energy Signals and Power Signals

Example 4

To examine whether the signal $x[n] = 0.5^n u[n]$ is an energy or power signal or both.

Answer: The signal energy $x[n]$ is calculated from the equation:

$$E_x = \sum_{n=-\infty}^{\infty} 0.5^n u[n] = \sum_{n=0}^{\infty} 0.5^n = \frac{1}{1-0.5} = \frac{1}{0.5} = 2$$

So it is an energy signal. The average signal strength is zero, as follows from the equation:

$$P = \lim_{N\to\infty} \frac{1}{2N+1} E_x = \lim_{N\to\infty} \frac{1}{2N+1} 2 = 0$$

4. Complex Exponential Sequence

📖 Example 5

Calculate for which values of the parameters β, Ω_0 και T_s a discrete-time signal $y[n] = a^n \cos(n\omega_0), n \geq 0$ is obtained from the continuous-time signal $x(t) = e^{-\beta t} \cos(\Omega_0 t)\, u(t)$.

Answer: If we sample the continuous time signal with a sampling period T_s we get:

$$x(nT_s) = x(t)|_{t=nT_s} = e^{-\beta n T_s} \cos(\Omega_0 n T_s)\, u[n] = \left(e^{-\beta T_s}\right)^n \cos\left((\Omega_0 T_s) n\right) u[n]$$

Comparing the sampled signal $x(nT_s)$ with $y[n]$ we find that they are equal when:

$$\alpha = e^{-\beta T_s} \ (1) \quad \text{and} \quad \Omega_0 T_s = \omega_0 \ (2)$$

In these equations we have two known parameters α *and* ω_0 and three unknown ones β, Ω_0 και T_s. So, no unique solution can emerge. However according to the Nyquist criterion the sampling period must satisfy the equation:

$$T_s \leq \frac{\pi}{\Omega_{max}}$$

Assuming that the maximum frequency is $\Omega_{max} = N\Omega_0$, για $N \geq 2$ we have:

$$T_s = \frac{\pi}{N\Omega_0}$$

Substituting it T_s into equations (1) and (2) we get:

$$\alpha = e^{-\beta\pi/N\Omega_0} \ (3) \quad \text{and} \quad \omega_0 = \frac{\pi}{N} \ (4)$$

From equation (4) we have $N = \pi/\omega_0$. Substituting N into equation (3), we solve for β and find:

$$\beta = -\frac{\Omega_0}{\omega_0} \log a$$

Considering $\Omega_0 = 2\pi, \omega_0 = \pi$ and $\alpha = 0.8$, we find $\beta = -2\log 0.8$ and $T_s = \omega_0/\Omega_0 = \pi/2\pi = 0.5$.

5. Sine Sequence

📖 Example 6

To examine whether sinusoidal sequences of infinite duration ($-\infty < n < \infty$) are periodic:

(a) $x_1[n] = sin(0.1\pi n)$ (b) $x_2[n] = sin(0.2\pi n)$

(c) $x_3[n] = sin(0.6\pi n)$ (d) $x_4[n] = sin(0.7\pi n)$

Can these sequences be sampled versions of the corresponding continuous-time functions?

Answer: The given sequences are written:

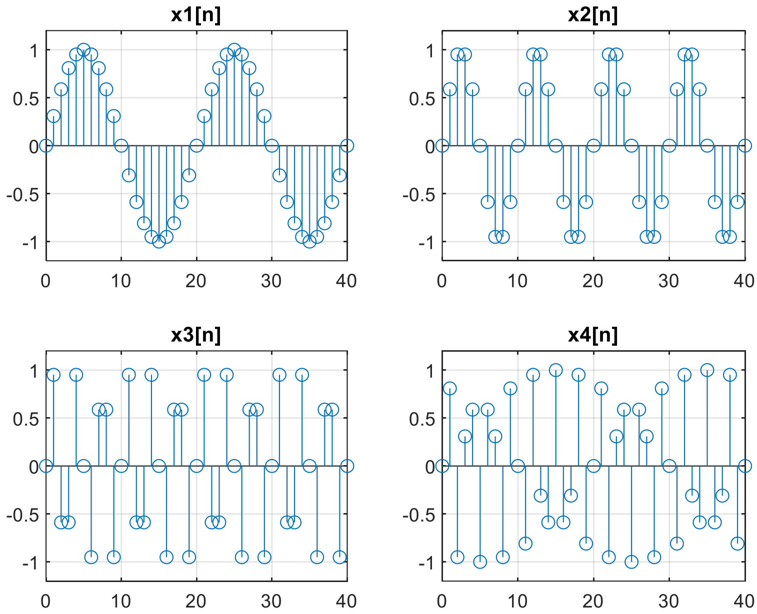
$$x_1[n] = sin(0.1\pi n) = \sin\left(\frac{2\pi}{20} n\right)$$

$$x_2[n] = sin(0.2\pi n) = \sin\left(\frac{2\pi}{20} 2n\right)$$

$$x_3[n] = sin(0.6\pi n) = \sin\left(\frac{2\pi}{20} 6n\right)$$

$$x_4[n] = sin(0.7\pi n) = \sin\left(\frac{2\pi}{20} 7n\right)$$

Therefore, the sequences are periodic and harmonically connected to each other.



Sine sequence for different frequency values of f_0

In the figure above, their graphical representations are shown. Plotting the four sequences shows that the first two, namely $x_1[n]$ and $x_2[n]$ are the sampled versions of the corresponding continuous-time functions. But this does not apply to the sequences $x_3[n]$ and $x_4[n]$. It would be wrong to assume that this is due to a violation of the Nyquist rule, i.e. due the incorrect sampling rate.

Let's explain why this happens: To obtain the discrete sequence we must $sin(\omega_0 n)$ sample the $T_s = 1$ continuous-time function with a sampling period $sin(\Omega_0 t)$ according to the Nyquist condition:

$$T_s = 1 \leq \frac{\pi}{\Omega_0}$$

where π/Ω_0 is the maximum allowed value of the sampling period for which the aliasing does not occur. For the sequence $x_3[n] = sin(0.6\pi n) = sin(0.6\pi t)|_{t=nT_s=n}$ when $T_s = 1$, it holds:

$$T_s = 1 \leq \frac{\pi}{0.6\,\pi} \approx 1{,}66$$

Conversely, in the case of the sequence $x_2[n]$ we have: $x_2[n] = sin(0.2\pi n) = sin(0.2\pi t)|_{t=nT_s=n}$when $T_s = 1$, then we have:

$$T_s = 1 \leq \frac{\pi}{0.2\,\pi} = 5$$

Therefore, generating the sequence $x_2[n]$ is done by taking a larger number of samples from the function $sin(0.2\pi t)$, than generating the sequence $x_3[n]$ from the function $sin(0.6\pi t)$, using in both cases the same sampling period. This results in the sequence $x_2[n]$being more like an analog sine wave than $x_3[n]$, but in both cases no aliased frequency occurs.

Comments:

- The analog frequency Ω of analog sines varies in the range $[0, \infty)$, while the discrete (digital) frequencies ω are radial and vary in the range $[0, \pi]$.
- Negative frequencies are needed in the analysis of real - valued signals and thus we end up with frequency ranges: (a) for continuous-time signals: $-\infty < \Omega < \infty$ and (b) for discrete-time signals: $-\pi < \omega \leq \pi$.

📖 Example 7

Check whether the following sequences of infinite duration are periodic:

(a) $x_1[n] = e^{j(\pi n/2+\pi/4)}$ (b) $x_2[n] = e^{-j\pi n/8} + e^{-jn/2}$ (c) $x_3[n] = e^{-j\pi n/2} + e^{-j\pi n/2}$

If positive, calculate the period.

Answer: (a) From the Euler equation we have:

$$x_1[n] = \cos(\pi n/2 + \pi/4) + j\sin(\pi n/2 + \pi/4)$$

The frequency of the cosine and the sine is $\omega_1 = \pi/2 = (1/4)2\pi$, that is, $m = 1$ and $N = 4$, so the frequency is expressed as a quotient of positive integers that are not divisible by each other. So the given sine and cosine and their sum are periodic sequences with fundamental period $N = 4$ samples.

(b) We have:

$$x_2[n] = e^{-j\pi n/8} + e^{-jn/2} = \cos(\pi n/8) - j\sin(\pi n/8) + \cos(n/2) - j\sin(n/2)$$

The frequency of the first cosine and sine pair can be expressed as a quotient of integers that are not divided by each other. But, this is not true for the frequency of the second cosine and sine, for which is $\omega_2 = n/2 = (1/4\pi)2\pi$, that is, $m = 1$and $N = 4\pi$. Since N is an irrational number, it follows that the second pair is not periodic, so the overall sequence is not periodic.

(c) The sequence is written:

$$x_3[n] = e^{-j\pi n/4} + e^{-j\pi n/4} = \frac{1}{2}\cos\left(\frac{\pi n}{4}\right)$$

Since $\omega_3 = \pi/4 = (1/8)\ 2\pi$, that is, $m = 1$and $N = 8$, the sequence is periodic with fundamental period $N = 16$ samples.