University of the Peloponnese
Electrical and Computer Engineering Department

DIGITAL SIGNAL PROCESSING

Solved Examples

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SET #1 – Analog to Digital Signal Conversion

- Sampling
- Quantization

1. Sampling

Example 1

If the Nyquist rate for the signal $x(t)$ is Ω_s, what is the Nyquist rate for the signals:

(a) $y(t) = dx(t)/dt$

(b) $y(t) = x(t)\cos(\Omega_0 t)$

(c) $y(t) = x(t)\, x(t)$

(d) $y(t) = x(t) * x(t)$

Answer: (a) From the derivative property of the Fourier transform (see Table 4.1) we know that:

$$Y(\Omega) = j\Omega\, X(\Omega)$$

Since no change in the frequency field results from the above equation, we conclude that the Nyquist rate of the signal $y(t)$ is the same as that of the signal $x(t)$.

(b) We know that when we multiply a signal $x(t)$ with a sinusoidal function $\cos(\Omega_0 t)$, a modulation occurs and the signal spectrum $x(t)$ is shifted by $\pm\Omega_0$, as its $y(t)$ Fourier transform is:

$$Y(\Omega) = \frac{1}{2}[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$$

Therefore, the Nyquist rate of the signal $y(t)$ is $\Omega_s + 2\Omega_0$.

(c) From the time multiplication property of the Fourier transform (see Table 4.1) we know that:

$$Y(\Omega) = \frac{1}{2\pi}\,[X(\Omega) * X(\Omega)]$$

We also know that the convolution of two functions defined on finite intervals of the independent variable (frequency Ω, in the present case) produces a function defined on an interval that is the sum of the edges of the intervals of definition of the original functions. Since signal $x(t)$ is defined in the interval $[-\Omega_s, \Omega_s]$, it follows that signal $x(t)$ is defined in the interval $[-2\Omega_s, 2\Omega_s]$. Consequently, the Nyquist rate of the signal $y(t)$ is four times that of the signal $x(t)$.

(d) From the convolution property of the Fourier transform (see Table 4.1) we know that:

$$Y(\Omega) = X(\Omega)X(\Omega)$$

Therefore, the Nyquist rate of the signal $y(t)$ is twice that of the signal $x(t)$.

Example 2

Calculate the Nyquist rate for the following signals:

(a) $x(t) = sinc^2(100\pi t)$ (b) $x(t) = sinc(100\pi t) * sinc(200\pi t)$

Answer: To answer the questions we need to calculate the maximum frequency of each signal. We will use the Fourier transform and the findings:

$$rect\left(\frac{t}{T}\right) \overset{F}{\leftrightarrow} T\, sinc(fT) \qquad T\, sinc(tT) \overset{F}{\leftrightarrow} rect\left(\frac{f}{T}\right)$$

$$tri\left(\frac{t}{T}\right) \overset{F}{\leftrightarrow} T\, sinc^2(fT) \qquad T\, sinc^2(tT) \overset{F}{\leftrightarrow} tri\left(\frac{f}{T}\right)$$

Its $x(t) = sinc^2(100\pi t)$ Fourier transform is:

$$sinc^2(100\pi t) \overset{F}{\leftrightarrow} tri\left(\frac{f}{100\pi}\right)$$

Therefore, the spectrum is triangular with range $[-100\pi, 100\pi]$. Since the maximum frequency of the signal is it follows that the 100π Nyquist rate is 200π.

Its $sinc(100\pi t) * sinc(200\pi t)$ Fourier transform is:

$$sinc(100\pi t) * sinc(200\pi t) \overset{F}{\leftrightarrow} \frac{1}{100\pi} rect\left(\frac{f}{100\pi}\right) \frac{1}{200\pi} rect\left(\frac{f}{200\pi}\right)$$

Therefore, the spectrum is the product of two square spectra with amplitudes $[-50\pi, 50\pi]$ the first and $[-100\pi, 100\pi]$ the second. So, the resulting spectrum width is $[-50\pi, 50\pi]$ and the maximum frequency of the signal is 50π and the Nyquist rate is 100π.

Example 3

An analog signal is created by combining sinusoidal signals at frequencies $f_1 = 15\ Hz, f_2 = 60\ Hz, f_3 = 220\ Hz$ και $f_4 = 310\ Hz$. It is sampled with a frequency of 100 Hz.

(a) Which are the aliased frequencies?

(b) The sampled signal is passed through an ideal low-pass filter with cutoff frequency $f_c = 35\ Hz$. What frequencies will appear in the reconstructed signal?

Answer: (a) Because the sampling frequency $f_s = 100\ Hz$ we used is less than the Nyquist frequency $f_N = 2f_{max} = 2\ x\ 310\ Hz = 620\ Hz$, i.e. the Nyquist criterion is not satisfied, it follows that the aliasing effect will appear. For sampling frequency $f_s = 100\ Hz$ the Nyquist criterion is satisfied only for the sine signal of frequency $f_1 = 15\ Hz$. Therefore, only for this frequency the aliasing phenomenon will not appear, as well as for any other frequencies that existed in the region $[-f_s/2, f_s/2] = [-50Hz, 50\ Hz]$. The frequencies of the remaining sine signals do not satisfy the Nyquist criterion, so they will generate aliased frequencies based on the equation $f_i' = f_i - kf_s$, where $k$an integer is chosen that leads to a result within the range $[-f_s/2, f_s/2]$. So is:

$$f_2' = f_2 - kf_s = 60 - k100 = 60 - 1x100 = -40Hz$$
$$f_3' = f_3 - kf_s = 220 - k100 = 220 - 2x100 = 20Hz$$
$$f_4' = f_4 - kf_s = 310 - k100 = 310 - 3x100 = 10Hz$$

Therefore, the frequencies $f_2 = 60\ Hz, f_3 = 140\ Hz$ και $f_4 = 310\ Hz$ will produce aliasing frequencies within the range $[-50Hz, 50\ Hz]$ namely the frequencies $f_2' = -40\ Hz, f_3' = 20\ Hz$ and $f_4{}' = 10\ Hz$ as well as their mirrors. So, in the frequency domain $[0Hz, 50\ Hz]$ there will be the frequencies $f_4' = 10\ Hz, f_1 = 15Hz, f_3' = 20\ Hz$ and $f_2' = 40\ Hz$.

(b) The ideal low-pass filter with a cut-off frequency $f_c = 35\ Hz$ will pass only the frequencies $f_4' = 10\ Hz, f_1 = 15Hz, f_3' = 20\ Hz$. We notice that only the frequency $f_1 = 15Hz$ was present in the original signal, while the frequencies $f_4' = 10\ Hz$ and $f_3' = 20\ Hz$ are aliased due to low sampling rate.

Example 4

The analog signal is given $x_a(t) = 2cos(200\pi t)$.

(a) Determine the Nyquist frequency and the minimum value of the sampling frequency.

(b) Write the discrete-time signal if the analog signal is sampled with a sampling frequency of 400 Hz. Calculate the frequency of the discrete time signal.

(c) Same as (b) but for a sampling frequency of 150 Hz.

Answer: (a) The frequency of the analog signal is $2\pi f = 200\pi \Rightarrow f = 100\ Hz$. Hence, the Nyquist frequency is $f_N = 2f = 200\ Hz$ and this is the minimum acceptable value of the sampling frequency, for which the frequency aliasing effect will not occur.

(b) For a sampling frequency $f_s = 400\ Hz$ (i.e., sampling period $T_s = 1/400\ sec$), the discrete-time signal is:

$$x(n) = x_a(t)|_{t=nTs} = 2cos\frac{200\pi}{400}n = 2\cos\frac{\pi}{2}n$$

The frequency of this signal can be calculated as follows:

$$\omega = \frac{\pi}{2} \Rightarrow \Omega T_s = \frac{\pi}{2} \Rightarrow 2\pi f\frac{1}{f_s} = \frac{\pi}{2} \Rightarrow f = \frac{f_s}{4} \Rightarrow f = \frac{400}{4} = 100\ Hz$$

We notice that the discrete-time signal has the same frequency as the continuous-time signal, which is due to the choice of sampling frequency that satisfies the Nyquist criterion.

(c) For a sampling frequency $f_s = 150\ Hz$ (i.e., sampling period $T_s = 1/150\ sec$), the discrete-time signal is:

$$x(n) = x_a(t)|_{t=nTs} = 2cos\frac{200\pi}{150}n = 2\cos\frac{4\pi}{3}n$$
$$= 2\cos\left(2\pi - \frac{2\pi}{3}\right)n = 2\cos\frac{2\pi}{3}n$$

The frequency of the signal is:

$$\omega = \frac{2\pi}{3} \Rightarrow \Omega T_s = \frac{2\pi}{3} \Rightarrow 2\pi f\frac{1}{f_s} = \frac{2\pi}{3} \Rightarrow f = \frac{f_s}{3} \Rightarrow f = \frac{150}{3} = 50\ Hz$$

We notice that the discrete-time signal has a different frequency (aliased) than the continuous-time signal, which is due to the improper sampling frequency.

📖 Example 5

(a) The analog signal $x_a(t) = 2cos(20\pi t)\cos(30\pi t) + \sin(40\pi t)$ is sampled at a rate of 20 samples per second. Determine the resulting discrete time signal.

(b) Repeat question (a) for a sampling rate of 50 samples per second.

Answer: (a) We will express the given signal as a sum of sinusoidal functions. The product $cos(20\pi t)\cos(30\pi t)$ is written[1]:

$$2cos(20\pi t)\cos(30\pi t) = \cos(50\pi t) + \cos(10\pi t)$$

So the analog signal is $x_a(t) = \cos(50\pi t) + \cos(10\pi t) + \sin(40\pi t)$ and contains the frequencies $f_1 = 25\ Hz, f_2 = 5\ Hz$ και $f_3 = 20\ Hz$. The Nyquist frequency is $f_N = 2x25\ Hz = 50\ Hz$. The discrete-time signal resulting from sampling with frequency $f_s = 20\ Hz$ ($T_s = 1/20\ sec$), is:

$$x(n) = x_a(t)|_{t=nTs}$$
$$= \cos\left(\frac{50\pi}{20}n\right) + \cos\left(\frac{10\pi}{20}n\right) + \sin\left(\frac{40\pi}{20}n\right)$$
$$= \cos\left(\frac{5\pi}{2}n\right) + \cos\left(\frac{\pi}{2}n\right) + \sin(2\pi n) = \cdots$$
$$= 0$$

The sampling frequency chosen does not satisfy the Nyquist criterion and the frequencies produced resulted in a zero-signal value.

(b) The discrete-time signal resulting from sampling with frequency $f_s = 50\ Hz$ ($T_s = 1/50\ sec$), is:

$$x(n) = x_a(t)|_{t=nTs} = \cos\left(\frac{50\pi}{50}n\right) + \cos\left(\frac{10\pi}{50}n\right) + \sin\left(\frac{40\pi}{50}n\right)$$
$$= \cos(\pi n) + \cos\left(\frac{\pi}{5}n\right) + \sin\left(\frac{4\pi}{5}n\right)$$

The frequency of the component $\cos(\pi n)$ is:

$$\omega_1 = \pi \Rightarrow \Omega_1 T_s = \pi \Rightarrow 2\pi f_1\frac{1}{f_s} = \pi \Rightarrow f_1 = \frac{f_s}{2} \Rightarrow f_1 = \frac{50}{2} = 25\ Hz$$

The frequency of the component $\cos(\pi n/5)$ is:

[1]We used the well-known equation: $cosA\ cosB = (1/2)[\cos(A+B) + \cos(A-B)]$

$$\omega_2 = \frac{\pi}{5} \Rightarrow \Omega_2 T_s = \frac{\pi}{5} \Rightarrow 2\pi f_2 \frac{1}{f_s} = \frac{\pi}{5} \Rightarrow f_2 = \frac{f_s}{10} \Rightarrow f_2 = \frac{50}{10} = 5\ Hz$$

The frequency of the component $\cos(4\pi n/5)$ is:

$$\omega_3 = \frac{4\pi}{5} \Rightarrow \Omega_3 T_s = \frac{4\pi}{5} \Rightarrow 2\pi f_3 \frac{1}{f_s} = \frac{4\pi}{5} \Rightarrow f_3 = \frac{2f_s}{5} \Rightarrow f_3 = \frac{100}{5} = 20\ Hz$$

We notice that the frequencies of the discrete-time signal are the same as the frequencies of the analog signal, which is because the sampling frequency chosen satisfies the Nyquist criterion.

2. Quantization

Example 6

The analog signal is given:

$$x_a(t) = -\frac{3}{2} + cos(100\pi t)\cos(200\pi t) + \frac{1}{2}\sin\left(200\pi t - \frac{\pi}{2}\right) + \cos(300\pi t)$$

(a) Determine the Nyquist frequency and the minimum acceptable value of the sampling frequency.

(b) What frequencies will result if the analog signal is sampled at a sampling frequency of 150 Hz.

(c) What is the discrete-time signal that will result from question (b)?

(d) If the signal amplitude is expressed in volts and each sample of the discrete signal is quantized to 8 bits, how many volts does the quantization step correspond to?

Answer: (a) To determine the Nyquist frequency, the maximum frequency of the signal must be found. For this reason, we will express the given signal as a sum of sinusoidal functions. The product $cos(100\pi t)\cos(200\pi t)$ is written[2]:

$$cos(100\pi t)\cos(200\pi t) = \frac{1}{2}[\cos(300\pi t) + \cos(100\pi t)]$$

So, the analog signal is written:

$$x_a(t) = -\frac{3}{2} + \frac{1}{2}\cos(300\pi t) + \frac{1}{2}\cos(100\pi t) + \frac{1}{2}\sin\left(200\pi t - \frac{\pi}{2}\right) + \cos(300\pi t)$$
$$= -\frac{3}{2} + \frac{1}{2}\cos(100\pi t) + \frac{1}{2}\sin\left(200\pi t + \frac{\pi}{2}\right) + \frac{3}{2}\cos(300\pi t) \quad (1)$$

Therefore, the frequencies of the signal are: $f_1 = 0\ Hz, f_2 = 50\ Hz, f_3 = 100\ Hz$ and $f_4 = 150\ Hz$. So the Nyquist frequency and minimum acceptable value of the sampling frequency is:

$$f_{s(min)} = f_N = 2f_4 = 300\ Hz$$

(b) For sampling frequency $f_s = 150\ Hz$, only frequencies $f_1 = 0\ Hz$ και $f_2 = 50\ Hz$, that lie within the range will be correctly represented $[-f_s/2, f_s/2] = [-75Hz, 75\ Hz]$. The frequencies $f_3 = 100\ Hz$ και $f_4 = 150\ Hz$will be convoluted and appear to correspond to the pseudo-labeled frequencies:

[2]We used the well-known relation:$cosA\ \cos B = (1/2)[\cos(A + B) + \cos(A - B)]$

$$f_3' = f_3 - kf_s = 100 - 150 = -50Hz$$
$$f_4' = f_4 - kf_s = 150 - 150 = 0Hz$$

Based on the above, it follows that the sampled signal will contain a continuous component ($0\ Hz$) and a sinusoidal frequency component 50 Hz, i.e. the frequencies 100 Hz and 150 Hz will no longer appear in the sampled signal.

(c) For a sampling frequency $f_s = 150\ Hz$ (i.e., sampling period $T_s = 1/150\ sec$), the discrete-time signal is:

$$x(n) = x_a(t)|_{t=nTs}$$
$$= -\frac{3}{2} + \frac{1}{2}\cos\left(\frac{100\pi}{150}n\right) + \frac{1}{2}\sin\left(\frac{200\pi}{150}n - \frac{\pi}{2}\right) + \frac{3}{2}\cos\left(\frac{300\pi}{150}n\right)$$
$$= -\frac{3}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{3}n\right) + \frac{1}{2}\cos\left(\frac{4\pi}{3}n\right) + \frac{3}{2}\cos(2\pi n)$$
$$= -\frac{3}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{3}n\right) + \frac{1}{2}\cos\left(2\pi - \frac{2\pi}{3}n\right) + \frac{3}{2}$$
$$= \frac{1}{2}\cos\left(\frac{2\pi}{3}n\right) + \frac{1}{2}\cos\left(\frac{2\pi}{3}n\right) = \cos\left(\frac{2\pi}{3}n\right)$$

The frequency of this signal can be calculated as follows:

$$\omega = \frac{2\pi}{3} \Rightarrow \Omega T_s = \frac{2\pi}{3} \Rightarrow 2\pi f\frac{1}{f_s} = \frac{2\pi}{3} \Rightarrow f = \frac{f_s}{3} \Rightarrow f = \frac{150}{3} = 50\ Hz$$

(d) From equation (1) it follows that the analog signal takes a maximum value of +1 Volt (when each trigonometric term takes a value of +1) and a minimum value of -4 Volts (when each trigonometric term takes a value of -1). Therefore, the dynamic range of the analog signal is 5 Volts and the quantization step Δ is calculated as:

$$\Delta = \frac{x_{max} - x_{min}}{2^L - 1} = \frac{1 - (-4)}{2^8 - 1} = \frac{5}{255} = 19{,}61\ mV$$

Usually quantizers work assuming that the amplitude values of the signal are symmetrical, i.e. $\pm 5\ V, \pm 10V, etc$. In the case of the above signal that the amplitude of the signal ranges from +1 Volt to -4 Volts we have to use a quantizer $\pm 5\ V$. So the quantization step for an 8 bits converter is:

$$\Delta = \frac{x_{max} - x_{min}}{2^L - 1} = \frac{10}{2^8 - 1} = \frac{10}{255} = 39{,}22\ mV$$

📖 Example 7

(a) An analog signal is sampled at the Nyquist rate f_s and quantized into L levels. Calculate the time duration (τ) of 1 bit of the signal encoded in binary.

(b) If each sample of a quantized analog signal must be known to within ±0.5% of the peak-to-peak value, how many bits must each sample be represented by?

Answer: (a) Let B is the number of bits per sample. Then, it holds:

$$B = [log_2 L]$$

Where $[log_2 L]$ indicates the next largest integer that will be taken if is $log_2 L$ not an integer value.

Binary pulses per sec must be transmitted nf_s. Thus, we will have:

$$\tau = \frac{1}{nf_s} = \frac{T_s}{n} = \frac{T_s}{[log_2 L]}$$

where T_s is the sampling period.

(b) Suppose that the peak - to - peak value of the signal is $2\ m_p$. Then the maximum error is $0.005\ (2\ m_p) = 0.01\ m_p$, and the peak error is $2(0.01\ m_p) = 0.02\ m_p$ and corresponds to the maximum quantization step size Δ. The required number of quantization stations is:

$$L = \frac{2m_p}{\Delta} = \frac{2\ m_p}{0{,}02\ m_p} = 100 \leq 2^n$$

Consequently, the number of bits needed for each sample is $n = 7$.

Example 8

The analog signal is given:

$$x(t) = \delta(t) - \frac{10}{2\pi} sinc^2\left(\frac{10}{2\pi}\right)$$

(a) Calculate the minimum sampling frequency for the signal $x(t)$.

The signal $x(t)$ is passed through an ideal low-pass filter $H_{LPF}(\Omega)$ and the output of the filter is the signal $y(t)$.

$$H_{LPF}(\Omega) = rect\left(\frac{\Omega}{20}\right)$$

(b1) Calculate the impulse response $h_{LPF}(t)$ of the filter.

(b2) Calculate and plot the amplitude spectrum $Y(\Omega)$.

(b3) Calculate the signal $y(t)$, without using convolution.

Next, the analog signal is converted $y(t)$ to digital through an A/D converter.

(c1) Calculate the minimum sampling frequency for the signal $y(t)$.

(c2) The signal $y(t)$ is sampled at a sampling frequency that is a multiple of $3\pi$the minimum sampling frequency and then quantized to 256 levels. Calculate the information rate at the output of the A/D converter and find the minimum bandwidth of the output signal in order to transmit the signal with PCM modulation.

(c3) Calculate the analytical relationship in the frequency domain $Y_s(\Omega)$ and in the discrete time domain of the sampled signal $y_s(n)$ for a sampling frequency the same as in question c2.

(c4) Calculate and plot the spectrum $Y_s(\Omega)$for $k = -1,0,1$ the same sampling frequency as in question c2.

(c5) Repeat question c4 for sampling frequency $\Omega_s = 4\pi$.

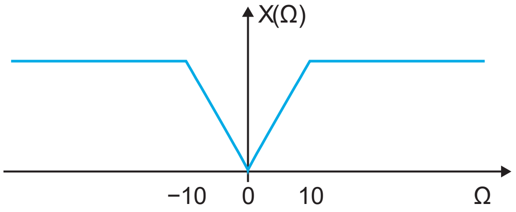
Answer: (a) Because the following Fourier transforms apply (see Table 4.2):

$$\delta(t) \overset{F}{\leftrightarrow} 1 \qquad \text{and} \qquad \frac{B}{2\pi} sinc^2\left(\frac{Bt}{2\pi}\right) \overset{F}{\leftrightarrow} tri\left(\frac{\Omega}{B}\right)$$

the amplitude $X(\Omega)$ spectrum of the signal is:

$$X(\Omega) = 1 - tri\left(\frac{\Omega}{10}\right)$$

The spectrum $X(\Omega)$ is shown in the following figure:



We observe that the spectrum extends to infinity, so the bandwidth of the signal is infinite, so it is not possible to determine any minimum sampling frequency according to the Nyquist criterion. So, this particular signal is not possible to sample.

(b1) From Table 4.2 we know that:

$$\frac{B}{2\pi} sinc\left(\frac{Bt}{2\pi}\right) \overset{F}{\leftrightarrow} rect\left(\frac{\Omega}{B}\right)$$

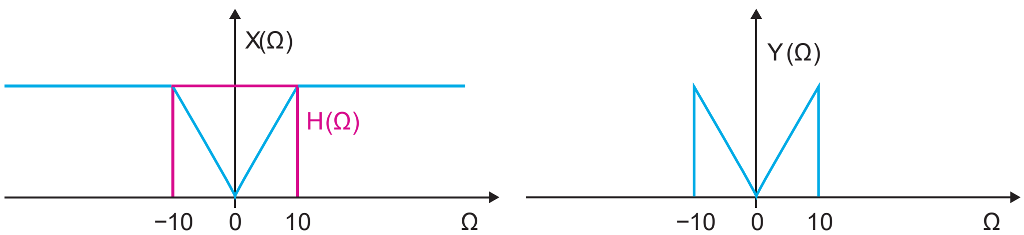
Therefore, the impulse response $h_{LPF}(t)$ of the filter can be calculated by inverse Fourier transform of the frequency response and is:

$$h_{LPF}(t) = \frac{20}{2\pi} sinc\left(\frac{20t}{2\pi}\right) = \frac{10}{\pi} sinc\left(\frac{10t}{\pi}\right)$$

(b2) The amplitude spectrum $Y(\Omega)$ is calculated from the equation:

$$Y(\Omega) = X(\Omega)H(\Omega) = \left[1 - tri\left(\frac{\Omega}{10}\right)\right] rect\left(\frac{\Omega}{20}\right) = rect\left(\frac{\Omega}{20}\right) - tri\left(\frac{\Omega}{10}\right)$$

The graphical representation of the spectrum $Y(\Omega)$ results from the product of the spectra $X(\Omega)$ and $H(\Omega)$ according to the figure:



(b3) The spectrum $Y(\Omega)$ is given by the equation:

$$Y(\Omega) = rect\left(\frac{\Omega}{20}\right) - tri\left(\frac{\Omega}{10}\right)$$

Using Table 4.2, it follows that the signal $y(t)$ is:

$$y(t) = \frac{20}{2\pi} sinc\left(\frac{20t}{2\pi}\right) - \frac{10}{2\pi} sinc^2\left(\frac{10t}{2\pi}\right) = \frac{10}{\pi} sinc\left(\frac{10t}{\pi}\right) - \frac{5}{\pi} sinc^2\left(\frac{5t}{\pi}\right)$$

(c1) The signal $y(t)$ has a finite frequency bandwidth and the maximum frequency is $\Omega_{max} = 10\, rad/sec$ or $f_{max} = 5/\pi\ (Hz)$ Therefore, the signal can be sampled. The Nyquist frequency (minimum sampling frequency) is:

$$f_{s(min)} = f_N = 2f_{max} = 10/\pi\ (Hz)$$

and in cyclic frequency is:

$$\Omega_N = 2\pi f_N = 20\ (rad/sec)$$

(c2) The selected sampling frequency is:

$$f_s = 3\pi f_{s(min)} = 3\pi(10/\pi) = 30\ (Hz)$$

The cyclic sampling frequency is:

$$\Omega_s = 2\pi f_s = 60\pi\ (Hz)$$

and the sampling period is:

$$T_s = \frac{1}{30}\ sec$$

Since quantization is done in 256 levels, the word length is:

$$B = [log_2 L] = [log_2 256] = 8\ bits$$

Therefore, the desired information rate at the output of the A / D converter is:

$$R = f_s B = 30\ samples/sec\ x\ 8\ bits/sample = 240\ bits/sec = 240\ bps$$

The minimum bandwidth of the output signal in order to transmit the PCM modulated signal is given by the equation:

$$W_{PCM} = \frac{1}{2} f_s B = 120\ Hz$$

(c3) The sampled signal in the frequency domain is given by equation (6.6) and is:

$$Y_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} Y(\Omega - k\Omega_s) = 30\sum_{k=-\infty}^{\infty}\left[rect\left(\frac{\Omega - k\Omega_s}{20}\right) - tri\left(\frac{\Omega - k\Omega_s}{10}\right)\right]$$

and in the time domain is:

$$y_s(n) = y(t)|_{t=nT_s} = \frac{10}{\pi} sinc\left(\frac{10n}{30\pi}\right) - \frac{5}{\pi} sinc^2\left(\frac{5n}{30\pi}\right) = \frac{10}{\pi} sinc\left(\frac{n}{3\pi}\right) - \frac{5}{\pi} sinc^2\left(\frac{n}{6\pi}\right)$$

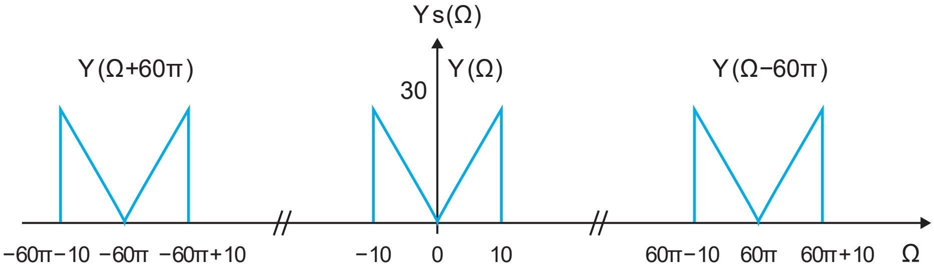
(c4) For $k = -1,0,1$ the analytical equation of the spectrum is:

$$Y_s(\Omega) = \frac{1}{T_s}\sum_{k=-1}^{1} Y(\Omega - k\Omega_s) = 30[Y(\Omega + 60\pi) + Y(\Omega) + Y(\Omega - 60\pi)] =$$

$$= 30\left[\left[rect\left(\frac{\Omega + 60\pi}{20}\right) - tri\left(\frac{\Omega + 60\pi}{10}\right)\right]\right.$$

$$\left. + \left[\left[rect\left(\frac{\Omega}{20}\right) - tri\left(\frac{\Omega}{10}\right)\right] + \left[rect\left(\frac{\Omega - 60\pi}{20}\right) - tri\left(\frac{\Omega - 60\pi}{10}\right)\right]\right]\right]$$

The graphical representation of the spectrum of the sampled signal is:



We note that there is sufficient empty spectral space between spectrum repetitions (due to sampling) so that the spectra do not overlap. We can easily recover the original signal by filtering the above spectrum with a deep-pass filter with a cutoff frequency that satisfies the equation: $10 < \Omega_c < 60\pi - 10$.

(c5) The given sampling frequency $\Omega_s = 4\pi$ rad / sec is less than the Nyquist cyclic frequency $\Omega_N = 2\pi f_N = 20\ rad/sec$, so we expect frequency aliasing. The sampling period is $T_s = 1/2\ sec$ and the requested range is:

$$Y_s(\Omega) = 2\sum_{k=-1}^{1} Y(\Omega - k4\pi) = 2[Y(\Omega + 4\pi) + Y(\Omega) + Y(\Omega - 4\pi)]$$