University of the Peloponnese
Electrical and Computer
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Digital Signal Processing

Lecture 10: Digital IIR Filters

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Introduction to Infinite Impulse Filters

(IIR) Filters

Impulse response:

$$h[n] \;= \sum_{m=0}^{M} b_m \delta[n-m] \;- \sum_{k=1}^{N} a_k h[n-k]$$

Transfer function:

$$H(z) = \frac{\sum_{m=0}^{M} b_m \, z^{-m}}{1 + \sum_{k=0}^{N} a_k \, z^{-k}} = \frac{B(z)}{A(z)} = C \, \frac{\prod_{m=1}^{M}(z - z_m)}{\prod_{n=1}^{N}(z - p_n)}$$

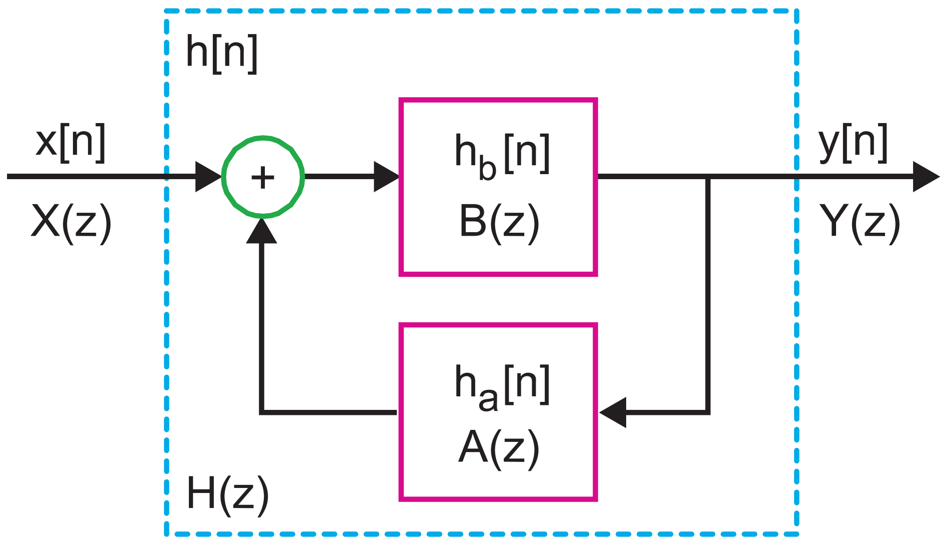
Order: the largest number of previous input and output values needed to compute the current output. Because it must $N \geq M$ for the filter to be causal, the order of the recursive filter is determined by the term N.

Design problem:

- The "specifications" are given, i.e. the desired frequency response.
- The calculation of the coefficients is requested $a[n], b[m]$

Characteristics of IIR filters

- They have an impact response **of infinite duration.**
- They are **recursive, i.e. samples** of the output are used to calculate the new values of the output at subsequent times.
- The simplest example of an analog IIR filter is an RC filter consisting of a resistor and a capacitor.



General diagram of a recursive (IIR) system

Characteristics of IIR filters

- IIR filters require <u>more</u> computation because there are previous output terms in the filter expression as well as input terms.
- In fact, the opposite is true. In particular, to achieve a desired frequency response with an IIR filter, **a lower filter order is generally required,** and thus **fewer terms** to be calculated by the processor, compared to a FIR filter.

Disadvantages of IIR filters

- They can become **unstable** if their coefficients are not chosen correctly, i.e. if even one pole of the transfer function is found outside the unit circle.
- **They do not have a linear phase response** in the passband, as is the case with FIR filters with symmetric or antisymmetric impulse response.

Design of IIR Filters

General method of designing IIR filters

To design an IIR filter we will rely, as an intermediate step, on a **standard analog filter.** Specifically, we follow the following steps:

- We convert the original digital specifications to analog.
- We identify the **standard low-pass analog filter** that satisfies the specification.
- We convert the analog filter to digital through a suitable **complex valued mapping of** the analog complex frequency s to the digital complex frequency z.

The design of a digital filter via analog is used because:

- Analog filter design techniques are **widely studied** and usually result in closed-form solutions.
- **Ready-made tables** for designing analog filters (but only for high-pass filters) are available in the literature.
- Analog complex frequency to digital complex frequency **conversion tables** are available.

Individual ways of designing IIR filters

Method A':

- Design of the standard low-pass analog filter
- Transforming the standard low-pass analog filter into the appropriate operating frequency band ($s \rightarrow s$)
- Convert the standard analog filter to digital ($s \rightarrow z$)

Method B':

- Design of the standard low-pass analog filter
- Convert the standard analog filter to digital ($s \rightarrow z$)
- Transforming the standard low-pass digital filter into the appropriate operating frequency band ($z \rightarrow z$)

These approaches do not provide complete control over the phase of the IIR filter.

We will implement the first way of drawing, for which Matlab offers several ready-made functions.

IIR filter design methods

Approaches to convert standard analog filter to digital:

- **Invariant Impulse Response** method

- **Bilinear Transform** method

Other techniques, in which the design is performed directly in the complex frequency plane z, such as the Pade approximation method, the Prony method, or the method of least squares for solving a set of linear or nonlinear equations.

Design with direct placement of poles and zeros

- Placing a zero near the unit circle reduces the magnitude of the response at that frequency.
- Placing a pole close to the unit circle has the opposite effect, i.e. it increases the magnitude of the response at that frequency.
- We can construct a desired transfer function of a filter by placing its poles and zeros at appropriate positions in the complex digital frequency plane with respect to the unit cycle.
- to the distance (r) of a pole from the center of the unit circle:

$$r \approx 1 - \frac{\Delta f_{3dB}}{F_s}\pi, \qquad 0.9 \leq r < 1$$

where Δf_{3dB} is the acceptable bandwidth of the filter at 3 dB and F_s is the sampling frequency.

Example 1

Design an IIR bandpass filter with the following specifications:

- Passband centered at frequency: 3,000 Hz
- Passbandwidth (3 dB): 1,000 Hz
- Zero response at: 0 Hz and 5,000 Hz
- Sampling frequency: 10. 000 Hz

Answer: Since zero magnitude of the frequency response at 0 Hz and 5000 Hz is desired, zeros should be placed at the corresponding points of the unit circle, i.e. at the points of the circle:

$$2\pi\ \frac{0\ Hz}{10.000\ Hz} = 0\pi \text{ ή } 0^{o}$$

$$2\pi\ \frac{5.000\ Hz}{10.000\ Hz} = \pi \text{ ή} 180^{o}$$

Since the filter is required to have a passband centered at 3000 Hz we will place a pole at the frequency:

$$2\pi\ \frac{3.000\ Hz}{10.000\ Hz} = \frac{3\pi}{5} \text{ ή } 108^{o}$$

Example 1 (continued)

and at a distance from the center of the unit circle:

$$r \approx 1 - \frac{\Delta f_{3dB}}{F_s}\pi = 1 - \frac{500}{10000}\pi = 1 - 0.05\pi = 0.95$$

For the coefficients of the transfer function to be real numbers, $H(z)$ the conjugate pole must also be placed in the appropriate position. Is:

$$H(z) = k\frac{(z-1)(z+1)}{(z - re^{j3\pi/5})(z - re^{-j3\pi/5})} = k\frac{z^2(1 - z^{-2})}{z^2(1 - 2rz^{-1}\cos(3\pi/5) + r^2 z^{-2})}$$

$$= k\frac{1 - z^{-2}}{1 + 0.717125z^{-1} + 0.9025\ z^{-2}}$$

The frequency response is:

$$H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega}} = \cdots = k\frac{1 - e^{-2j\omega}}{1 + 0.717125e^{-j\omega} + 0.9025\ e^{-2j\omega}}$$

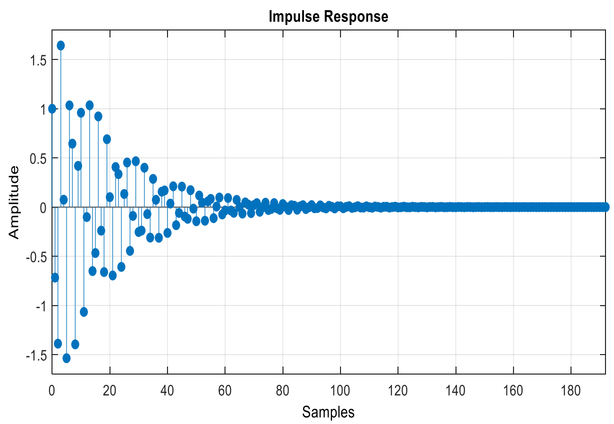
For the transit zone:

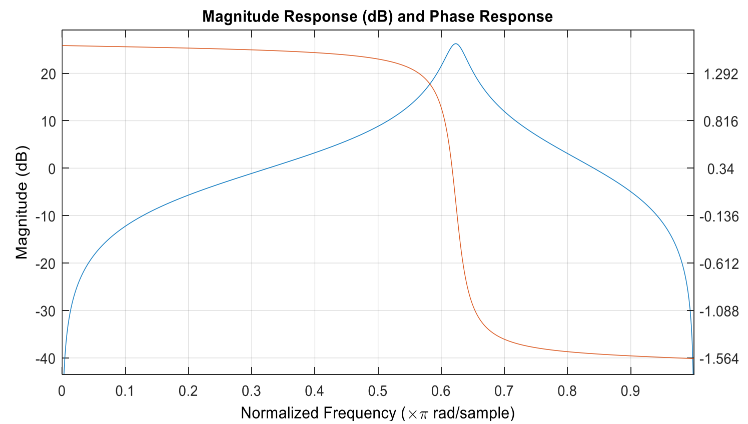
$$H\left(\frac{3\pi}{5}\right) = 1 \Rightarrow \ldots \Rightarrow k\frac{1.8090 + j\ 0.5878}{0.0483 - j\ 0.1516} = 1 \Rightarrow k = -0.0005 - j\ 0.0836$$

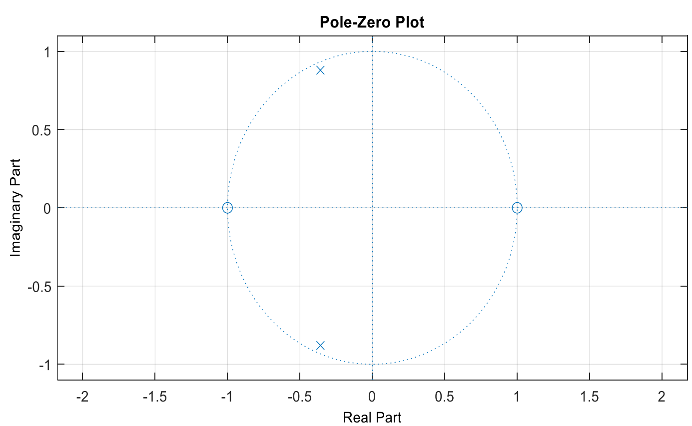
Example 1 (continued)

Based on the transfer function $H(z)$ we calculated and in order to draw the Impulse Response, the frequency response and the pole-zero diagram we write the following program in Matlab:

b = [1, 0, -1]; a = [1, 0.717125, 0.9025]; fvtool (b,a)







Standard Low-Pass Analog Filters

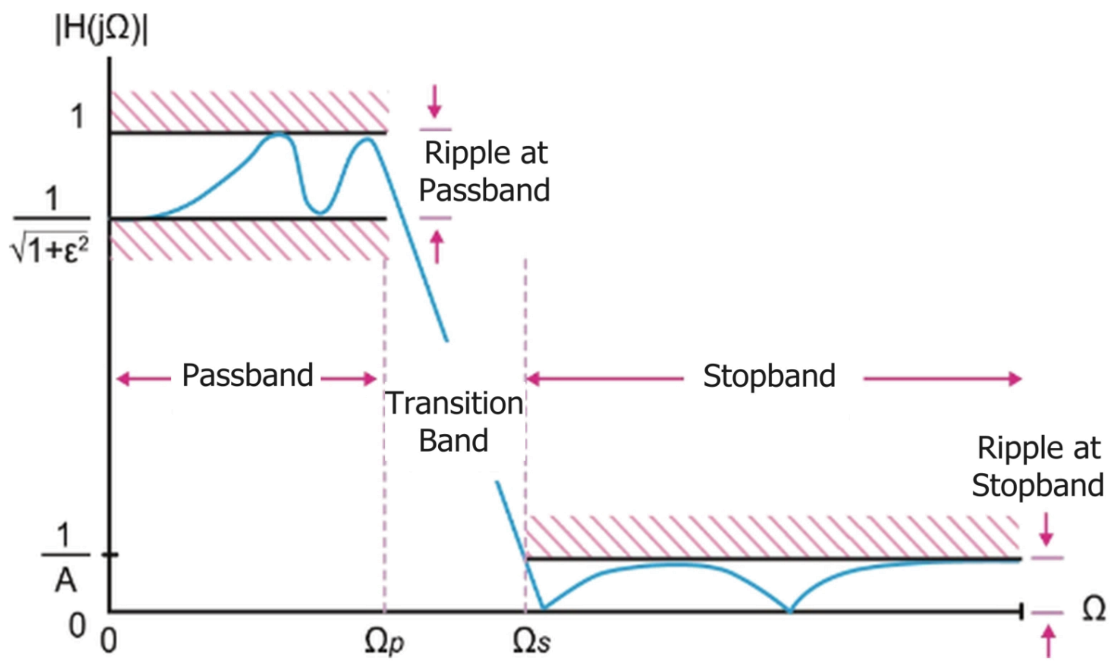
Standard analog low-pass filter

The squared magnitude of the frequency response satisfies the relations:

$$\frac{1}{1+\varepsilon^2} \le |H(j\Omega)|^2 \le 1, \qquad |\Omega| \le \Omega_p$$

$$0 \le |H(j\Omega)|^2 \le \frac{1}{A^2}, \qquad |\Omega| \ge \Omega_s$$

- **e:** ripple in the transition zone
- $\boldsymbol{\Omega_p}$: passband cutoff frequency
- $\boldsymbol{A}$: attenuation in the stopband zone
- $\boldsymbol{\Omega_s}$: cutoff frequency of the stopband



Standard analog low-pass filter

The parameters ε and A are related to the corresponding ripple A_s and attenuation R_p parameters, we have seen in FIR filters, through the relations:

$$R_p = -10log_{10}\left(\frac{1}{1+\varepsilon^2}\right) \Rightarrow \varepsilon = \sqrt{10^{R_p/10} - 1}$$

$$A_s = -10log_{10}\left(\frac{1}{A^2}\right) \Rightarrow A = 10^{A_s/10}$$

and with the magnitude deviation parameters δ_p and δ_s through the relations:

$$\frac{1-\delta_p}{1+\delta_\pi} = \sqrt{\frac{1}{1+\varepsilon^2}} \Rightarrow \varepsilon = \frac{2\sqrt{\delta_p}}{1-\delta_p} \text{ and } \frac{\delta_s}{1+\delta_p} = \frac{1}{A} \Rightarrow A = \frac{1+\delta_p}{\delta_s}$$

For a frequency equal to Ω_p και Ω_s, the squared magnitude of the frequency response is:

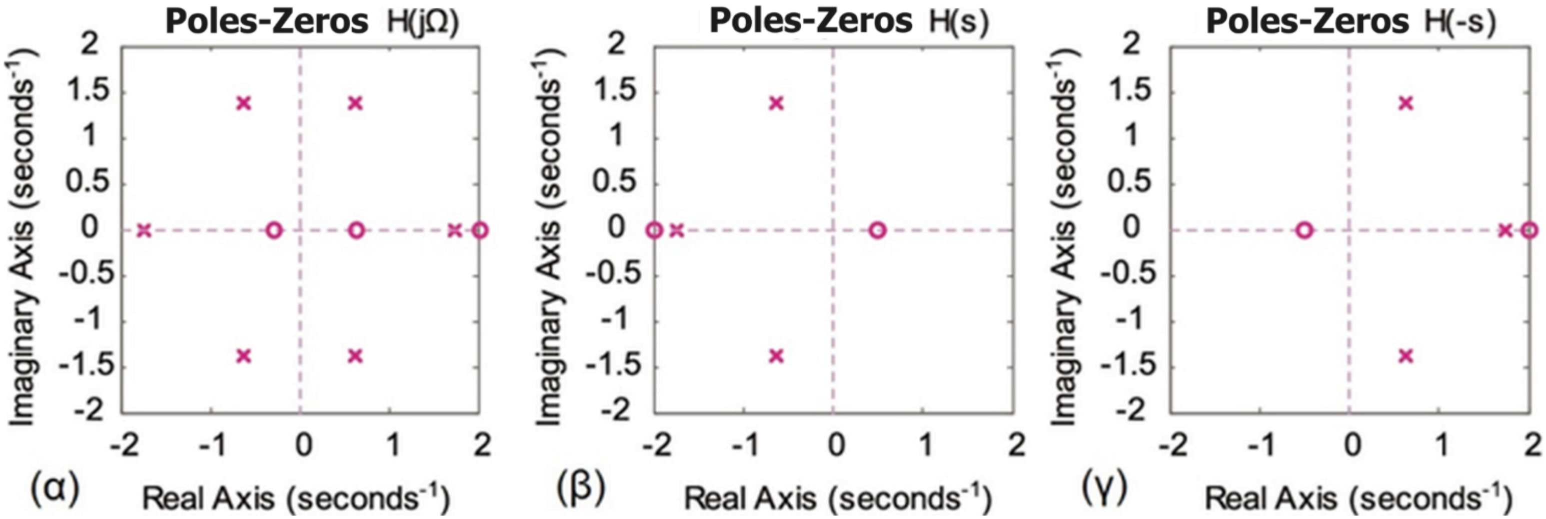
$$\left|H(j\Omega_p)\right|^2 = \frac{1}{1+\varepsilon^2} \text{ and} |H(j\Omega_s)|^2 = \frac{1}{A^2}$$

Filter IIR pole-zero diagram

For the squared magnitude of the frequency response:

$$|H(j\Omega)|^2 = H(j\Omega)H(-j\Omega) = H(s)\,H(-s)\Big|_{s=j\Omega} \Rightarrow H(s)\mathrm{H}(-\mathrm{s}) = |\mathrm{H}(j\Omega)|^2\Big|_{\Omega=\mathrm{s/j}}$$

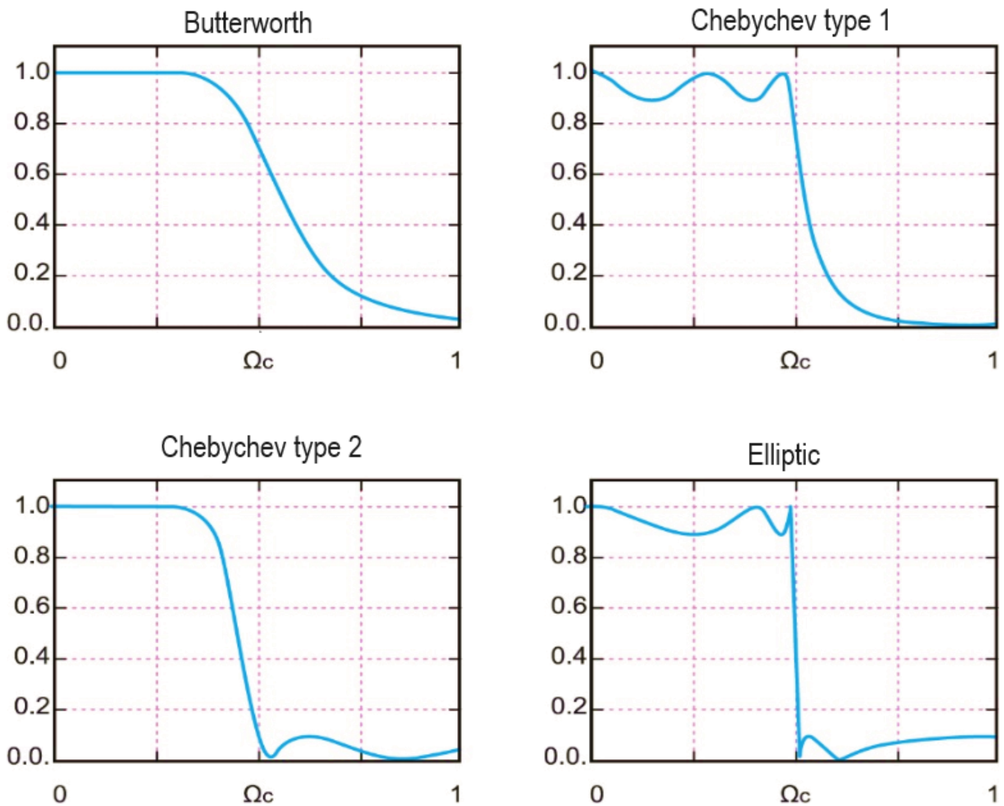
So, its poles and zeros $|H(j\Omega)|^2$ are placed symmetrically about the axis $j\Omega$.



Pole-zero diagrams of the functions:
(a) $|H(j\Omega)|^2 = \mathrm{H}(s)H(-s)$, (b) $H(s)$ and (c) $H(-s)$

Standard low-pass analog filters

- **Butterworth Low-Pass Filter**
- **Chebyshev type I and II Low-Pass Filter**
- **Elliptic Low-Pass Filter**



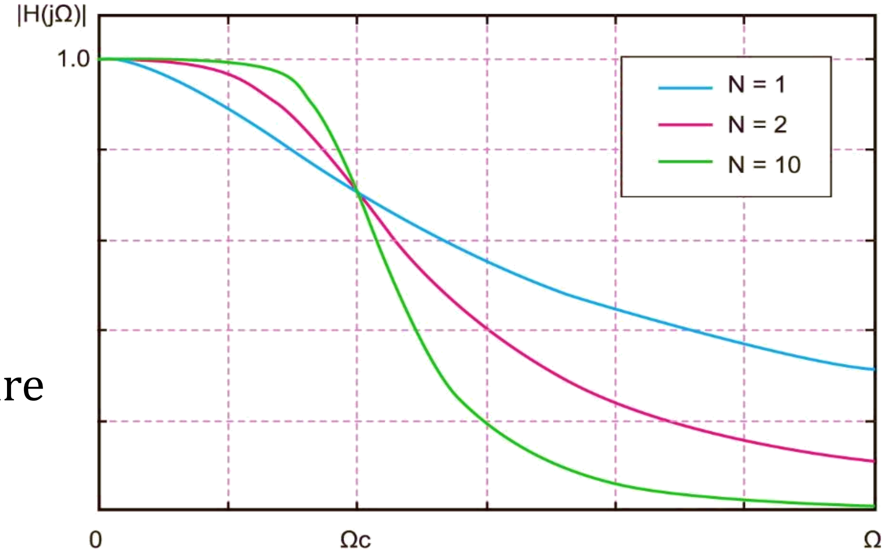
Frequency responses of standard analog deep-pass amplifiers
Butterworth, Chebyshev I, Chebyshev II and Elliptic filters

Butterworth Low-Pass Filter

Squared measure of the frequency response:

$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$$

- Ω_c: cutoff frequency
- N: order of Butterworth function



Butterworth frequency response measure for various order values

Apply:

- $|H(j0)|^2 = 1 = 0$dBfor all values of the order N.
- $|H(j\Omega_c)|^2 = 1/2 = 3dB$, regardless of the value of N.
- For $\Omega_c = 1\ rad/sec$we get the standard depth filter.
- For $N \to \infty$the filter it approaches the ideal depth filter.

Butterworth Low-Pass Filter

Based on specifications:

- Ω_p: cutoff frequency at the passband boundary
- Minimum value of magnitude response $1/\sqrt{1+\varepsilon^2}$ in the passband zone.
- Ω_s: frequency at stopband boundary
- Maximum value of the frequency response measure $1/A$ in the stopband

We calculate:

- The **order** of the filter:

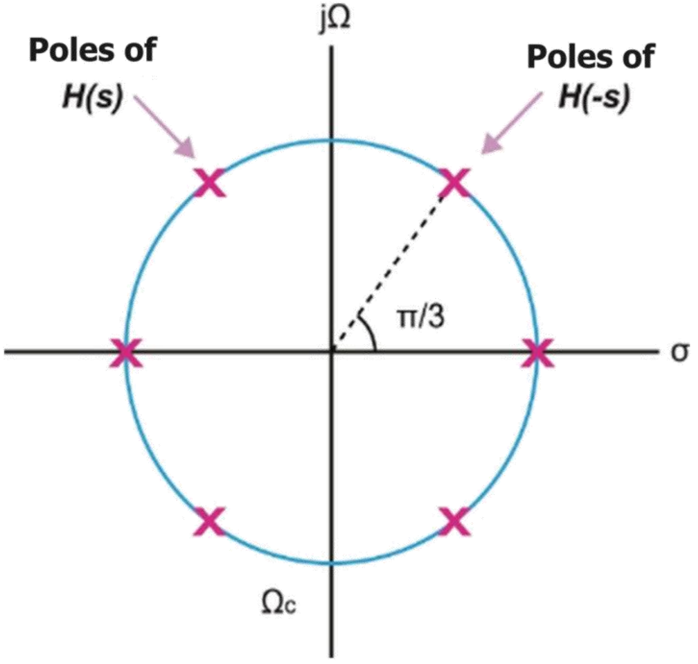
$$N = \left\lceil \frac{\log_{10}\left[(A^2-1)/\varepsilon^2\right]}{2\log_{10}\left[\Omega_s/\Omega_p\right]} \right\rceil \text{the} N = \left\lceil \frac{\log_{10}\left[(10^{\mathrm{R}p/10}-1)/(10^{\mathrm{A}s/10}-1)\right]}{2\log_{10}\left[\Omega_p/\Omega_s\right]} \right\rceil$$

- The **cutoff frequency:**

$$\Omega_c = \frac{\Omega_p}{\sqrt[2N]{10^{Rp/10}-1}} \text{ the } \Omega_c = \frac{\Omega_s}{\sqrt[2N]{10^{As/10}-1}}$$

Poles and zeros of a Low-Pass Butterworth filter

- For filter order N, the poles are $2N$ in the crowd.
- The poles are placed on a circle of radius Ω_c and equidistant from each other by angle $2\pi/2N = \pi/N$.
- The filter is causal and stable when all its poles lie in the left half-plane of the frequency s.
- The poles in the left half-plane correspond to $H(s)$ while on the right they correspond to $H(-s)$.



Poles of the product $H(s)H(-s)$ for N=3

Transfer Function of a Standard Low-Pass Butterworth Filter

Order (N)	Transfer function $H(s)$ (for $\Omega_c = 1rad/sec$)
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+\sqrt{2}s+1}$
3	$\frac{1}{s^3+2s^2+2s+1}$
4	$\frac{1}{s^4+2.6131s^3+3.4142s^2++2.6131s+1}$

Butterworth filter design in Matlab /Octave

Butterworth analog filter:

- **[N, Wn] = buttord (Wp, Ws, Rp, As):** Returns the smallest order N of an analog Butterworth filter satisfying the specification. Wn is the cutoff frequency Ω_c in normalized scale, i.e. $0 < W_n < 1$, where 1 corresponds to half the sampling frequency.
- **[b, a] = butter (N, Wn):** Returns the coefficients $a[n], b[m]$ of the transfer function, order N and cutoff frequency Wn. Alternatively it is written **[z, p, k] = butter (N, Wn)** so it returns the poles, zeros and gain factor of the transfer function.

Butterworth analog filter:

- **[z, p, k] = buttap (N):** Returns the poles, zeros, and gain of a normalized standard analog Butterworth filter of order N. The resulting filter has N poles about the unit circle in the left complex half-plane and no zeros.

Chebyshev Filter type I Low-Pass Filter

Chebyshev I filter has only poles, exhibits uniform ripple in the passband ($0 \leq \Omega < 1$) and monotonically decays in the cutoff ($\Omega > 1$).

The squared measure of its frequency response is:

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \, T_N^2(\Omega/\Omega_c)}$$

ε: ripple in the passband,
Ω_c: cut-off frequency

$T_N(\Omega)$Chebyshev polynomial of order N:

$$T_N(x) = \begin{cases} \cos(N \cos^{-1} x), & 0 \leq x \leq 1 \\ \cosh(N \cosh^{-1} x), & 1 < x < \infty \end{cases}, \qquad \text{όπου } x = \Omega/\Omega_c$$

- $\Omega = 0$ and N even: $|H(j0)|^2 = 1$
- $\Omega = 0$ and N unnecessary: $|H(j0)|^2 = 1/(1 + \varepsilon^2)$
- $\Omega = \Omega_c$: $|H(j\Omega_c)|^2 = 1/(1 + \varepsilon^2)$ regardless of$N$
- $0 \leq \Omega \leq \Omega_c$: $|H(j\Omega)|^2$ oscillates between 0 and$1/(1 + \varepsilon^2)$
- $\Omega > \Omega_c$: $|H(j\Omega)|^2$decreases monotonically.

Chebyshev Filter type I Low-Pass Filter

Cutoff frequencyΩ_c:

$$\Omega_c = \Omega_p \quad \text{και} \quad \Omega_r = \frac{\Omega_s}{\Omega_p}$$

order N:

$$N = \left\lceil \frac{\log_{10}\left[g + \sqrt{g^2 - 1}\right]}{\log_{10}\left[\Omega_r + \sqrt{\Omega_r^2 - 1}\right]} \right\rceil$$

Chebyshev type I deep-pass analog filter:

- **[N, Wn] = cheb1ord (Wp, Ws, Rp, Rs):** Returns the smallest order N of a Chebyshev-I analog filter satisfying the specifications. Wn is the $\Omega_c$normalized scale cutoff frequency.
- **[b, a] = cheby1 (N, R, Wn):** Returns the coefficients $a[n], b[m]$of the transfer function, order N, cutoff frequency Wn and ripple R. Alternatively it is written **[z, p, k] = butter (N, Wn)** so it returns the poles, zeros and gain factor of the transfer function.

Design of a model Chebyshev type I deep-pass analog filter:

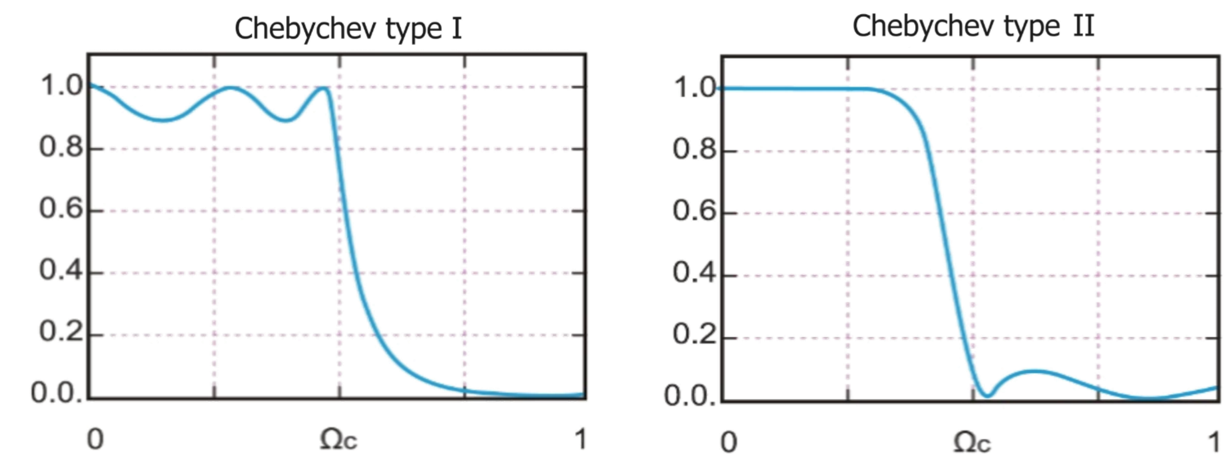
- **[z, p, k] = cheb1app (N, Rp):** Returns the poles, zeros, and gain of a normalized standard analog Chebyshev type I filter of order N, with Rp ripple in the passband.

Chebyshev Filter Type II Low-Pass Filter

- Chebyshev type II filter has poles and zeros, exhibits uniform ripple in the stopband, ($\Omega > 1$)and monotonically decays in the passband ($0 \leq \Omega < 1$).
- Its zeros lie on the imaginary axis of the s-plane.
- The squared measure of its frequency response is:

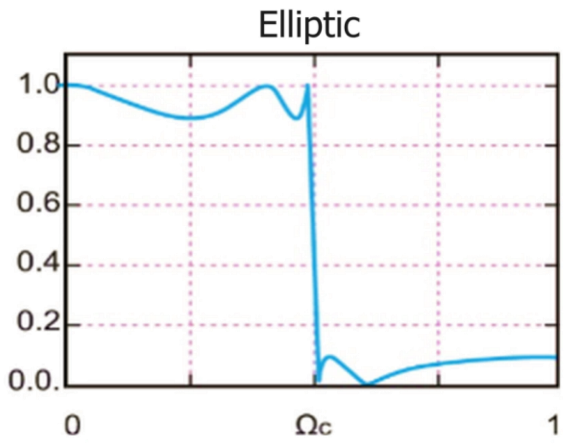
$$|H(j\Omega)|^2 = \frac{1}{1 + [\varepsilon^2\, T_N^2(\Omega_c/\Omega)]^{-1}}$$

- This filter exhibits more linear phase and improved group delay compared to the Chebyshev type I filter.
- Implementation with the functions **cheb 2 ord (), cheby 2() and cheb 2 ap ().**



Elliptic Low-Pass Filter

- The low-pass elliptic filter exhibits uniform ripple in both the passband and the stopband, like the isowave FIR filter.
- The elliptic filter is better than the Butterworth and Chebychev filters because it requires a smaller order to satisfy the same specifications.
- However, the response of the elliptic filter shows more non-linearities in the passband than the Butterworth or Chebychev filters.



- The squared measure of the frequency response of the deep-pass elliptic filter is:

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2\, U_N^2(\Omega/\Omega_c)}$$

- $\boldsymbol{\varepsilon}$: ripple in the passband, $\boldsymbol{\Omega_c}$: cutoff frequency,: elliptic $\boldsymbol{U_N(\Omega)}$ Jacobi function of order N.
- To design a deep-pass analog elliptic filter Matlab offers the function **ellip ()** and the function **ellipord ()** for the calculation of the optimal order, while for the design of a model deep-pass analog elliptic filter it offers the function **ellipp ().**

Invariant Impulse Response Method

Invariant Impulse Response Method

Converting an analog filter with desired specifications to a digital one by sampling the impulse response of the analog filter.

- We calculate the impulse response of the filter, by inverse Laplace transform of the transfer function $H(s)$of the analog filter:

$$h(t) = \mathrm{L}^{-1}\{H(s)\}$$

- We sample with sampling period T_d, the impulse response $h(t)$and obtain:

$$h[nT_d] = h(t)\Big|_{t=nT_s}, \qquad n \geq 0$$

- We calculate the transfer function $H(z)$ of the digital filter from the Z-transform of the sequence $h[nT_d]$:

$$H(z) = Z\{h[nT_d]\}$$

Invariant Impulse Response Method

Disadvantages:

- Sampling the impulse response causes the phenomenon of spectrum folding, hence spectrum distortion.
- The effect can be reduced by sufficiently increasing the sampling frequency, but not completely eliminated.
- It becomes necessary to filter the impulse response $h(t)$ with a low pass filter to remove the high frequencies.
- This fact constitutes an additional weakness of the method as it cannot be applied to the design of high-pass digital filters, only low-pass and band-pass.

Mapping between complex analog and digital frequency planes

- Analog (Ω) and digital (ω) frequency are related by the equation:

$$\omega = \Omega T_s \Rightarrow e^{j\omega} = e^{j\Omega T_d}$$

- Because in the unit circle it holds $z = e^{j\omega}$ and on the imaginary axis $s = j\Omega$, we have:

$$z = e^{s\,T_d}$$

- This equation describes a mapping relationship of the analog complex frequency $s$to the digital complex frequency z.

- Taking into account that $z = re^{j\omega}$and $s = \sigma + j\Omega$, we conclude:

$$r = e^{\sigma\,T_d}$$

Remarks:

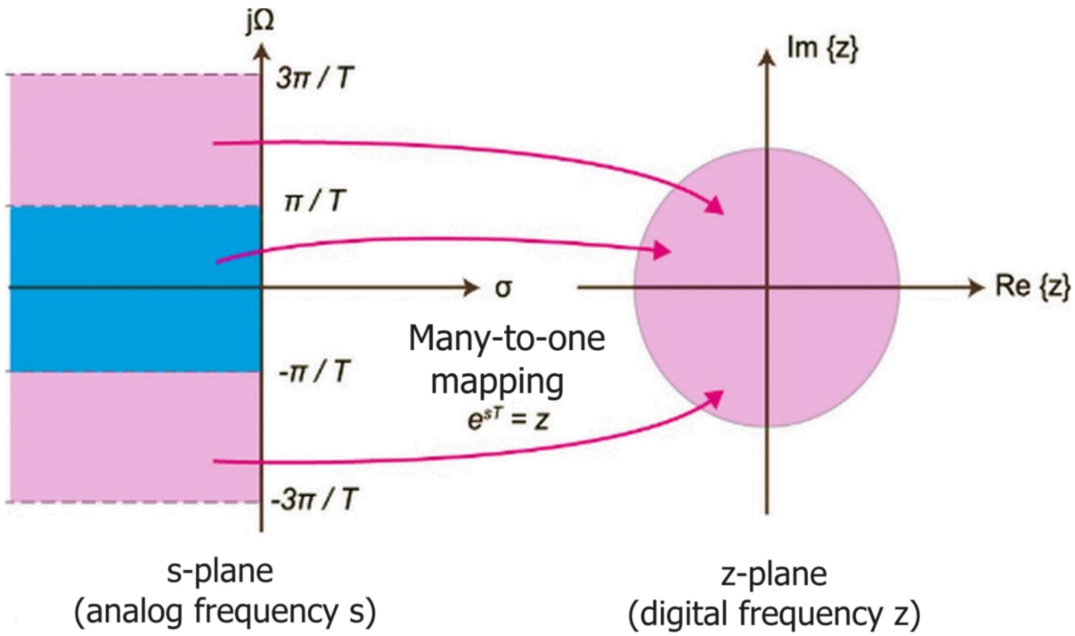
- For $\sigma = 0 \Rightarrow r = 1$, i.e. the imaginary axis $j\Omega$ of the plane $s$is projected onto the unit circle of the plane z.
- For $\sigma < 0 \Rightarrow r < 1$, i.e. the left half-plane $s$is mapped inside the unit circle of the plane z. However, the mapping between the points of the planes $s$and $z$is not one-to-one.
- For $\sigma > 0 \Rightarrow r > 1$, that is, the right half-plane $s$is mapped to the outside of the unit circle of the plane z.

Mapping between complex analog and digital frequency levels

Between$H(z)$ and $H(s)$ the relation holds:

$$H(z) = \frac{1}{T_d} \sum_{k=-\infty}^{\infty} H\left(s - j\frac{2\pi}{T_d}k\right)$$

Note: The display between layers$s$ and$z$ is not one-to-one, but many-to-one, as $2\pi/T_d$plane -width strips $s$are mapped to points on the plane's unit circle z. This fact is due to insufficient sampling frequency.



Design methodology

Given the desired specifications ω_p, ω_s, R_p and A_s we proceed to the design of the filter with the invariant impact method, with the following steps:

- We select it $T_d$and convert the digital frequencies ω_p, ω_s in analog:

$$\Omega_p = \omega_p / T_d \text{ and } \Omega_s = \omega_s / T_d$$

- We design the equivalent analog filter with one of the standard analog deep-pass filters and calculate its transfer function $H(s)$from the specifications Ω_p, Ω_s, $R_p$and A_s.

- If its poles $H(s)$ have multiplicity 1, then we can express it $H(s)$ in expansion of some fractions:

$$H(s) = \sum_{k=1}^{N} \frac{R_k}{s - p_k}$$

- The transfer function $H(z)$ of the digital filter is found to be:

$$H(z) = \sum_{k=1}^{N} \frac{R_k}{1 - e^{p_k T_d} z^{-1}}$$

- Solving in Matlab: **[bz, az] = impinvar (bs, as, Fd),** where bz, az, the coefficients of the digital filter, bs, as the coefficients of the analog filter, and Fd =1/Td.

Example 2

Transform the transfer function below $H(s)$ of the analog filter in a transfer function $H(z)$ of the digital filter.

$$H(s) = \frac{3s + 7}{s^2 + 4s + 3}$$

Answer: We write $H(s)$ in fractional expansion:

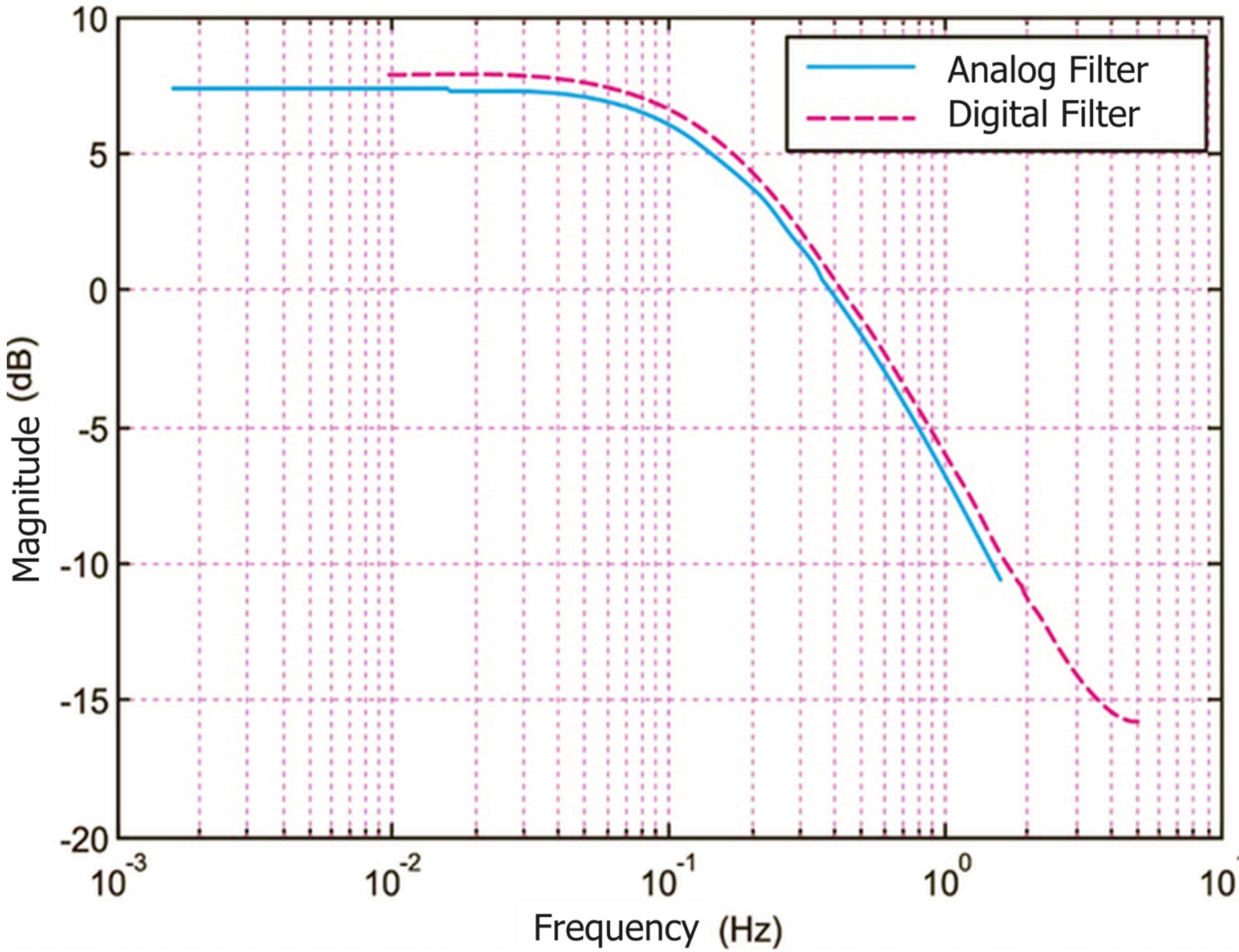
$$H(s) = \frac{3s + 7}{s^2 + 4s + 3} = \frac{3s + 7}{(s + 1)(s + 3)} = \cdots = \frac{1}{s + 3} + \frac{2}{s + 1}$$

Poles of $H(s)$ are: $p_1 = -1$and $p_2 = -3$.

For $T_d = 0.1$ the transfer function of the digital filter is:

$$H(z) = \frac{1}{1 - e^{-0.3} z^{-1}} + \frac{2}{1 - e^{-0.1} z^{-1}} = \frac{3 - 2.3865\, z^{-1}}{1 - 1.6457\, z^{-1} + 0.6703\, z^{-2}}$$

Example 2 (continued)



Comparison of analog and digital filter magnitude response

Conclusions

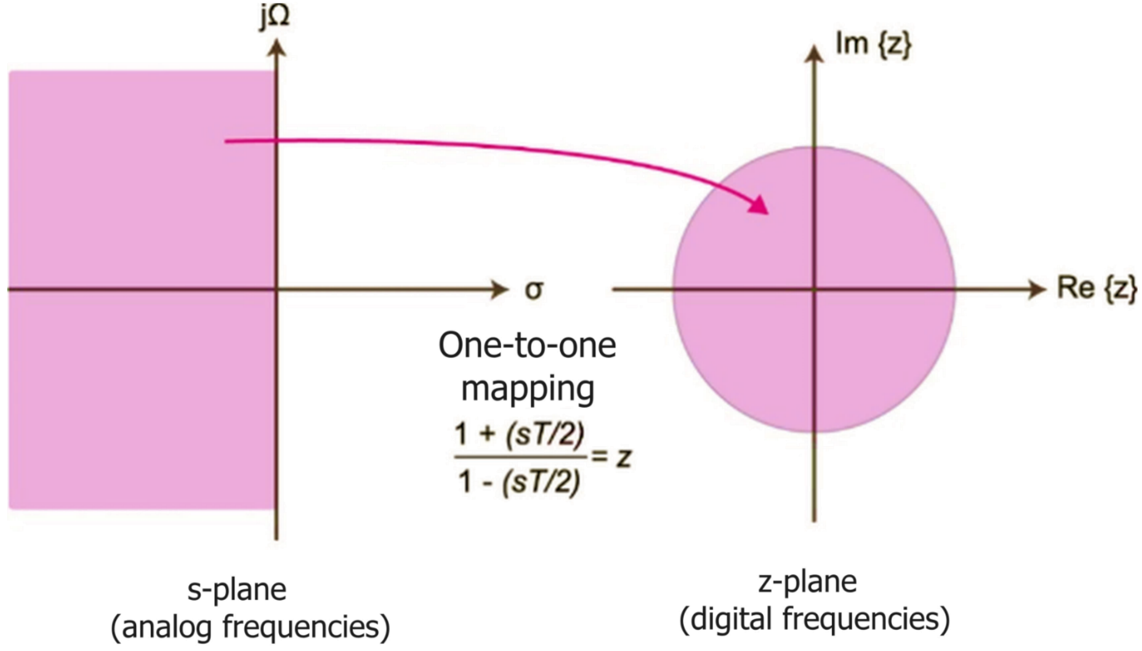
- The invariant impulse response method **generally fails** to accurately estimate the digital filter.
- The method is only suitable for designing low-pass filters because undersampling the impulse response creates the spectral folding effect due to the **many-to-one mapping between the** s and z planes.
- Particularly in the case of the standard Chebyshev filter of type II, the frequency response in the stopband of the generated digital filter differs significantly from the corresponding frequency response of the analog filter.
- This is because the Type II Chebyshev filter has isowave behavior in the stopband, so its frequency response does not tend to zero for high frequencies.
- Thus, the effect of spectrum distortion that occurs due to frequency sampling is even more pronounced and obvious.

Bilinear-Transform Method

Bilinear-Transform method

- The bilinear-transform method ensures a **one-to-one mapping** of the points of the planes $s$and z.
- The mapping relationship between the variables $s$and $z$turns out to be given by the relations:

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \text{and } z = \frac{1+(T/2)s}{1-(T/2)s}$$



Bilinear-Transform method

<u>Remarks:</u>

- The positive part of the imaginary axis $j\Omega$ of the plane s is mapped to the positive half of the circumference of the unit circle of the plane z.
- The negative part of the imaginary axis $j\Omega$ is mapped to the negative half of the circumference of the unit circle of the plane z.
- The left half-plane $s$is depicted inside the unit circle of the plane z. Therefore, a stable analog filter produces an also stable digital filter.
- The right half plane s is depicted on the outside of the unit circle of the plane z.

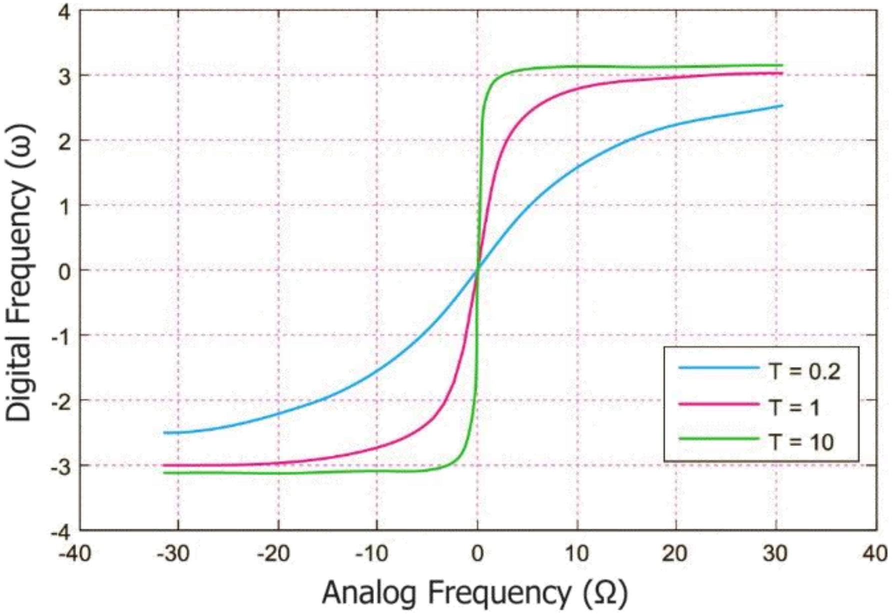
Mapping between analog and digital frequency

The mapping relationship between analog frequency Ω and digital frequency ω,is:

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right) \text{ and } \Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

Remarks:

- The entire set of analog frequencies is displayed in the digital frequency range $-\pi \leq \omega \leq \pi$.
- This representation is non-linear, i.e. it results in frequency wrapping (frequency wrapping), which is due to the existence of the function $\tan^{-1}(\ )$.



Solving in Matlab: **[bz, az] = bilinear(bs, as, Fd),** where bz, az, the coefficients of the digital filter, bs, as the coefficients of the analog filter and Fd =1/Td.

Example 3

Using the bilinear transform method, plot the following points of the s plane on the z plane:

(α) $s_1 = -1 + j$ (β) $s_2 = 1 - j$

(γ) $s_3 = 2j$ (δ) $s_4 = -2j$

<u>Answer:</u> We set a price $T = 2$ in the conversion relation (13.101), and we have:

$$(\alpha)\ z_1 = \frac{1+s_1}{1-s_1} = \frac{1-1+j}{1+1-j} = \frac{j}{2-j} = -0.2 + 0.4j = 0.447 \measuredangle\, 7.2^0$$

since $|z_1| < 1$, the point z_1 lies inside the unit circle.

$$(\beta)\ z_2 = \frac{1+s_1}{1-s_1} = \frac{1+1-j}{1-1+j} = \frac{2-j}{j} = -1 + 2j = 2.236 \measuredangle - 7.2^0$$

since $|z_2| > 1$, the point z_2 is outside the unit circle.

Example 3 (continued)

$(\gamma)\ z_3 = \frac{1+s_1}{1-s_1} = \frac{1+2j}{1-2j} = -0.6 + 0.8j = 1\measuredangle\ 38.6^0$

since $|z_3| = 1$, the point z_3 lies on the positive half of the circumference of the unit circle.

$$(\delta)\ z_4 = \frac{1+s_1}{1-s_1} = \frac{1-2j}{1+2j} = \frac{2-j}{-j} = -0.2 - 0.8j = 1\measuredangle - 38.6^0$$

since $|z_4| < 1$, the point $z_4$lies on the negative half of the circumference of the unit circle.

Example 4

Using the bilinear transform to convert the analog transfer function filter to digital (Example 1):

$$H(s) = \frac{3s+7}{s^2+4s+3}$$

<u>**Answer:**</u> We set a value $T = 2$to the relationship:

$$\mathrm{s} = \frac{2}{\mathrm{T}}\frac{z-1}{z+1}\bigg|_{T=2} = \frac{z-1}{z+1}$$

The required transfer function $H(z)$is given by the relation:

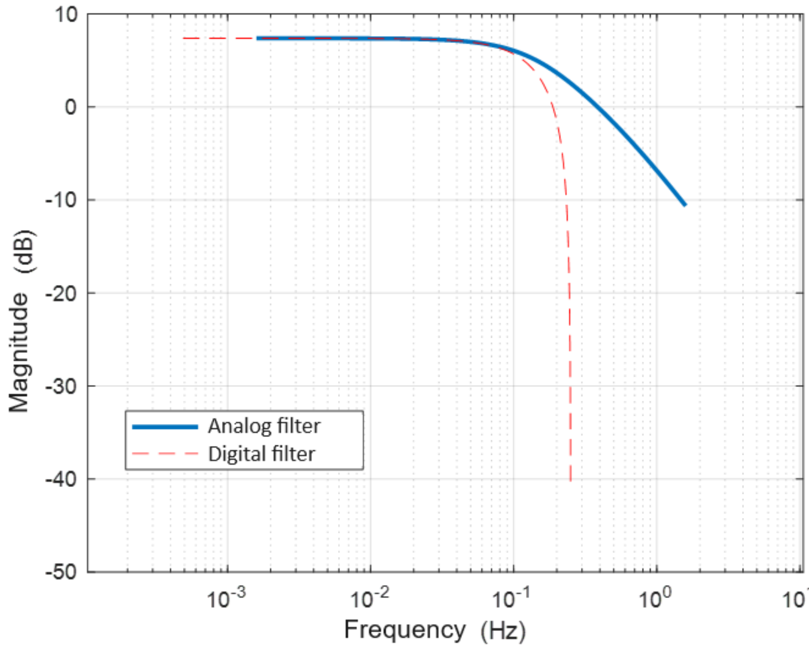
$$H(z) = H(s)\Big|_{s=\frac{z-1}{z+1}} = \frac{3\left(\frac{z-1}{z+1}\right)+7}{\left(\frac{z-1}{z+1}\right)^2+4\left(\frac{z-1}{z+1}\right)+3}$$

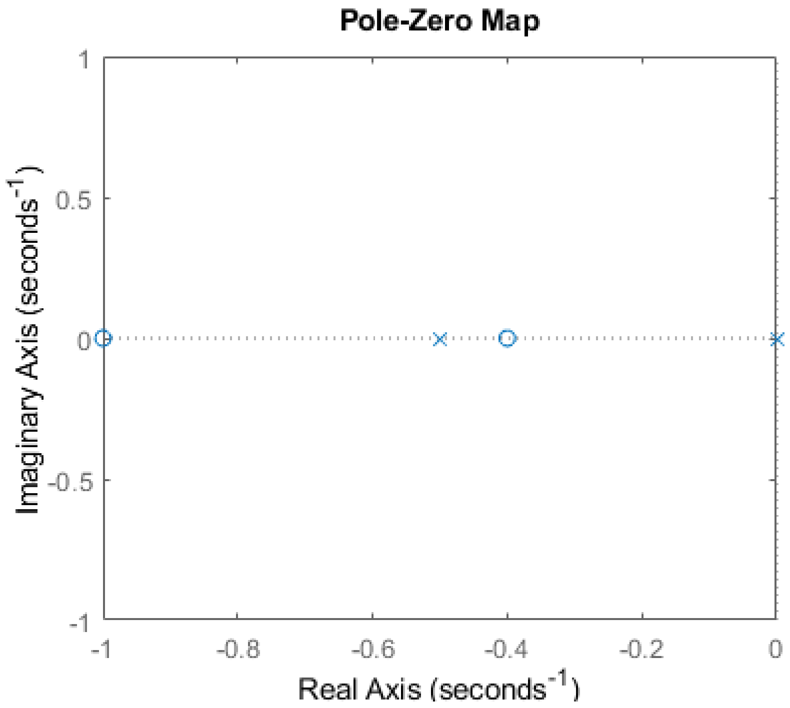
By simplifying we get:

$$H(z) = \frac{5z^2+7z+2}{4z^2+2z} = \frac{1.25+1.75z^{-1}+0.5z^{-2}}{1+0.5z^{-1}}$$

Comparing the result with Example 1 we notice that the transfer functions of the digital filters produced by the invariant impulse response and bilinear transformation methods are **different** from each other.

Example 4 (continued)





Frequency transformations

Method A'

- We transform the standard low-pass analog filter in the appropriate operating frequency band ($s \rightarrow s$).
- We convert the standard analog filter to digital ($s \rightarrow z$).

Method B’

- We convert the standard analog to digital ($s \rightarrow z$).
- We transform the standard deep-pass digital filter in the appropriate operating frequency band ($z \rightarrow z$).

From Low-Pass Analog to Frequency Selective Analog Filter

- We first design the standard low-pass analog filter with a 3- dB cutoff frequency Ω_p.
- We then convert this into the appropriate frequency selection filter based on the table's frequency representation:

Transformation	Mapping function	New cut-off frequencies
Low-Pass	$s \to \frac{\Omega_p}{\Omega'_p} s$	Ω'_p
High-Pass	$s \to \frac{\Omega_p \, \Omega'_p}{s}$	Ω'_p
Band-Pass	$s \to \frac{\Omega_p (s^2 + \Omega_l \Omega_u)}{s(\Omega_u - \Omega_l)}$	Ω_l, Ω_u
Band-Stop	$s \to \frac{\Omega_p (\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$	Ω_l, Ω_u

From Low-Pass Digital to Frequency Selective Digital

- We first design the standard low-pass digital filter based on the desired cut-off frequency ω_c'
- We then convert this into the appropriate frequency selection filter based on the table's frequency representation:

Transformation	Mapping function	Parameters
Low-Pass	$z^{-1} \rightarrow \frac{z^{-1} - a}{1 - az^{-1}}$	ω_c' $a = \frac{\sin[(\omega_c' - \omega_c)/2]}{\sin[(\omega_c' + \omega_c)/2]}$
High-Pass	$z^{-1} \rightarrow -\frac{z^{-1} + a}{1 + az^{-1}}$	ω_c' $a = -\frac{\cos[(\omega_c' + \omega_c)/2]}{\cos[(\omega_c' - \omega_c)/2]}$

From Low-PassDigital to Frequency Selective Digital

Transformation	Mapping function	Parameters
Band-Pass	$z^{-1} \rightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l, ω_u $\alpha_1 = -2\beta K/(K+1)$ $\alpha_2 = (K-1)/(K+1)$ $\beta = \frac{cos[(\omega_u + \omega_l)/2]}{cos[(\omega_u - \omega_l)/2]}$ $K = cot \frac{\omega_u - \omega_l}{2} tan\left(\frac{\omega_c'}{2}\right)$
Band-Stop	$z^{-1} \rightarrow \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l, ω_u $\alpha_{1,} \alpha_2, \beta$ όμοια με παραπάνω $K = \tan \frac{\omega_u - \omega_l}{2} \tan\left(\frac{\omega_c'}{2}\right)$

Effect of Finite Word Length on Filter Accuracy

Effect of finite factor precision on the filter implementation

The implementation accuracy of FIR and IIR filters is reduced by the finite accuracy of their coefficient representation, due to:

- The **binary representation of the numbers** in the computing system that implements the filter.
- Of **quantization** of the filter **coefficients.**
- Of **rounding of intermediate results** when calculating the filter output.

These phenomena result in the deviation of the characteristics of the digital filter from the desired specifications, so the filter ceases to be optimal.

IIR filters there is the danger of moving one or more poles onto the unit circle or even off it, so the filter becomes unstable.

Example 5

(a) To test for causality and stability the IIR digital filter with a transfer function:

$$H(z) = \frac{1 - 0.816z^{-1}}{1 - 1.927z^{-1} + 0.928176z^{-2}}$$

Answer: By factoring the denominator the transfer function is written:

$$H(z) = \frac{1 - 0.816z^{-1}}{(1 - 0.951z^{-1})(1 - 0.976z^{-1})}$$

The IIR filter has a zero at $z = 0.816$and two real poles at $p_1 = 0.951$and $p_2 = 0.976$. Both poles are inside the unit circle, so the filter is stable.

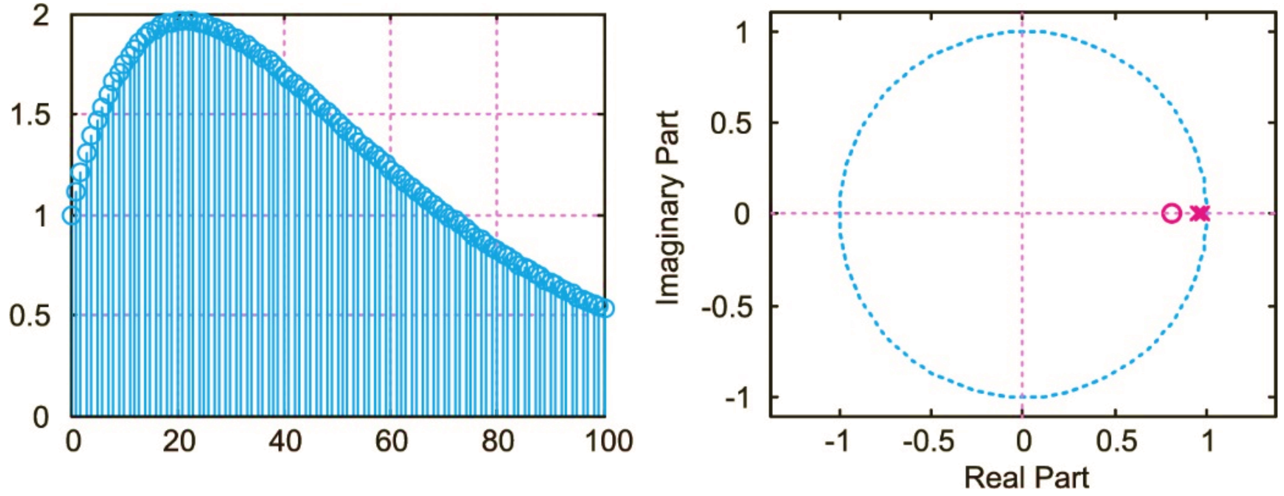
To calculate the Impulse Response we write the transfer function in simple fraction expansion:

$$H(z) = \frac{6.4}{1 - 0.976z^{-1}} - \frac{5.4}{1 - 0.951z^{-1}}$$

The impulse response is:

$$h[n] = [6.4\ x\ 0.976^{n} - 5.4\ x\ 0.951^{n}]\ u[n]$$

Example 5 (continued)



Impulse response and pole-zero diagram

Therefore, the system is causal and stable as both poles are inside the unit circle, as it also emerged from the theoretical solution.

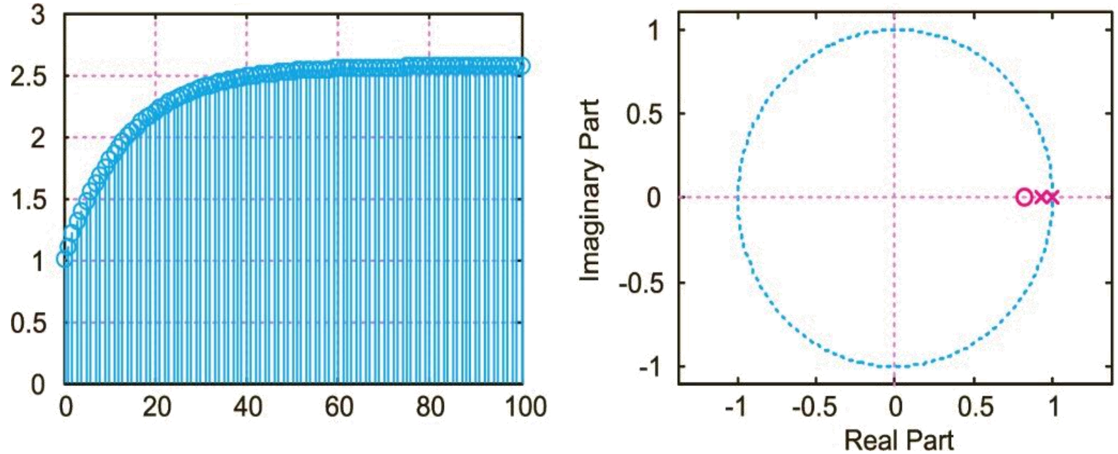
Example 5 (continued)

(b) Plot the filter coefficients to two decimal places and repeat the investigation.

Answer: In the case that the filter coefficients are given with a precision of two decimal places, then the transfer function is written:

$$H(z) = \frac{1 - 0.82z^{-1}}{1 - 1.93z^{-1} + 0.93z^{-2}}$$

The impulse response is: $h[n] = [2.5714\ x\ 1^n - 1.5714\ x\ 0.93^n]\ u[n]$



We notice that one pole has moved onto the unit circle. Therefore the system is now marginally stable.

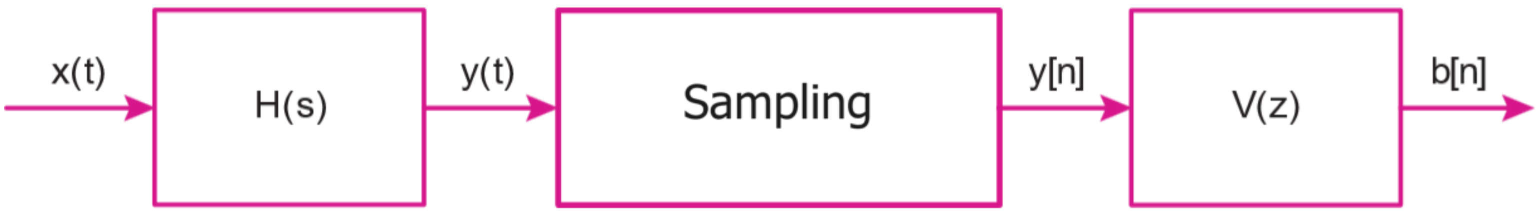
Conclusions

- An IIR filter is vulnerable to the accuracy of its coefficient representation, because small changes in the coefficient representation can create a large change in frequency response, as it is possible for the filter poles to move on or off the unit circle, making the filter unstable.
- An FIR filter has no poles, so the reduced precision of its coefficients may make it less optimal, but it does not make it unstable.
- The fact that FIR filters are always stable is an important characteristic of them.
- Another advantage of them is that they always have a linear phase response.
- It is shown that the effects of finite precision on the representation of coefficients and the computation of operations in a digital filter can be reduced if the higher-than-second-order filter can be implemented as a combination of first- and/or second-order structures in series or parallel.

Other uses of the Bilinear-Transform

Example 6

In the following signal processing system, the transfer function $H(s)$ of the continuous-time system is known.



If we wish it to be valid $b[n] = x[n]$ what should be the transfer function $V(z)$ of the discrete-time system?

<u>Answer:</u> The signal $y(t)$ is calculated by convolution: $y(t) = x(t) * h(t)$ (1)

where: $h(t) = F^{-1}\{H(s)\}$ (2)

Accordingly, in the plane of analog complex frequency (s):

$$Y(s) = X(s)\, H(s) \qquad (3)$$

Therefore:

$$X(s) = \frac{Y(s)}{H(s)} \qquad (4)$$

Example 6 (continued)

Since in relations (1) and (3) the only known functions are $H(s)$ και $h(t)$, we will need to "connect" the signal $y(t)$ to the output $b[n]$ of the overall system, in order to formulate a relation between and $x[n]$, $b[n]$ which will lead us to find the function $V(z)$. Due to sampling, it applies:

$$y[n] = y(t)\Big|_{t=nT_s} \qquad (5)$$

We will use the bilinear-transform in order to express relation (5) at the complex digital frequency level z and calculate the function $Y(z)$. Is:

$$Y(z) = Y(s)\Big|_{s=\frac{z-1}{z+1}} \qquad (6)$$

As for the discrete-time system $V(z)$, its output in the time and frequency domains is given by the equations:

$$b[n] = y[n] * v[n] \qquad (7)$$
$$B(z) = Y(z)\,V(z) \qquad (8)$$

Example 6 (continued)

In relation (8) the functions $Y(z)$ και $V(z)$ are known. The $Y(z)$ was calculated from the relation (6), while the $V(z)$ is given. Since it is given that

$$b[n] = x[n] \qquad (9)$$

resulting:

$$B(z) = X(z) \qquad (10)$$

The function $X(z)$ is connected to the function $X(s)$ through the bilinear-transform, i.e. it is:

$$X(z) = X(s)\Big|_{s=\frac{z-1}{z+1}} \qquad (11)$$

Solving relation (8) in terms of $V(z)$, and taking into account equations (6), (11) and (4) we have:

$$V(z) = \frac{B(z)}{Y(z)} = \frac{X(z)}{Y(z)} = \frac{X(s)|_{s=\frac{z-1}{z+1}}}{Y(s)|_{s=\frac{z-1}{z+1}}} = \frac{Y(s)/H(s)|_{s=\frac{z-1}{z+1}}}{Y(s)|_{s=\frac{z-1}{z+1}}} = \frac{1}{H(s)|_{s=\frac{z-1}{z+1}}} \qquad (12)$$

The relation (12) describes the desired transfer function $V(z)$ of the discrete-time system. We observe that it is the inverse of the transfer function $H(s)$of the discrete-time system, computed appropriately with the bilinear-transform due to the sampling that took place between the two systems.

Example 7

A continuous-time LTI system is described by LDECC:

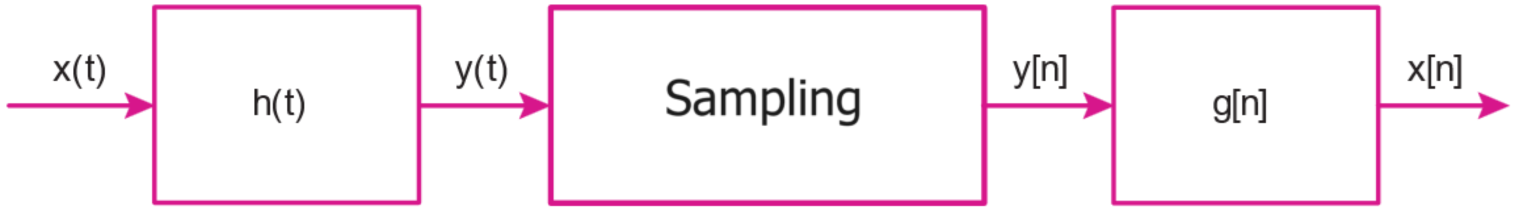
$$\frac{dy^2(t)}{dt^2} + c_1\frac{dy(t)}{d(t)} = c_2\frac{dy(t)}{d(t)} + c_3x(t)$$

and is initially relaxed (zero initial conditions).

(a) Calculate the impulse response of the system

(b) If we sample every T sec what is the difference equation?

(c) What must be the function $g[n]$ in the circuit below to restore the effect of $h(t)$?



Example 7 (continued)

Answer: (a) We can solve the problem either in the time domain (by solving for eigenvalues) or in the frequency domain with Laplace transform. We will prefer the second solution due to simplicity. We apply Laplace constants to both members of the LDECC and since the initial conditions are zero, we have:

$$s^2Y(s) + c_1sY(s) = c_2sX(s) + c_3X(s) \Rightarrow$$

$$Y(s)[s^2 + c_1s] = X(s)[c_2s + c_3] \Rightarrow$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{c_2s + c_3}{s^2 + c_1s} = \frac{c_2s + c_3}{s(s + c_1)}$$

The system has a zero at frequency $s = -c_3/c_2$ and two simple poles at frequencies $s_1 = 0$ and $s_2 = -c_1$.

To calculate the Impulse Response $h(t) = F^{-1}\{H(s)\}$ we will apply the method of partial fractions. Is:

$$H(s) = \frac{c_2s + c_3}{s(s + c_1)} = \frac{R_1}{s} + \frac{R_2}{s + c_1}$$

Example 7 (continued)

The coefficients R_1 and R_2 are calculated from the relations:

$$R_1 = s\,H(s)\Big|_{s=0} = \left.\frac{c_2 s + c_3}{(s + c_1)}\right|_{s=0} = \frac{c_3}{c_1}$$

$$R_2 = (s + c_1)\,H(s)\Big|_{s=-c_1} = \left.\frac{c_2 s + c_3}{s}\right|_{s=-c_1} = \frac{c_2(-c_1) + c_3}{-c_1} = \frac{c_1 c_2 - c_3}{c_1}$$

Therefore the transfer function is written:

$$H(s) = \left(\frac{c_3}{c_1}\right)\frac{1}{s} + \left(\frac{c_1 c_2 - c_3}{c_1}\right)\frac{1}{s + c_1}$$

So the impulse response is:

$$h(t) = \left(\frac{c_3}{c_1}\right)u(t) + \left(\frac{c_1 c_2 - c_3}{c_1}\right)e^{-t}u(t) = \frac{1}{c_1}\left[c_3 + (c_1 c_2 - c_3)e^{-t}\right]u(t)$$

Example 7 (continued)

(b) To calculate the difference equation we will need to calculate the $H(z)$ discrete-time transfer function. We use the bilinear-transform to express $H(z)$ in terms of $H(s)$, which we calculated in question (a). In the end, by inverse Z-transform we will find the requested difference equation.

Applying a bilinear-transform to $H(s)$ and we have:

$$H(z) = H(s)\Big|_{s=\frac{z-1}{z+1}} = \frac{c_2 s + c_3}{s(s+c_1)}\Bigg|_{s=\frac{z-1}{z+1}} = \frac{c_2 \frac{z-1}{z+1} + c_3}{\frac{z-1}{z+1}\left(\frac{z-1}{z+1} + c_1\right)}$$

$$= \frac{\frac{c_2(z-1) + c_3(z+1)}{z+1}}{\frac{(z-1)^2 + c_1(z-1)(z+1)}{(z+1)^2}} = \frac{(z+1)[c_2(z-1) + c_3(z+1)]}{(z-1)^2 + c_1(z^2-1)}$$

$$= \frac{z^2(c_2+c_3) + 2c_3 z + (c_3 - c_2)}{z^2(1+c_1) - 2z + (1-c_1)} = \frac{(c_2+c_3) + 2c_3 z^{-1} + (c_3-c_2)z^{-2}}{(1+c_1) - 2z^{-1} + (1-c_1)z^{-2}} = \frac{Y(z)}{X(z)}$$

Example 7 (continued)

Therefore we have:

$$Y(z)[(1 + c_1) - 2z^{-1} + (1 - c_1)z^{-2}] = X(z)[(c_2 + c_3) + 2c_3z^{-1} + (c_3 - c_2)z^{-2}]$$

$$\Rightarrow (1 + c_1)Y(z) - 2z^{-1}Y(z) + (1 - c_1)z^{-2}Y(z)$$

$$= (c_2 + c_3)X(z) + 2c_3z^{-1}X(z) + (c_3 - c_2)z^{-2}X(z)$$

By inverse Z-transform we find the difference equation:

$$(1 + c_1)y[n] - 2y[n - 1] + (1 - c_1)y[n - 2]$$

$$= (c_2 + c_3)x[n] + 2c_3x[n - 1] + (c_3 - c_2)x[n - 2]$$

Example 7 (continued)

(c) The relationships apply:

$$y(t) = x(t) * h(t) \Rightarrow Y(s) = X(s)H(s) \qquad (1)$$

$$X(s) = \frac{Y(s)}{H(s)} \qquad (2)$$

Due to sampling, it applies:

$$y[n] = y(t)\Big|_{t=nT_s} \qquad (3)$$

The output $x[n]$ is given by the convolution:

$$x[n] = y[n] * g[n] \Rightarrow X(z) = Y(z)\, G(s) \Rightarrow G(s) = \frac{X(s)}{Y(s)} \qquad (4)$$

The function $Y(z)$is calculated by bilinear-transform on the function $Y(s)$and is:

$$Y(z) = Y(s)\Big|_{s=\frac{z-1}{z+1}} = Y\left(\frac{z-1}{z+1}\right) \qquad (5)$$

Example 7 (continued)

Likewise for the function $X(z)$:

$$X(z) = X(s)\Big|_{s=\frac{z-1}{z+1}} = \frac{Y(s)}{H(s)}\Bigg|_{s=\frac{z-1}{z+1}} = \frac{Y\left(\frac{z-1}{z+1}\right)}{H\left(\frac{z-1}{z+1}\right)} \qquad (6)$$

Substituting equations (5) and (6) into equation (4), we obtain:

$$G(z) = \frac{X(z)}{Y(z)} = \frac{Y\left(\frac{z-1}{z+1}\right)/H\left(\frac{z-1}{z+1}\right)}{Y\left(\frac{z-1}{z+1}\right)} = \frac{1}{H\left(\frac{z-1}{z+1}\right)} \qquad (7)$$