University of the Peloponnese

Electrical and Computer
Engineering Department

Digital Signal Processing

Unit 07: Study of Discrete-Time Systems in the Frequency Domain

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Lecture Contents

- Description of a Discrete-Time System in the Z-plane

    o Transfer function

    o Relationship between Transfer Function and Difference Equation

    o Frequency Response

    o Poles and Zeros of the Transfer Function

    o All-Pole and All-Zero Systems

    o Theorems of Causality and Stability of Systems

Lecture Contents

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- Solving Differential Equations
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- DTFT Applications:
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Discrete-Time System modelling in complex frequencies (z-plane)

- Transfer Function
- Derivation of Transfer Function from Difference Equation
- Frequency Response
- Poles and Zeros of the Transfer Function
- All-Pole and All-Zero Systems
- Theorems of Causality and Stability of Systems

LSI System Transfer Function

For an LSI discrete-time system according to the convolution theorem we have:

$$y[n] = h[n] * x[n] \xleftrightarrow{\;Z\;} Y(z) = H(z)\,X(z)$$

where $h[n]$ the impulse response of the system, $x[n]$ the signal applied to its input and $H(z)$ the $X(z)$corresponding transformations Z. Solving for $H(z)$ we have:

$$H(z) = \frac{Y(z)}{X(z)}$$

The function $H(z)$ is called **the** transfer function of the system and is unique for each system.

LSI System Transfer Function

The transfer function of a LSI system is the Z-transform of the impulse response $h[n]$ of the system, that is:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

from $h[n]$ to be complete, $H(z)$ the area of convergence should also be determined. If this is not possible, then other characteristics of the system, such as causality and stability, should be known.

Example 1

A discrete-time LSI system accepts input $x[n] = (0.5)^n \, u[n]$ and produces output given by $y[n] = [(-1)^n + 1] \, u[n]$.

(a) Calculate the impulse response $h[n]$ of the system.

(b) Calculate the output of the system for input $x[n] = (-0.2)^n \, u[n]$.

Answer: (a) For the signals $x[n]$ and $y[n]$ we calculate the functions $X(z)$ and $Y(z)$:

$$X(z) = \frac{1}{1 - 0.5^n z^{-1}} = \frac{z}{z - 0.5}, \qquad\qquad R_x: |z| > 0.5$$

$$Y(z) = \frac{z}{z+1} + \frac{z}{z-1} = \frac{2z^2}{(z-1)(z+1)}, \qquad R_y: |z| > 1$$

The transfer function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\dfrac{2z^2}{(z-1)(z+1)}}{\dfrac{z}{z - 0.25}} = \frac{2z(z - 0.5)}{(z-1)(z+1)}, \qquad R_h: |z| > 1$$

Z-transform we calculate the impulse response $h[n]$:

$$h[n] = [0.5 + 1.5(-1)^n] \, u[n]$$

Example 1 (continued)

(b) First we calculate the function $X(z)$, then the product $H(z) X(z) = Y(z)$ and finally with inverse Z-transform the sequence $y[n]$.

$$X(z) = \frac{1}{1 - 0.2^n z^{-1}} = \frac{z}{z - 0.2}, \qquad R_x: |z| > 0.2$$

$$Y(z) = H(z) X(z) = \frac{2z^2(z - 0.5)}{(z - 1)(z + 1)(z - 0.2)}, \qquad R_y = R_h \cap R_x: |z| > 1$$

Since η $Y(z)$ is expressed in factors of the term z, we calculate the expansion of the expression $\tilde{Y}(z) = Y(z)/z$. Is:

$$\frac{Y(z)}{z} = \tilde{Y}(z) = \frac{2z(z - 0.5)}{(z - 1)(z + 1)(z - 0.2)} = \frac{R_1}{z - 1} + \frac{R_2}{z + 1} + \frac{R_3}{z - 0.2}$$

where the rest R_1, R_2 and R_3 are given by the relations:

$$R_1 = [(z - 1)\tilde{Y}(z)]_{z=1} = \left[\frac{2z(z - 0.5)}{(z + 1)(z - 0.2)}\right]_{z=1} = \frac{5}{8}$$

Example 1 (continued)

$$R_2 = [(z + 1)\tilde{Y}(z)]_{z=-1} = \left[\frac{2z(z - 0.5)}{(z - 1)(z - 0.2)}\right]_{z=-1} = \frac{5}{4}$$

$$R_3 = [(z - 0.2)\tilde{Y}(z)]_{z=0.2} = \left[\frac{2z(z - 0.5)}{(z - 1)(z + 1)}\right]_{z=0.2} = \frac{1}{8}$$

Therefore, the expansion of the function $Y(z)$ is:

$$Y(z) = \frac{5}{8}\frac{z}{z - 1} + \frac{5}{4}\frac{z}{z + 1} + \frac{1}{8}\frac{z}{z - 0.2}$$

Since the region of convergence is R_y: $|z| > 1$ it follows that the subsequences are right-sided, so the output of the system is:

$$y[n] = \left[\frac{5}{8} + \frac{5}{4}(-1)^n + \frac{1}{8}(0.2)^n\right]u[n]$$

Relationship between Transfer Function and Difference Equation

- The Linear Difference Equation with Constant Coefficients (LDECC) that describes a LSI system, which is initially relaxed (zero state), is:

$$y[n] = \sum_{m=0}^{M} b_m \, x[n-m] - \sum_{k=1}^{N} a_k \, y[n-k]$$

- Calculating the Z-transform of both terms and using the time-shift property we obtain:

$$Y(z) = \sum_{m=0}^{M} b_m \, z^{-m} \, X(z) - \sum_{k=1}^{N} a_k \, z^{-k} \, Y(z)$$

- Solving for $Y(z)/X(z)$, gives the transfer function $H(z)$ which is given by the equation:

$$H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m \, z^{-m}}{1 + \sum_{k=1}^{N} a_k \, z^{-k}} = \frac{B(z)}{A(z)}$$

Example 2

A LSI system is described by LDECC:

$$y[n] = 0.5y[n-1] + x[n-1]$$

Calculate: (a) the Transfer Function and (b) the impulse response of the system.

Answer: (a) We write the LDECC, transferring the terms $y[\ ]$ to the right-hand member:

$$y[n] - 0.5\ y[n-1] = x[n-1]$$

We calculate the Z-transform of both members of the LDECC. Using the time shift property we get:

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = z^{-1}X(z) \Rightarrow Y(z)[1 - 0.5\ z^{-1}] = z^{-1}X(z) \Rightarrow$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - 0.5\ z^{-1}} = \frac{1}{z - 0.5}$$

Example 2 (continued)

(b) Because the impulse response is calculated with an inverse Z-transform in the transfer function ($h[n] = Z^{-1}[H(z)]$), we develop in a sum of some fractions the expression:

$$\frac{H(z)}{z} = \frac{1}{z(z - 0.5)} = \frac{R_1}{z} + \frac{R_2}{z - \frac{1}{2}}$$

We find the rest: $R_1 = -2$ και $R_2 = 2$.

The Tansfer Function is:

$$H(z) = -2 + \frac{2z}{z - 0.5}$$

Therefore the impulse response is:

$$h[n] = -2\delta[n] + 2(0.5)^n u[n]$$

Example 3

Write the LDECC implemented by the system with a transfer function:

$$H(z) = \frac{1 - 0.5z^{-1} + z^{-3}}{1 + 0.5z^{-1} + 0.75z^{-2}}$$

Answer: Because:

$$H(z) = \frac{Y(z)}{X(z)}$$

by cross multiplication of the right and left members we have:

$$[1 + 0.5z^{-1} + 0.75z^{-2}]\,Y(z) = [1 - 0.5z^{-1} + z^{-3}]\,X(z)$$

Calculating the inverse Z-transform of each term, we find the LDECC:

$$y[n] + 0.5y[n-1] + 0.75y[n-2] = x[n] - 0.5x[n-1] + x[n-3] \Rightarrow$$
$$y[n] = -0.5y[n-1] - 0.75y[n-2] + x[n] - 0.5x[n-1] + x[n-3]$$

Example 4

For an LSI system is initially relaxed an has a transfer function, calculate:

$$H(z) = \frac{z - 1}{z^2 - z + 0.25}$$

(a) The LDECC describing the system.

(b) The impulse response of the system.

(c) The system output for input $x[n] = u[n]$

Answer: (a) We calculate the transfer function from: $H(z) = Y(z)/X(z)$, so we have:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z - 1}{z^2 - z + 0.25} = \frac{z^{-1} - z^{-2}}{1 - z^{-1} + 0.25z^{-2}}$$

We cross-multiply the fractions and get:

$$Y(z) - z^{-1}Y(z) + 0.25z^{-2}Y(z) = z^{-1}X(z) - z^{-2}X(z)$$

We apply inverse Z-transform and get the LDECC:

$$y[n] - y[n - 1] + 0.25y[n - 2] = x[n - 1] - x[n - 2] \Rightarrow$$
$$y[n] = y[n - 1] - 0.25y[n - 2] + x[n - 1] - x[n - 2]$$

Example 4 (continued)

(b) Because in the expression of the transfer function $H(z)$ in terms z^{-n} the degree of the numerator is the same as the degree of the denominator we will develop the function in some fractions $\widetilde{H}(z)$:

$$\widetilde{H}(z) = \frac{H(z)}{z} = \frac{z-1}{z(z^2 - z + 0.25)} = \frac{(z-1)}{z(z-0.5)^2}$$

The development is:

$$\widetilde{H}(z) = \frac{H(z)}{z} = \frac{R_1}{z} + \frac{R_2}{(z-0.5)^2} + \frac{R_3}{(z-0.5)}$$

We find the rest R_1 and R_2:

$$R_1 = \left[z\,\widetilde{H}(z)\right]_{z=0} = \left[\frac{(z-1)}{(z-0.5)^2}\right]_{z=0} = -4$$

$$R_2 = \left[(z-0.5)^2\,\widetilde{H}(z)\right]_{z=0.5} = \left[\frac{(z-1)}{z}\right]_{z=0.5} = -1$$

Example 4 (continued)

To find the remainder R_3 we substitute random values of z, which must not be poles. We put $z = 1$ in equation (1) and we have:

$$\widetilde{H}(1) = \frac{H(1)}{1} = \frac{(1-1)}{1(1-0.5)^2} = 0 \Rightarrow \frac{-4}{1} + \frac{-1}{(1-0.5)^2} + \frac{R_3}{(1-0.5)} = 0 \Rightarrow$$

$$= -4 - 4 + \frac{R_3}{0.5} \Rightarrow R_3 = 4$$

Therefore:

$$H(z) = \frac{R_1\, z}{z} + \frac{R_2\, z}{(z-0.5)^2} + \frac{R_3\, z}{(z-0.5)} = -4 - 2\frac{0.5\, z}{(z-0.5)^2} + 4\frac{z}{(z-0.5)}$$

Based on Table 9.1 (p. 421 of the book), the impulse response is:

$$h[n] = -4\delta[n] - 2n(0.5)^n\, u[n] + 4(0.5)^n\, u[n]$$

Example 4 (continued)

(c) We can compute the output by either convolution $y[n] = x[n] * h[n]$ or inverse transformation $y[n] = Z^{-1}\{X(z)H(z)\}$. We will follow the second way. The Z-transform of the input is:

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{1}{z - 1}, \qquad |z| > 1$$

Therefore:

$$Y(z) = X(z)H(z) = \frac{1}{z-1} \cdot \frac{z-1}{z^2 - z + 0.25} = \frac{1}{z^2 - z + 0.25} = \frac{z^{-2}}{(1 - z^{-1} + 0.25z^{-2})}$$

$$= \frac{z^{-2}}{(1 - 0.5z^{-1})^2}$$

Since the expression in the denominator is quadratic (double root) we use equation (9.39) and obtain the expansion:

$$Y(z) = \frac{R_1}{1 - 0.5z^{-1}} + \frac{R_2 z^{-1}}{(1 - 0.5z^{-1})^2} \qquad (1)$$

We calculate it R_2 from the equation (9.38):

$$R_2 = [(1 - 0.5z^{-1})^2 \, Y(z)]_{z=2} = [z^{-2}]_{z=2} = \frac{1}{4}$$

Example 4 (continued)

To calculate it R_1 we choose a value of z which is not a pole of the function $Y(z)$. Let's say that $z = 1$. We calculate it $Y(1)$ from the definition:

$$Y(1) = \frac{1^{-2}}{(1 - 0.5 \, 1^{-1})^2} = \frac{1}{0.5^2} = 4$$

We calculate it $Y(1)$ from the expansion (equation 1). Is:

$$Y(1) = \frac{R_1}{1 - 0.5 \, 1^{-1}} + \frac{R_2 \, 1^{-1}}{(1 - 0.5 \, 1^{-1})^2} = \frac{R_1}{0.5} + \frac{0.25}{0.25} = \frac{R_1}{0.5} + 1$$

Apply:

$$\frac{R_1}{0.5} + 1 = 4 \Rightarrow R_1 = \frac{3}{2}$$

So the development is:

$$Y(z) = \left(\frac{3}{2}\right) \frac{1}{1 - 0.5z^{-1}} + \left(\frac{1}{2}\right) \frac{0.5 \, z^{-1}}{(1 - 0.5z^{-1})^2}$$

From Table 9.1 we have:

$$y[n] = \frac{3}{2}(0.5)^n \, u[n] + \frac{1}{2}(0.5)^n \, nu[n] = \frac{1}{2}[3 + n] \, (0.5)^n \, u[n]$$

Example 5

To solve the LDECC with zero initial conditions and input $x[n] = u[n]$:

$$y[n] = 1.6\, y[n - 1] - 0.64\, y[n - 2] + x[n]$$

Answer: We write the LDECC, transferring the terms $y[\ ]$ to the left member:

$$y[n] - 1.6\, y[n - 1] - 0.64\, y[n - 2] = x[n]$$

The Z-transform of the input is:

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{1}{z - 1}, \qquad |z| > 1$$

We calculate the Z-transform of both members of the LDECC. Using the time shift property of the Z-transform we get:

$$Y(z) - 1.6\, z^{-1}Y(z) - 0.64\, z^{-2}Y(z) = X(z) \Rightarrow Y(z)[1 - 1.6z^{-1} - 0.64z^{-2}] = \frac{1}{1 - z^{-1}} \Rightarrow$$

$$Y(z) = \frac{1}{(1 - z^{-1})(1 - 1.6z^{-1} - 0.64z^{-2})} = \frac{1}{(1 - z^{-1})(1 - 0.8z^{-1})^2}$$

Example 5 (continued)

Due to the existence of a multiple pole we use equation (9.39) and obtain the expansion:

$$Y(z) = \frac{R_1}{1 - z^{-1}} + \frac{R_2}{1 - 0.8z^{-1}} + \frac{R_3 z^{-1}}{(1 - 0.8z^{-1})^2}, \qquad (1)$$

We calculate R_1 and R_3 from the equation (9.38):

$$R_1 = [(1 - z^{-1})^2 \, Y(z)]_{z^{-1}=1} = \left[\frac{1}{(1 - 0.8z^{-1})^2} \right]_{z^{-1}=1} = \cdots = 25$$

$$R_3 = [(1 - 0.8z^{-1})^2 \, Y(z)]_{z^{-1}=1/0.8=1.25} = \left[\frac{1}{(1 - z^{-1})} \right]_{z^{-1}=1.25} = \cdots = -4$$

To calculate it R_2 we choose a value of z which is not a pole of the function $Y(z)$. Let's say that $z = 2$. We calculate it $Y(2)$ from the definition. Is:

$$Y(2) = \frac{1}{(1 - 2^{-1})(1 - 0.8 \, 2^{-1})} = \cdots = \frac{1}{0.18}$$

Example 5 (continued)

We calculate it $Y(2)$ from the expansion (equation 1). Is:

$$Y(2) = \frac{25}{1 - 2^{-1}} + \frac{R_2}{(1 - 0.8\,2^{-1})} - \frac{4\,2^{-1}}{(1 - 0.8\,2^{-1})^2} = \frac{25}{0.5} + \frac{R_2}{0.6} - \frac{2}{0.36}$$

Doing the operations, we find:

$$R_2 = -\frac{5}{6}$$

So the development is:

$$Y(z) = 25\frac{1}{1 - z^{-1}} + \left(-\frac{5}{6}\right)\frac{1}{1 - 0.8z^{-1}} + (-5)\frac{0.8\,z^{-1}}{(1 - 0.8z^{-1})^2}$$

and from Table 9.2 (p. 421) we find:

$$y[n] = 25(1)^n u[n] - \frac{5}{6}(0.8)^n u[n] - 5(0.8)^n n\,u[n] = \left[25 - 5\left[n + \frac{1}{6}\right](0.8)^n\right] u[n]$$

Frequency Response

- If the unit circle ($z = e^{j\omega}$) is included in the region of convergence, then we can calculate the transfer function $H(z)$ on the unit circle. The result is the **frequency response** of the system:

$$H\left(e^{j\omega}\right) = b_0 \, e^{j(N-M)\omega} \frac{\prod_{m=1}^{M}(e^{j\omega} - z_m)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

- The frequency response is a complex function, so in polar coordinates it can be expressed in a **magnitude response** and a **phase response.**

- The **magnitude response** is:

$$\left|H\left(e^{j\omega}\right)\right| = |b_0| \frac{\prod_{m=1}^{M}\left|e^{j\omega} - z_m\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - p_k\right|}$$

- The **phase response** is:

$$\sphericalangle H\left(e^{j\omega}\right) = [0 \, \eta \, \pi] + [(N-M)\omega] + \sum_{m=1}^{M} \sphericalangle(e^{j\omega} - z_m) - \sum_{k=1}^{N}(\sphericalangle e^{j\omega} - p_k)$$

Frequency Response

- Since each term $\left|e^{j\omega} - z_m\right|$ represents the distance of the null z_m from the unit circle for frequency ω, and likewise each term $\left|e^{j\omega} - p_k\right|$ represents the distance of the pole p_k from the unit circle for frequency ω, it follows that the meter response translates as the quotient of the sum of the distances of the nulls from unit circle, to the sum of the distances of the poles from the unit circle, for each value of the digital frequency ω.

- The phase response is the sum of a constant term [0 or π], a linear coefficient $(N - M)$ with respect to frequency ω and a non-linear factor. The non-linear factor is the difference of the sum of the pole vectors from the sum of the zero vectors.

Poles and Zeros of the Transfer Function

- Factoring the numerator and denominator polynomials of the transfer function, we have:

$$H(z) = b_0 \, z^{N-M} \frac{\prod_{m=1}^{M}(z - z_m)}{\prod_{k=1}^{N}(z - p_k)}$$

- The poles p_k are not included in the convergence region R_H because the transfer function tends to infinity: $\lim_{z \to p_k} H(z) = \infty$.

- The graphical representation of poles and zeros forms the **pole - zero map.**

- If there is a pole and a zero in the same position of the diagram, then they cancel each other out.

- Because of the term z^{N-M} we will have $|N - M|$ zeros in the $z = 0$ if position $N > M$ or $|N - M|$ poles if $N < M$.

- The pole-zero diagram is useful in studying critical properties of a system, such as stability and causality, and in the design of digital systems and filters.

All-Zero Systems

- If $\alpha_k = 0$ για $1 \leq k \leq N$, then the transfer function, is written:

$$H(z) = \sum_{m=0}^{M} b_m \, z^{-m} = \frac{1}{z^M} \sum_{m=0}^{M} b_m \, z^{M-m}$$

- **All-Zero system** because it has only zeros and no poles. Alternatively, we can consider that all its poles are concentrated at the point $z = 0$ (with multiplicity M).

- By inverse Z-transform we find that the impulse response of the All-Zero system is given by the following equation, which describes a **non-recursive system:**

$$h[n] = \sum_{m=0}^{M} b_m \, \delta[n-m]$$

- Non-recursive systems have **a finite impulse response** (FIR).

- FIR systems are always stable, which is a major advantage.

All-Pole Systems

- In the case that $b_k = 0$ για $1 < k \leq M$, then the transfer function is written:

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a[k] \, z^{-k}}$$

- **All-Pole system** because it has only poles and no zeros. The impulse response of the all-pole system is calculated from the equation:

$$h[n] = Z^{-1}\{H(z)\}$$

- All-Pole systems are a subcase of recursive systems.

- Recursive systems are also called **infinite impulse response** (IIR).

- If all its poles $H(z)$ lie inside the unit circle, then the system is stable.

Theorems of Causality and Stability of Systems

System Stability Theorem: An LSI system is stable if and only if the Region of Convergence (ROC) of the transfer function $H(z)$ includes the unit circle $|z| = 1$.

Theorem of Causality and Stability of Systems: An causal LSI system is stable if and only if all its poles lie <u>inside</u> the unit circle.

- A **stable** system includes in the ROC the unit circle. The reverse is also true.

- If the system is **causal** and has a ROC R_h: $|z| > a$, where $\alpha < 1$ then the ROC includes the unit circle.

- Since the ROC cannot contain poles, it follows that a causal LSI system is stable when all its poles lie inside the unit circle. The reverse is also true.

Depending on the value of α, the pole can be either inside the unit circle ($|\alpha| < 1$) or outside ($|\alpha| > 1$) or on the circle ($|\alpha| = 1$). More specifically, it can take six different positions: $\alpha < -1$, $\alpha = -1$, $1 < a \leq 0$, $0 \leq a < 1$, $\alpha = 1$ και $\alpha > 1$, as shown in the following figures.

Stability Study of Systems using Z-transform

Let be system with impulse response $h[n] = a^n u[n]$ and transfer function:
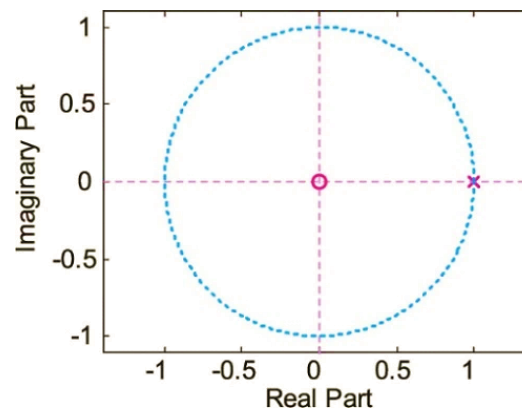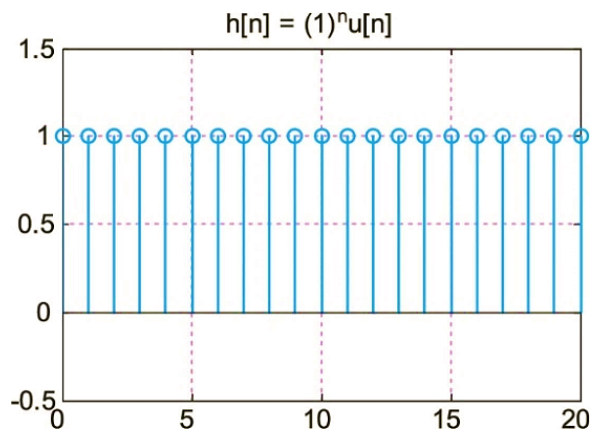
$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad R_x: |z| > a$$

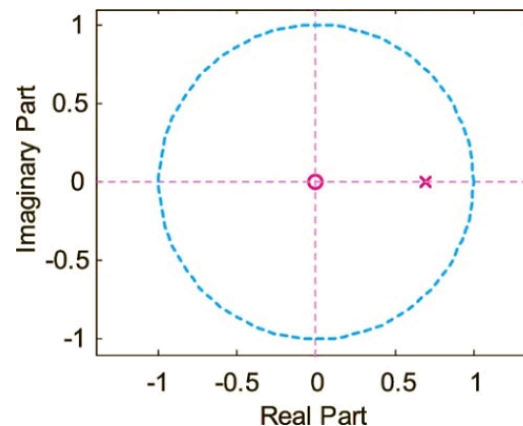The system has a pole at $z = a$ and a zero at $z = 0$.
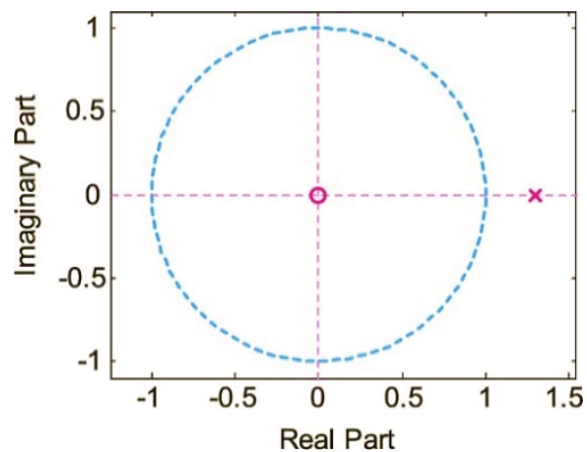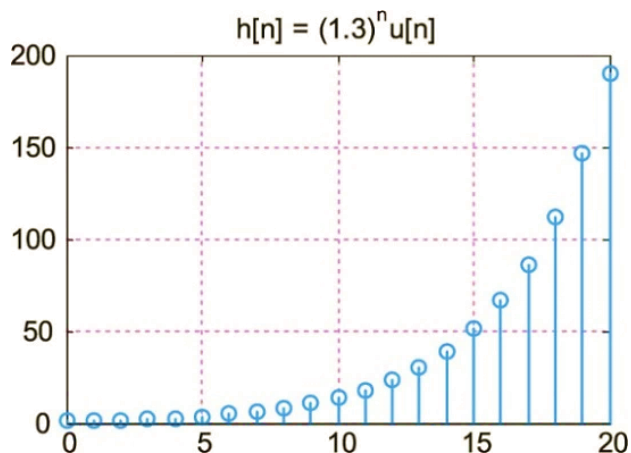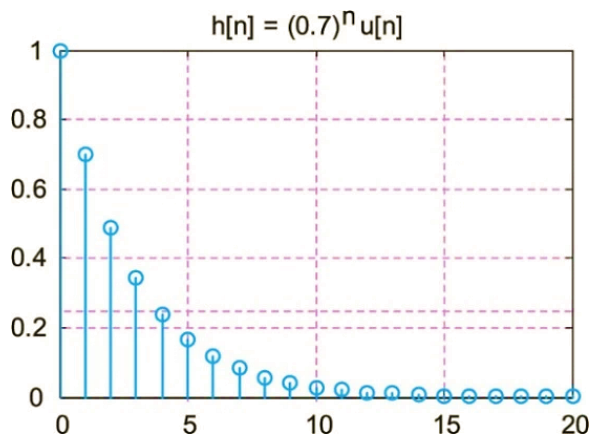

(a) $a > 1$: Pole outside the unit circle (right half-plane)
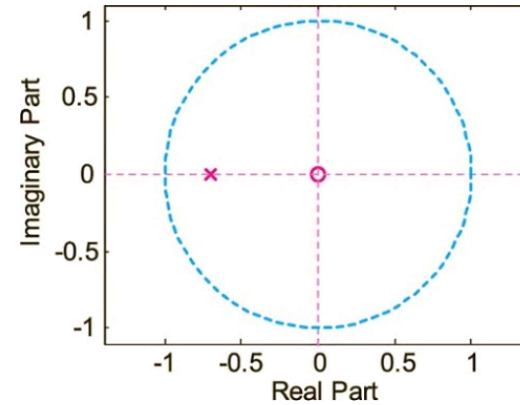
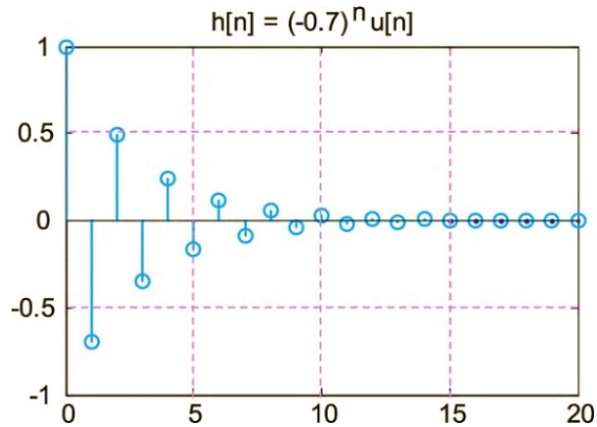# Stability Study of Systems using Z-transform

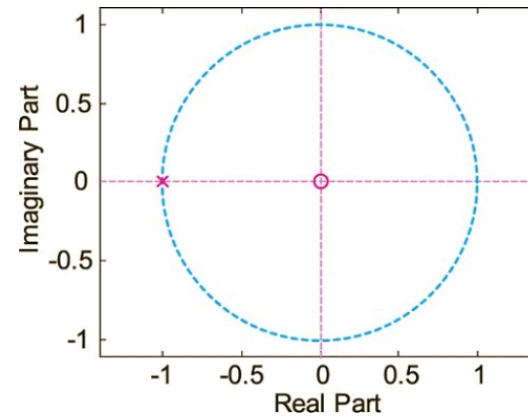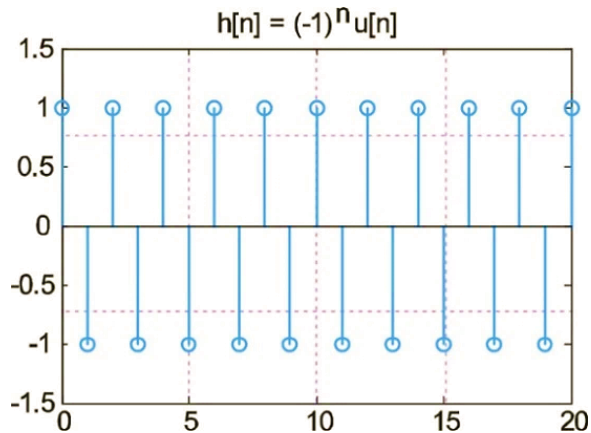(b) $a = 1$: Pole on the unit circle (right half-plane)

(c) $0 \leq a < 1$: Pole inside the unit circle (right half-plane)

Stability Study of Systems using Z-transform

(d) $1 < a \le 0$: Pole inside the unit circle (left half plane)

(e) $a = -1$: Pole on the unit circle (left half plane)

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Stability Study of Systems using Z-transform

h[n] = $(-1.3)^n$u[n]
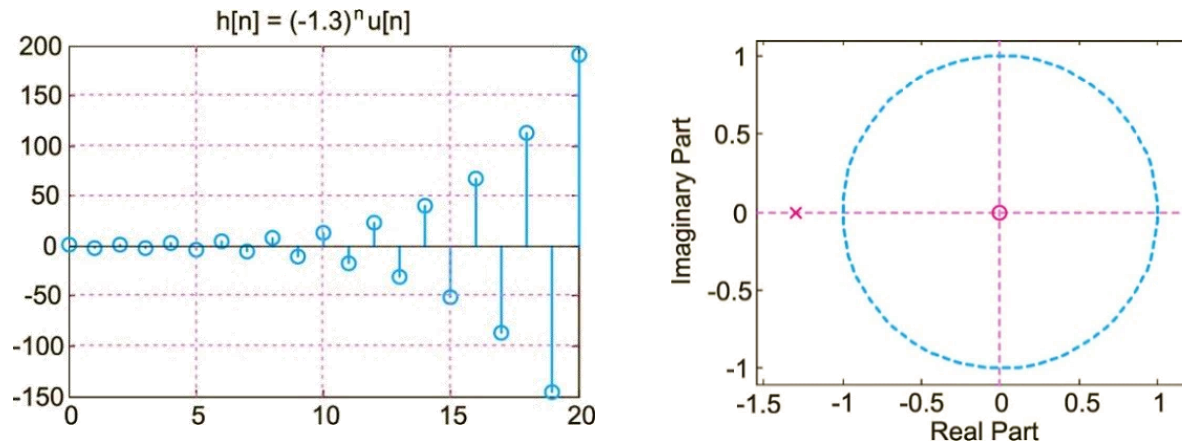
(f) $a < -1$: Pole outside the unit circle (left half plane)

- When the pole is in the right half-plane the sequence is monotonic.

- When the pole is inside the unit circle $|\alpha| < 1$, then $h[n]$ tends to zero for $n \to \infty$, so the system is **stable.**

- When the pole lies on the unit circle $|\alpha| = 1$, then $h[n]$ has a constant value and infinite duration, so the system is **marginally stable.**

- When the pole is outside the unit circle $|\alpha| > 1$, then, $h[n]$ tends to infinity for $n \to \infty$, so the system is **unstable.**

The same applies to the case of complex poles.

Example 6

An LSI system is described by the transfer function:

$$H(z) = \frac{3}{1 - 0.5z^{-1}} + \frac{4}{1 - 2z^{-1}}$$

Determine the region of convergence R_h and calculate the impulse response $h[n]$ so that the system is: (a) Stable, (b) Causal.

Answer: The poles of the system are $p_1 = 0.5$ and $p_2 = 2$.

(a) For the system to be stable, according to the stability theorem, the ROC R_h must include the unit circle.
So, the ROC must satisfy the equation:

$$R_h: \; 0.5 < |z| < 2 \; \text{ or } |z| > 0.5 \text{ also } |z| < 2$$

Since $|z| > 0.5$, the first part of the transfer function $H(z)$, that is, the function $H_1(z) = 3/(1 - 0.5z^{-1})$ has an inverse transformation Z the right-hand side sequence $h_1[n] = 3(0.5)^n u[n]$, which is causal and stable.

Example 6 (continued)

Since $|z| < 2$, the function $H_2(z) = 4/(1 - 2z^{-1})$ has an inverse Z-transform the left-hand side sequence $h_2[n] = -4(2)^n u[-n-1]$, which is stable but not causal. Therefore, the total impulse response is:

$$h[n] = h_1[n] + h_2[n] = 3(0.5)^n u[n] - 4(2)^n u[-n-1]$$

We observe that the impulse response for $n \to \infty$ tends to zero, so the system is stable. But it is not causal, because of the left side sequence $h_2[n]$.

Example 6 (continued)

(b) For the system to be causal, the ROC must be a region outside the circle that has a radius greater than the farthest pole. So, the ROC must satisfy the equation R_h: $|z| > 2$.

Since $|z| > 2$ and $|z| > 0.5$. So, its inverse transformation $H_1(z) = 3/(1 - 0.5z^{-1})$ is the right-hand side sequence $h_1[n] = 3(0.5)^n u[n]$, which is causal and stable.

Since $|z| > 2$, the function $H_2(z) = 4/(1 - 2z^{-1})$ has an inverse Z-transform the right-hand side sequence $h_2[n] = 4(2)^n u[n]$, which is causal but unstable because it tends to infinity for $n \to \infty$. Therefore, the total impulse response is:

$$h[n] = h_1[n] + h_2[n] = 3(0.5)^n u[n] + 4(2)^n u[n]$$

We observe that the impulse response is causal, but not stable due to the sequence $h_2[n]$.

Solving Difference Equations

Solving LDECC with Z^{+} (1/4)

- In practical discrete-time systems the input signals are causal, while it is common for the system not to be in an initial quiescent state.

- We will present a methodology for solving a Linear Difference Equation with Constant Coefficients (LDECC) with initial conditions that describes a LSI system that is not initially relaxed, making use of the one-sided transformation Z^{+}.

- For the solution we will use the property of the displacement in time of Z^{+}, which is described by:

$$x[n - n_0]\, u[n] \xrightarrow{\ Z^{+}\ } z^{-n_0} X^{+}(z) + \sum_{n=1}^{n_0} x[-n]\, z^{n-n_0}$$

- We consider a discrete-time LSI system described by LDECC:

$$y[n] = \sum_{m=0}^{M} b_m x[n-m] - \sum_{k=1}^{N} a_k y[n-k] \qquad (1)$$

with initial conditions $y[-1], y[-2], \dots, y[-N]$. The signal $x[n]$ is causal.

- Calculating the one-sided transformation Z^+ of both members of the LDECC and from the time shift property of Z^+, we have:

$$Y^+(z) = \sum_{m=0}^{M} b_m\, z^{-m}\, X^+(z) - \sum_{k=1}^{N} a_k\, z^{-k}\, Y^+(z) + \sum_{k=1}^{N} z^{-k} \sum_{n=0}^{N-1} y[-n] z^n$$

- We transfer to the left member the terms $Y^+(z)$, and we have:

$$Y^+(z) \sum_{k=0}^{N} a_k\, z^{-k} = X^+(z) \sum_{m=0}^{M} b_m\, z^{-m} + \sum_{k=1}^{N} z^{-k} \sum_{n=0}^{N-1} y[-n]\, z^n$$

Solving LDECC with Z^+ (3/4)

- Calling the polynomials $A(z) = \sum_{k=0}^{N} a_k\, z^{-k}$, $B(z) = \sum_{m=0}^{M} b_m z^{-m}$ and $I(z) = \sum_{k=1}^{N} z^{-k} \sum_{n=0}^{N-1} y[-n]z^n$, the previous equation is written:

$$Y^+(z)A(z) = X^+(z)B(z) + I(z)$$

- Solving for $Y^+(z)$ and because $H(z) = B(z)/A(z)$ we get:

$$Y^+(z) = \frac{B(z)}{A(z)}X^+(z) + \frac{I(z)}{A(z)} = H(z)\, X^+(z) + \frac{I(z)}{A(z)} \qquad (2)$$

- We notice that in the case of a system that <u>is not initially relaxed,</u> the response also has the term $I(z)/A(z)$, where the polynomial $I(z)$ is due to the initial conditions of the system.

- We recall that the response of a system at initial rest is:

$$Y^+(z) = H(z)\, X^+(z)$$

Solving LDECC with Z $^+$ (4/4)

- The inverse Z-transform of the first fraction of (2) gives the **zero initial state response,** while the inverse Z -transform of the second fraction gives the **zero input response,** i.e.

  o Zero-state response:

$$y_{zs}[n] = Z^{-1}\{H(z)\, X^+(z)\}$$

  o Zero-input response:

$$y_{zi}[n] = Z^{-1}\left\{\frac{I(z)}{A(z)}\right\}$$

- Because the polynomial $I(z)$ describes the initial conditions of the system, we conclude that the zero-input response is the output due solely to the (non-zero) initial state of the system.

- Therefore, the total output of an LSI system that is not at initial rest and receives a causal signal at its input is:

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

Example 7

To find the response of the LSI system described by LDECC $y[n] = 0.2y[n-1] + 0.8y[n-2] + x[n]$, for entry $x[n] = (0.5)^n u[n]$ and initial conditions $y[-1] = 5$ and $y[-2] = 10$.

Answer: We calculate Z^+ each of the terms of the LDECC:

$$Y^+(z) = 0.2\left[z^{-1}Y^+(z) + y[-1]\right] + 0.8\left[z^{-2}Y^+(z) + z^{-1}y[-1] + y[-2]\right] + X^+(z)$$

Substituting the values of the initial conditions, we have:

$$Y^+(z) = 0.2\left[z^{-1}Y^+(z) + 5\right] + 0.8\left[z^{-2}Y^+(z) + 5z^{-1} + 10\right] + X^+(z) \Rightarrow$$

$$Y^+(z) = 0.2\,z^{-1}Y^+(z) + 1 + 0.8\,z^{-2}Y^+(z) + 4z^{-1} + 8 + X^+(z)$$

Transferring the terms containing the $Y^+(z)$ to the left member, we have:

$$Y^+(z)[1 - 0.2z^{-1} - 0.8z^{-2}] = 9 + 4z^{-1} + X^+(z)$$

Example 7 (continued)

Because $X^+(z)$ its one-sided transformation $x[n] = (0.5)^n u[n]$ is:
$X^+(z) = 1/(1 - 0.5z^{-1})$, we get:

$$Y^+(z)[1 - 0.2z^{-1} - 0.8z^{-2}] = (9 + 4z^{-1}) + \frac{1}{1 - 0.5z^{-1}}$$

Solving for $Y^+(z)$ we have:

$$Y^+(z) = \frac{(9 + 4z^{-1})}{1 - 0.2z^{-1} - 0.8z^{-2}} + \frac{\dfrac{1}{1 - 0.5z^{-1}}}{1 - 0.2z^{-1} - 0.8z^{-2}} \qquad (1)$$

By adding the fractions and then factoring the denominator, we have:

$$Y^+(z) = \frac{10 - 0.5z^{-1} - 2z^{-2}}{(1 - 0.2z^{-1} - 0.8z^{-2})(1 - 0.5z^{-1})} =$$

$$= \frac{10 - 0.5\, z^{-1} - 2z^{-2}}{(1 - z^{-1})(1 + 0.8z^{-1})(1 - 0.5z^{-1})}$$

Example 7 (continued)

The ROC is $|z| > 1$ and the poles of the system are $z_1 = 1,\ \ z_2 = -0.8,\ z_3 = 0.5$. Because one pole lies on the unit circle (and the rest are inside the unit circle) the system is marginally stable.

To expand it $Y^+(z)$ into some fractions, we will calculate the remainders R_1, R_2 and R_3:

$$Y^+(z) = \frac{R_1}{1 - z^{-1}} + \frac{R_2}{1 + 0.8z^{-1}} + \frac{R_3}{1 - 0.5z^{-1}}$$

Since the poles are simple and distinct ($z_1 = 1,\ \ z_2 = -0.8,\ z_3 = 0.5$), we calculate the residues R_1, R_2, R_3 calculate residuals from the equation:

$$R_k = \left. \frac{\tilde{b}[0] + \tilde{b}[1]z^{-1} + \cdots + \tilde{b}[N-1]z^{-(N-1)}}{1 + a[1]z^{-1} + \cdots + a[N]z^{-N}} (1 - p_k z^{-1}) \right|_{z = p_k}$$

and we have:

Example 7 (continued)

- $R_1 = [Y^+(z)(1 - z^{-1})]_{z=1} = \left[\frac{10 - 0.5\, z^{-1} - 2z^{-2}}{(1 + 0.8z^{-1})(1 - 0.5z^{-1})}\right]_{z=1} = \frac{25}{3}$

- $R_2 = [Y^+(z)(1 + 0.8z^{-1})]_{z=-0.8} = \left[\frac{10 - 0.5\, z^{-1} - 2z^{-2}}{(1 - z^{-1})(1 - 0.5z^{-1})}\right]_{z=-0.8} = \frac{80}{39}$

- $R_3 = [Y^+(z)(1 - 0.5z^{-1})]_{z=0.5} = \left[\frac{10 - 0.5\, z^{-1} - 2z^{-2}}{(1 - z^{-1})(1 + 0.8z^{-1})}\right]_{z=0.5} = -\frac{10}{26}$

Therefore, its expansion $Y^+(z)$ into some fractions is:

$$Y^+(z) = \left(\frac{25}{3}\right)\frac{1}{1 - z^{-1}} + \left(\frac{80}{39}\right)\frac{1}{1 + 0.8z^{-1}} + \left(-\frac{10}{26}\right)\frac{1}{1 - 0.5z^{-1}}$$

Performing an inverse Z-transform yields the desired solution:

$$y[n] = \left(\frac{25}{3}\right)(1)^n u[n] + \left(\frac{80}{39}\right)(-0.8)^n u[n] + \left(-\frac{10}{26}\right)(0.5)^n u[n]$$

$$= \left[\left(\frac{25}{3}\right) + \left(\frac{80}{39}\right)(-0.8)^n \left(-\frac{10}{26}\right)(0.5)^n\right] u[n]$$

Example 7 (continued)

The overall solution can be expressed in the following ways:

- As the sum of the **homogeneous solution** and the **partial solution:**

$$y[n] = \left[\left(\frac{25}{3}\right) + \left(\frac{80}{39}\right)(-0.8)^n\right]u[n] + \left(-\frac{10}{26}\right)(0.5)^n u[n]$$

The homogeneous solution is due to the poles and the partial solution to the zeros of the input signal.

- As the sum of the **transient state** and the **permanent state:**

$$y[n] = \left[\left(\frac{80}{39}\right)(-0.8)^n + \left(-\frac{10}{26}\right)(0.5)^n\right]u[n] + \left(\frac{25}{3}\right)u[n]$$

The transient state is attributed to poles located inside the unit circle, while the steady state is attributed to poles located on the unit circle. If there are poles outside the unit circle, then the response tends to infinity, and the system becomes unstable.

Example 7 (continued)

- As the sum of the **response zero input** (or initial state) and **response zero initial state.** Specifically, the above equation (1) is a sum of two terms. The first term can be written as:

$$Y_{zi}(z) = H(z)\, X_{ic}(z)$$

represents the response for the given input, assuming a zero initial state and is called **the zero initial state response.**

The function $X_{ic}(z)$ can be thought of as an equivalent initial state input which produces the same output $Y_{zi}(z)$ that is created by the initial conditions. In our example and based on equation (1) it follows: $x_{ic}[n] = \{\hat{9}, 4\}$.

The second term can be written as:

$$Y_{zs}(z) = H(z)\, X(z)$$

represents the response for zero input, with only the initial state applied, and is called **the zero input response.**

Example 7 (continued)

- Calculating the inverse transformation Z of equation (1), we have:

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

where $y_{zs}[n]$ is the zero input response given by:

$$y_{zi}[n] = \left(\frac{65}{9}\right)(1)^n u[n] + \left(\frac{116}{45}\right)(-0.8)^n u[n]$$

$$= \left[\left(\frac{65}{9}\right) + \left(\frac{116}{45}\right)(-0.8)^n\right]u[n]$$

and $y_{zs}[n]$ is the zero-state response, which is given by:

$$y_{zs}[n] = \left(\frac{10}{9}\right)(1)^n u[n] + \left(\frac{32}{117}\right)(-0.8)^n u[n] + \left(\frac{5}{13}\right)(0.5)^n u[n]$$

$$= \left[\left(\frac{32}{117}\right)(-0.8)^n + \left(\frac{5}{13}\right)(0.5)^n + \left(\frac{10}{9}\right)\right]u[n]$$

Solving Differential Equations

The solution of linear differential equations with constant coefficients that we studied with the Laplace transform can also be performed with the Z-transform if we appropriately transform the derivatives into differences taking into account the sampling frequency. For this purpose we use the approximate relations:

$$\frac{dy(t)}{dt} \approx \frac{y(t) - y(t - T_s)}{T_s}$$

$$\frac{d^2 y(t)}{dt^2} = \frac{d}{dt}\left(\frac{dy(t)}{dt}\right) \approx \frac{d}{dt}\left(\frac{y(t) - y(t - T_s)}{T_s}\right) = \frac{y(t) - 2y(t - T_s) + y(t - 2T_s)}{T_s^2}$$

In a similar way we can calculate the derivatives of higher order.

Example 8

A linear time-invariant LTI continuous-time system that is initially relaxed is described by the Linear Differential Equation with Constant Coefficients:

$$\frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + 6y(t) = x(t)$$

To find the response of the system for input $x(t) = u[t]$.

<u>Answer</u>: Applying Laplace transform to the differential equation we find:

$$s^2 Y(s) + 5sY(s) + 6Y(s) = X(s) = \frac{1}{s} \Rightarrow Y(s) = \frac{1}{s(s^2 + 5s + 6)} = \cdots$$

$$= \left(\frac{1}{6}\right)\frac{1}{s} - \left(\frac{1}{2}\right)\frac{1}{s+2} + \left(\frac{1}{3}\right)\frac{1}{s+3}$$

Therefore the response for step input is:

$$y(t) = \left[\frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}\right]u(t)$$

Example 8 (continued)

Applying the first and second derivative approximations given in relations (11.29) and (11.30) to the differential equation, we then set $t = nTs$ and obtain:

$$\frac{y(t) - 2y(t - T_s) + y(t - 2T_s)}{T_s^2} + 5\frac{y(t) - y(t - T_s)}{T_s} + 6y(t) = x(t)$$

$$\Rightarrow \left(6 + \frac{5}{T_s} + \frac{1}{T_s^2}\right)y(t) + \left(-\frac{5}{T_s} - \frac{2}{T_s^2}\right)y(t - T_s) + \left(\frac{1}{T_s^2}\right)y(t - 2T_s) = x(t)$$

$$\Rightarrow \left(6 + \frac{5}{T_s} + \frac{1}{T_s^2}\right)y(nT_s) + \left(-\frac{5}{T_s} - \frac{2}{T_s^2}\right)y((n-1)T_s) + \left(\frac{1}{T_s^2}\right)y((n-2)T_s) = x(nT_s)$$

$$\Rightarrow a_1 y(nT_s) + a_2 y((n-1)T_s) + a_3 y((n-2)T_s) = b_1 x(nT_s)$$

where:

$$a_1 = \left(6 + \frac{5}{T_s} + \frac{1}{T_s^2}\right), \qquad a_2 = -\left(\frac{5}{T_s} + \frac{2}{T_s^2}\right), \qquad a_3 = \left(\frac{1}{T_s^2}\right), \qquad b_1 = 1$$

Example 8 (continued)

The sampling period must have a sufficiently small value calculated by the Nyquist criterion. For simplicity we set $T_s = 1$ and get:

$$a_1 = 12, \qquad a_2 = -7, \qquad a_3 = 1, \qquad b_1 = 1$$

Therefore, the given continuous-time system when sampled is described by the difference equation:

$$12y[n] - 7y[n-1] + y[n-2] = x[n], \ \ n > 0$$

For zero initial conditions we have $y[0] = 1/12$. From the final value theorem we find $y[n] = 1/6$ for $n \to \infty$. The Z-transform of the output is:

$$Y(z)(12 - 7z^{-1} + z^{-2}) = \frac{1}{1 - z^{-1}} \Rightarrow \ldots \Rightarrow Y(z) = \frac{1}{(1 - z^{-1})(3 - z^{-1})(4 - z^{-1})} = \cdots$$

$$= \frac{1}{6}\frac{1}{1 - z^{-1}} - \frac{1}{2}\frac{1}{3 - z^{-1}} + \frac{1}{3}\frac{1}{4 - z^{-1}}$$

Calculating the inverse of Z using Table 9.2, we find:

$$y[n] = \left[\frac{1}{6} - \frac{1}{2}(3)^n + \frac{1}{3}(4)^n\right]u[n]$$

This solution is approximate and not exact because we chose a large value for the sampling period $T_s = 1$. With a smaller value, the solution becomes more accurate.

Frequency Response

- Definition
- Properties

Frequency Response

- The DTFT of the impulse response $h[n]$ of an LSI stable system is called the **frequency response** and is calculated by:

$$H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{+\infty} h[k]\, e^{-j\omega k}$$

- **Magnitude:**

$$\left|H\left(e^{j\omega}\right)\right| = \sqrt{H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})}$$

- **Phase:**

$$\varphi_H(\omega) = \tan^{-1}\left[\frac{H_I\left(e^{j\omega}\right)}{H_R(e^{j\omega})}\right]$$

- **Gain diagram:** $A(\omega) = 20 log\left|H(e^{j\omega})\right|(\text{dB})$

- Apply: $\left|H(e^{j\omega})\right| = 1 \rightarrow 0\ dB,\ \left|H(e^{j\omega})\right| = 10 \rightarrow 20\ dB,\ \left|H(e^{j\omega})\right| = 0.1 \rightarrow -20\ dB$

- **Group delay:**

$$\tau_H(\omega) = -\frac{d\varphi_H(\omega)}{d\omega}$$

Frequency Response Properties

- It has all the properties of DTFT. Particularly:

- **Periodicity:** The frequency response is a periodic function with **a period 2π**, ie it is valid $H\left(e^{j\omega_0}\right) = H\left(e^{j(\omega_0+2\pi)}\right)$.

- **Spectral Symmetry:** If $h[n]$ is a real sequence, then $H\left(e^{j\omega}\right)$ is a conjugate symmetric function of frequency, that is:

$$H\left(e^{-j\omega}\right) = H^*\left(e^{j\omega}\right)$$

- <u>Perfect symmetry</u> **-** Actual part and width:

$$H_R\left(e^{j\omega}\right) = H_R\left(e^{-j\omega}\right) \text{ and } \left|H_R\left(e^{j\omega}\right)\right| = \left|H_R\left(e^{-j\omega}\right)\right|$$

- <u>Unnecessary symmetry</u> **-** Fantasy part, phase and group delay:

$$H_I\left(e^{j\omega}\right) = -H_I\left(e^{-j\omega}\right), \ \varphi_H(\omega) = -\varphi_H(-\omega) \text{ and } \tau_H(\omega) = -\tau_H(-\omega)$$

- Based on these properties, it follows that for the design of the frequency response $H\left(e^{j\omega}\right)$ **only half a period** is sufficient, we usually choose $[\mathbf{0}, \boldsymbol{\pi}]$.

Frequency Response Properties

- **Frequency Response Inversion:** If the frequency response of an LSI system is:

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-jn\omega}$$

- The impulse response can be retrieved by integration over any interval of length 2π:

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} H\left(e^{j\omega}\right) e^{jn\omega} d\omega$$

Example 9

To find the amplitude, phase and group delay of a system with impulse response $h[n] = \delta[n] - \alpha\delta[n-1]$, where $\alpha \in \mathbb{R}$.

Answer: The frequency response is:

$$H\left(e^{j\omega}\right) = 1 - ae^{-j\omega} = 1 - a\cos\omega + ja\sin\omega$$

The magnitude is:

$$\left|H\left(e^{j\omega}\right)\right|^2 = (1 - \alpha\cos\omega)^2 + (\alpha\sin\omega)^2$$
$$= 1 - 2\alpha\cos\omega + \alpha^2\cos^2\omega + \alpha^2\sin^2\omega$$
$$= 1 - 2\alpha\cos\omega + a^2$$

The phase is:

$$\varphi_H(\omega) = tan^{-1}\left[\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right] = tan^{-1}\left[\frac{a\sin\omega}{1 - \alpha\cos\omega}\right]$$

The group delay is:

$$\tau_H(\omega) = \frac{d\varphi_H\omega}{d\omega} = \cdots = \frac{\alpha^2 - \alpha\cos\omega}{1 - 2\alpha\cos\omega + a^2}$$

Applications of DTFT

- Calculation of Frequency Response

- Solving Differential Equations

- Design of Inverse Systems

- Systems Connections

Calculation of Frequency Response
by Difference Equation

- We know that the following LDECC describes the input-output relationship of a LSI system:

$$y[n] = \sum_{m=0}^{M} b_m x[n-m] - \sum_{k=1}^{N} a_k y[n-k]$$

- Applying DTFT to both members of the LDECC and from the linearity and time-shift properties of the DTFT, the frequency response is obtained:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{m=0}^{M} b_m e^{-jm\omega}}{1 + \sum_{k=1}^{N} a_k e^{-jk\omega}} = \frac{\sum_{m=0}^{M} b_m e^{-jm\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

- Because the LDECC is unique to each system and describes it uniqualy, so is the frequency response.

Example 10

To find the frequency response of a discrete-time LSI system with LDECC: $y[n] - 0.5y[n-1] = x[n] + 2x[n-1] + x[n-2]$ and with zero initial conditions.

Answer: We calculate the DTFT of each member of the difference equation:

$$Y(e^{j\omega}) - 0.5\,Y(e^{j\omega})\,e^{-j\omega} = X(e^{j\omega}) + 2e^{-j\omega}X(e^{j\omega}) + e^{-j2\omega}X(e^{j\omega}) \Rightarrow$$

$$\Rightarrow \left(1 - 0.5e^{-j\omega}\right)Y(e^{j\omega}) = \left(1 + 2e^{-j\omega} + e^{-j2\omega}\right)X(e^{j\omega})$$

Solving for $Y(e^{j\omega})/X(e^{j\omega})$, we find the frequency response:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - 0.5e^{-j\omega}}$$

Solving Difference Equations

- Solving the LDECC for $Y(e^{j\omega})$, we find:

$$Y(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_m \, e^{-jm\omega}}{1 + \sum_{k=1}^{N} a_k \, e^{-jk\omega}} X(e^{j\omega})$$

- For <u>zero initial conditions,</u> a LDECC is solved with the following steps:

  - Transferring the problem to the frequency domain, calculating the DTFT of each term of the LDECC.

  - Solving for $Y(e^{j\omega})$.

  - Returning to the time domain and finding $y[n]$, by computing the inverse DTFT of $Y(e^{j\omega})$.

- It is a computationally simpler way than the method of finding a homogeneous and partial solution in the time domain.

- It is only applicable to the case of a system described by LDECC with zero initial conditions.

- If the system is not initially at rest, then the complete solution in the frequency domain is given by the one-sided Z-transform.

Example 11

To find the impulse response of a LSI system described by LDECC:
$y[n] - 0.25y[n-1] = x[n] - x[n-2]$ (zero initial conditions).

Answer: We calculate the DFTF of each LDECC member:

$$Y(e^{j\omega}) - 0.25\, Y(e^{j\omega})\, e^{-j\omega} = X(e^{j\omega}) - e^{-2j\omega}X(e^{j\omega})$$

Because $X(e^{j\omega}) = \Delta(e^{j\omega}) = 1$ we have:

$$(1 - 0.25e^{-j\omega})\, Y(e^{j\omega}) = (1 - e^{-2j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{1 - e^{-2j\omega}}{1 - 0.25\, e^{-j\omega}} = \frac{1}{1 - 0.25\, e^{-j\omega}} - \frac{e^{-2j\omega}}{1 - 0.25\, e^{-j\omega}}$$
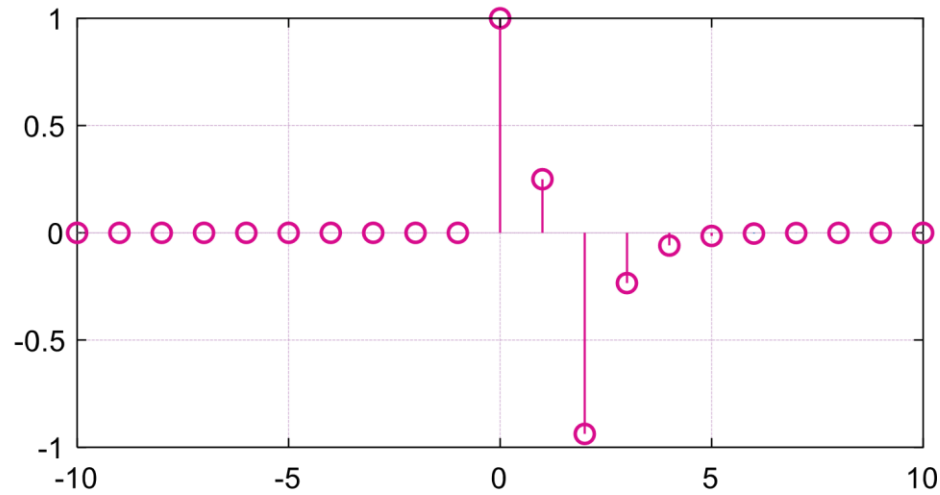
Because:

$$(0.25)^n u[n] \xleftrightarrow{DTFT} 1/(1 - 0.25\, e^{-j\omega})$$

and with the linearity and time-shift properties of the DTFT we find the inverse DTFT, i.e. the impulse response:

Example 11 (continued)

Impulse response:

$$h[n] = (0.25)^n \, u[n] - (0.25)^{n-2} u[n-2]$$

Impulse response $h[n]$

Example 12

Find the LDECC of the LSI system with frequency response:

$$H(e^{j\omega}) = e^{j\omega} \frac{1}{1.1 + \cos \omega}$$

Answer: We express the $H(e^{j\omega})$ function of complex exponential terms:

$$H(e^{j\omega}) = \frac{e^{j\omega}}{1.1 + 0.5e^{-j\omega} + 0.5e^{j\omega}}$$

Multiplying numerator and denominator by the quantity $2e^{j\omega}$ we get:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1 + 2.2e^{-j\omega} + e^{-2j\omega}}$$

Cross multiplying the terms of the right and left sides, we have:

$$\left[1 + 2.2e^{-j\omega} + e^{-2j\omega}\right] Y(e^{j\omega}) = 2X(e^{j\omega})$$

By inverse DTFT of each term we obtain the difference equation:

$$y[n] + 2.2y[n-1] + y[n-2] = 2x[n]$$

Inverse Systems

- Given a discrete-time LSI system with impulse response $h[n]$, **inverse system** has a impulse response $g[n]$ that satisfies the equation:

$$h[n] * g[n] = \delta[n]$$

- The frequency response of the inverse system is:

$$H(e^{j\omega})\, G(e^{j\omega}) = 1 \Rightarrow G(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

- After inverting a practical system we should check whether the resulting inverse system is causal, i.e. practically realizable, as well as stable. For the latter to hold, the poles and zeros of the original system must lie within the unit circle.

Example 13

To calculate the impulse response of the inverse system of an LSI system with frequency response:

$$H\left(e^{j\omega}\right) = \frac{1 - 0.4\, e^{-j\omega}}{1 + 0.7\, e^{-j\omega}}$$

<u>Answer</u>: The frequency response of the inverse system is:

$$G\left(e^{j\omega}\right) = \frac{1}{H(e^{j\omega})} = \frac{1 + 0.7\, e^{-j\omega}}{1 - 0.4\, e^{-j\omega}} = \frac{1}{1 - 0.4\, e^{-j\omega}} + \frac{0.7\, e^{-j\omega}}{1 - 0.4\, e^{-j\omega}}$$

Using the DTFT pair:

$$\frac{1}{1 - \alpha\, e^{-j\omega}} \xleftarrow{\quad DTFT \quad} \alpha^n u[n], \qquad |\alpha| < 1$$

and the time shift property:

$$x[n - n_0] \xleftarrow{\quad DTFT \quad} e^{-jn_0\omega} X\left(e^{j\omega}\right)$$
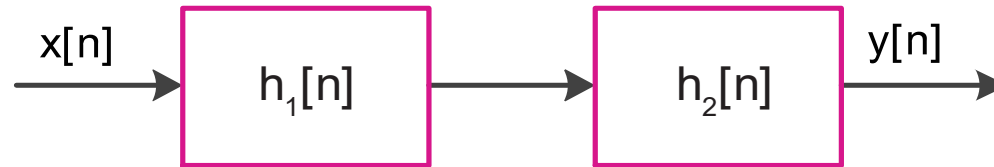
we find the impulse response:

$$g[n] = (0.4)^n\, u[n] + 0.7\, (0.4)^{n-1} u[n - 1]$$

Systems Connections

- Serial Connection
- Parallel Connection
- Feedback Connection

Serial Connection

Serial connection of systems

- Total System Impulse Response:
$$h[n] = h_1[n] * h_2[n]$$

- Total System Frequency Response:
$$H(e^{j\omega}) = H_1(e^{j\omega}) \, H_2(e^{j\omega})$$

- Relationships apply:

  ○ $20log|H(e^{j\omega})| = 20log|H_1(e^{j\omega})| + 20log|H_2(e^{j\omega})|$

  ○ $\varphi_H(\omega) = \varphi_{H1}(\omega) + \varphi_{H2}(\omega)$

  ○ $\tau_H(\omega) = \tau_{H1}(\omega) + \tau_{H2}(\omega)$

Parallel Connection

Parallel connection of systems

- Total System Impulse Response:

$$h[n] = h_1[n] + h_2[n]$$

- Total System Frequency Response:

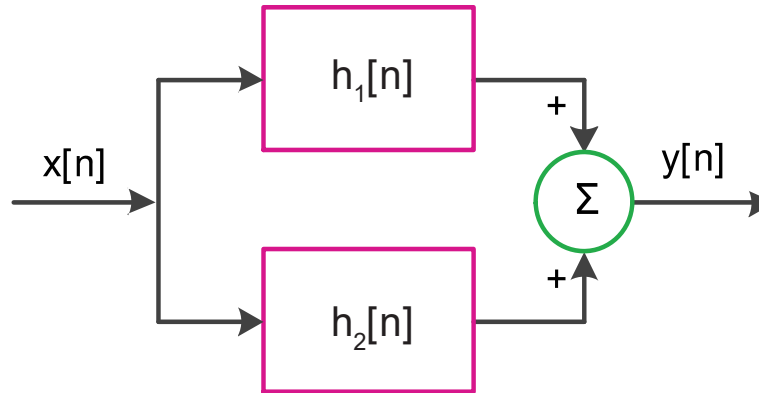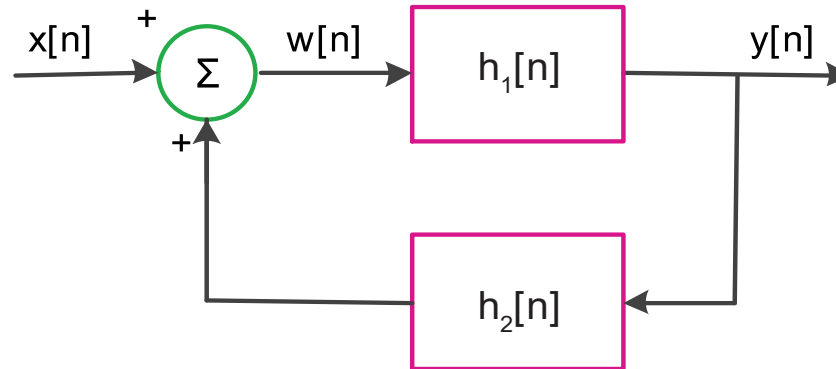$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

Feedback Connection

Feedback (positive) connection of systems
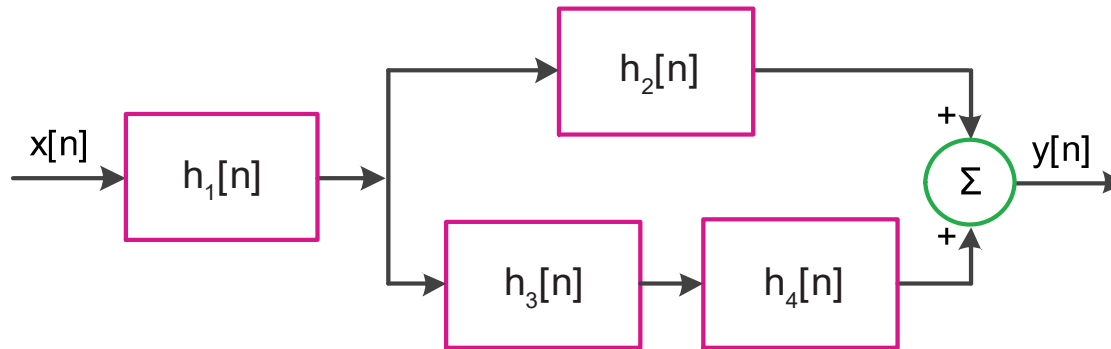
- Total System Impulse Response:

$$h[n] = \frac{h_1[n]}{1 - h_1[n] * h_2[n]}$$

- Total System Frequency Response:

$$H(e^{j\omega}) = \frac{H_1(e^{j\omega})}{1 - H_1(e^{j\omega}) \, H_2(e^{j\omega})}$$

Example 14

(a) For the following system wiring, calculate the overall frequency response, as a function of the responses $H_1(e^{j\omega})$, $H_2(e^{j\omega})$, $H_3(e^{j\omega})$ and $H_4(e^{j\omega})$.

(b) Calculate the overall frequency response given that:

- $h_1[n] = \delta[n] + 2\delta[n-1] + \delta[n-4]$

- $h_2[n] = h_3[n] = (0.2)^n u[n]$

- $h_4[n] = \delta[n-2]$

Example 14 (continued)

Answer: (a) $H(e^{j\omega}) = H_1(e^{j\omega})\left[H_2(e^{j\omega}) + H_3(e^{j\omega})H_4(e^{j\omega})\right]$

(b) The individual responses of each system individually are:

$$H_1(e^{j\omega}) = 1 + 2e^{-j2\omega} + e^{-j4\omega} = (1 + e^{-j2\omega})^2$$

$$H_2(e^{j\omega}) = H_3(e^{j\omega}) = \frac{1}{1 - 0.2\, e^{-j\omega}}$$

$$H_4(e^{j\omega}) = e^{-j2\omega}$$

Therefore, the total frequency response is:

$$H(e^{j\omega}) = H_1(e^{j\omega})\left[H_2(e^{j\omega}) + H_3(e^{j\omega})H_4(e^{j\omega})\right]$$
$$= H_1(e^{j\omega})H_2(e^{j\omega})\left[1 + H_4(e^{j\omega})\right]$$

$$= \frac{(1 + e^{-j2\omega})^3}{1 - 0.2\, e^{-j\omega}}$$

Example 15

If a filter with impulse response $h(n)$ is implemented with a LDECC of the form:

$$y[n] = \sum_{m=1}^{M} b_m x[n-m] - \sum_{k=1}^{N} a_k y[n-k]$$

in what way should the LDECC be modified, in order to implement the impulse response system $g[n] = (-1)^n h[n]$?

Answer: The frequency response of the impulse response filter $h[n]$ is:

$$H(e^{j\omega}) = \frac{\sum_{m=0}^{M} b_m e^{-jm\omega}}{1 - \sum_{k=1}^{N} a_k e^{-jk\omega}}$$

Multiplying $h[n]$ by the term $(-1)^n$ results in a system with a frequency response:

$$G(e^{j\omega}) = H(e^{j(\omega-\pi)}) = \frac{\sum_{m=0}^{M} b_m e^{-jm(\omega-\pi)}}{1 - \sum_{k=1}^{N} a_k e^{-jk(\omega-\pi)}}$$

Excercise 15 (continued)

Because $e^{jk\pi} = (-1)^k$ resulting:

$$G(e^{j\omega}) = \frac{\sum_{m=0}^{M}(-1)^m b_m\, e^{-jm\omega}}{1 - \sum_{k=1}^{N}(-1)^k a_k\, e^{-jk\omega}}$$

and the difference equation becomes:

$$y[n] = \sum_{m=1}^{M}(-1)^m\, b_m x[n-m] - \sum_{k=1}^{N}(-1)^k a_k y[n-k]$$

Thus, the coefficients a_k και b_m for odd values of m και k take a negative sign.

State-Space modelling of Discrete Time Systems

Solving Dynamic Equations

Methodology

A discrete-time LSI system is described by LDECC:

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M], \qquad n \geq 0$$

with initial conditions $y[-1], y[-2], \ldots, y[-N]$. The signal $x[n]$ is causal and $M \leq N$. We describe the output lags of the above relationship with the variables:

$$v_1[n] = y[n-1]$$
$$v_2[n] = y[n-2]$$
$$\vdots$$
$$v_N[n] = y[n-N]$$

and we get the **State-Space equations:**

$$v_1[n+1] = y[n] = -a_1 v_1[n] - \cdots - a_N v_N[n] + b_0 x[n] + \cdots + b_M x[n-M]$$
$$v_2[n+1] = v_1[n]$$
$$\vdots$$
$$v_N[n+1] = v_{N-1}[n]$$

Methodology

as well as the **output equation:**

$$y[n] = -a_1 v_1[n] - \cdots - a_N v_N[n] + b_0 x[n] + \cdots + b_M\, x[n-M]$$

The initial conditions of the equations of state are related to the initial conditions of the system through the equation:

$$v_1[0] = y[-1], \qquad v_2[0] = y[-2], \qquad \dots, \qquad v_N[0] = y[-N]$$

The above equations (of state and output) are also known as **dynamic equations** and can be written in tabular form:

$$\boldsymbol{v}[n+1] = \boldsymbol{A}\boldsymbol{v}[n] + \boldsymbol{B}\boldsymbol{x}[n]$$

$$y[n] = \boldsymbol{c}^T \boldsymbol{v}[n] + \boldsymbol{d}^T \boldsymbol{x}[n], \qquad n \geq 0$$

by appropriately defining the (system) tables $\boldsymbol{A}$ and $\boldsymbol{B}$ (input), the vectors $\boldsymbol{c}$ (measurement) and $\boldsymbol{d}$ (output), as well as the state vector $\boldsymbol{v}[n]$ and the input vector $\boldsymbol{x}[n]$.

Solving Dynamic Equations

Solving the equation of state:

$$\boldsymbol{v}[n+1] = \boldsymbol{A}\boldsymbol{v}[n] + \boldsymbol{B}\boldsymbol{x}[n], \qquad n \geq 0$$

can be done by an iterative process:

$$\boldsymbol{v}[1] = \boldsymbol{A}\boldsymbol{v}[0] + \boldsymbol{B}\boldsymbol{x}[0]$$

$$\boldsymbol{v}[2] = \boldsymbol{A}\boldsymbol{v}[1] + \boldsymbol{B}\boldsymbol{x}[1] = \boldsymbol{A}^2\boldsymbol{v}[0] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{x}[0] + \boldsymbol{B}\boldsymbol{x}[1]$$

$$\vdots$$

$$\boldsymbol{v}[n] = \boldsymbol{A}^n\boldsymbol{v}[0] + \sum_{k=0}^{n-1} \boldsymbol{A}^{n-1-k}\boldsymbol{B}\boldsymbol{x}[k]$$

where $\boldsymbol{A}^0 = \boldsymbol{I}$ is the unit matrix. The complete solution is given by the equation:

$$y[n] = \boldsymbol{c}^T\boldsymbol{A}^n\boldsymbol{v}[0] + \sum_{k=0}^{n-1} \boldsymbol{c}^T\boldsymbol{A}^{n-1-k}\boldsymbol{B}\boldsymbol{x}[k] + \boldsymbol{d}\boldsymbol{x}[n]$$

The first term in the above equation is the zero input response:

$$y_{zi}[n] = \boldsymbol{c}^T\boldsymbol{A}^n\boldsymbol{v}[0]$$

Solving Dynamic Equations

and the second is the zero-state response:

$$y_{zs}[n] = \sum_{k=0}^{n-1} \boldsymbol{c}^T \boldsymbol{A}^{n-1-k} \boldsymbol{B} \boldsymbol{x}[k] + \boldsymbol{d} \boldsymbol{x}[n]$$

Instead of the iterative solution we can use the one-sided transformation Z^+ to obtain a closed expression of the solution. Specifically, calculating Z^+ the state variables $V_i(z) = Z^+\{v_i[n]\}, \ i = 1, \dots, N$, the input $X_m(z) = Z\{x[n-m]\}, \ m = 0, \dots, M$ and the output $Y(z) = Z\{y[n]\}$, we arrive at the following transformation expression Z^+:

$$z\boldsymbol{V}(z) - z\boldsymbol{v}[0] = \boldsymbol{A}\boldsymbol{V}(z) + \boldsymbol{B}\boldsymbol{X}(z) \Rightarrow (z\boldsymbol{I} - \boldsymbol{A})\boldsymbol{V}(z) = z\boldsymbol{v}[0] + \boldsymbol{B}\boldsymbol{X}(z)$$

Assuming that the inverse matrix of the term $(z\boldsymbol{I} - \boldsymbol{A})$ can be computed, (i.e. $\det(z\boldsymbol{I} - \boldsymbol{A}) \neq 0$), we solve for $\boldsymbol{V}(z)$ and find:

$$\boldsymbol{V}(z) = (z\boldsymbol{I} - \boldsymbol{A})^{-1} z\boldsymbol{v}[0] + (z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{X}(z)$$

Similarly, the transformation Z^+ of the output is:

$$\boldsymbol{Y}(z) = \boldsymbol{c}^T(z\boldsymbol{I} - \boldsymbol{A})^{-1}z\boldsymbol{v}[0] + [\boldsymbol{c}^T(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{d}]\boldsymbol{X}(z)$$

If the initial conditions $\boldsymbol{v}[0]$ are zero, then the transfer function is given by the equation:

$$H(z) = \frac{Y(z)}{X(z)} = \boldsymbol{c}^T(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{d}$$

Solving Dynamic Equations

We recall that the inverse Matrix $\boldsymbol{R}^{-1}$ of a matrix $\boldsymbol{R}_{nxn}$ is given by the equation (Cramer's rule):

$$\boldsymbol{R}^{-1} = \frac{adj(\boldsymbol{R})}{det(\boldsymbol{R})}$$

where:

- $det(\boldsymbol{R})$ is the matrix determinant, and

- $adj(\boldsymbol{R})$ is the complementary matrix of dimensions $n \; x \; n$, which has elements $(-1)^{i+j} detR_{ij}$, where R_{ij} is the matrix resulting from the matrix $\boldsymbol{R}$ if we delete the $i-$row and the $j-$column.

Example 16

Find the State-Space description of the discrete-time system described by the difference equation:

$$y[n] = y[n-1] - 0.5y[n-2] + 0.25x[n]$$

Answer: In the given difference equation we replace the output delays with the following state variables:

$$v_1[n] = y[n-2]$$
$$v_2[n] = y[n-1]$$

This results in the description of the system in the State-Space:

$$v_1[n+1] = v_2[n]$$
$$v_2[n+1] = -0.5v_1[n] + v_2[n] + 0.25x[n]$$

which in tabular form is written:

$$\begin{bmatrix} v_1[n+1] \\ v_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0.25 \end{bmatrix} x[n]$$

Example 16 (continued)

Therefore the (system) matrices A and B (input), is:

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix}$$

The difference equation is written:

$$y[n] = -0.5v_1[n] + v_2[n] + 0.25x[n]$$

and in tabular form are:

$$y[n] = [-0.5 \quad 1] \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + [0.25]x[n]$$

Therefore, the vectors c (measurement) and d (output), is:

$$c^T = [-0.5 \quad 1] \quad d = [0.25]$$