7 ΑΟΡΙΣΤΑ ΟΛΟΚΛΗΡΩΜΑΤΑ

7.1 Ορισμοί

Μία συνάρτηση $f(x)$ έχει *αόριστο ολοκλήρωμα* (ή παράγουσα ή αντιπαράγωγο)

$$F(x) = \int f(x)\,dx$$

στο διάστημα (a, b) αν και μόνο αν υπάρχει μια συνάρηση $F(x)$ τέτοια ώστε να υπάρχει η παράγωγος $F'(x)$ και να είναι $F'(x) = f(x)$. Αν η $F(x)$ είναι αόριστο ολοκλήρωμα της $f(x)$, τότε και κάθε συνάρτηση της μορφής $F(x) + c$, όπου c αυθαίρετη σταθερή, είναι αόριστο ολοκλήρωμα της $f(x)$. Η $f(x)$ καλείται *ολοκληρωτέα συνάρτηση* και η *x μεταβλητή ολοκλήρωσης*.

Εφόσον $F'(x) = f(x)$, μπορούμε να γράψουμε και την ταυτοτική σχέση

$$F(x) = \int F'(x)\,dx = \int \frac{dF}{dx}\,dx = \int dF$$

7.2 Γενικοί Κανόνες

$$\int f'(x)\,dx = f(x)$$

$$\int af(x)dx = a\int f(x)dx$$

$$\int (u \pm \upsilon)dx = \int u dx \pm \int \upsilon dx$$

$$\left.\begin{aligned}
&\int u\,d\upsilon = u\upsilon - \int \upsilon\,du \\
&\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx \\
&\int f^{(n)} g dx = f^{(n-1)} g - f^{(n-2)} g' + f^{(n-3)} g'' - \cdots (-1)^n \int f g^{(n)} dx
\end{aligned}\right\} \begin{array}{c}\text{Ολοκλήρωση}\\ \text{κατά}\\ \text{παράγοντες}\end{array}$$

$$\int \frac{u'\upsilon - u\upsilon'}{\upsilon^2}\,dx = \frac{u}{\upsilon}$$

$$\int \frac{u'\upsilon - u\upsilon'}{u\upsilon}\,dx = \ln\frac{u}{\upsilon}$$

$$\int f(ax)dx = \frac{1}{a}\int f(u)du \qquad \text{όπου } u = ax$$

$$\int g\{f(x)\}dx = \int g(u)\frac{dx}{du}du = \int \frac{g(u)}{f'(x)}du \qquad \text{όπου } u = f(x)$$

7.3 Μέθοδοι Υπολογισμού Αόριστων Ολοκληρωμάτων

Στην πράξη μπορεί ένα ολοκλήρωμα να αναχθεί σε άλλο απλούστερο με μια αλλαγή της μεταβλητής ολοκλήρωσης ή της ολοκληρωτέας συνάρτησης, δηλαδή με ένα μετασχηματισμό του ολοκληρώματος.

Αν η μεταβλητή ολοκλήρωσης εμφανίζεται με μία συγκεκριμένη παράσταση u στην ολοκληρωτέα συνάρτηση, τότε δοκιμάζουμε ως νέα μεταβλητή ολοκλήρωσης την u. Έτσι έχουμε τις ακόλουθες σχέσεις:

$$\int f(ax+b)dx = \frac{1}{a}\int f(u)du \qquad \text{με } u = ax + b$$

$$\int f\left(\sqrt{ax+b}\right)dx = \frac{2}{a}\int uf(u)du \qquad \text{με } u = \sqrt{ax+b}$$

$$\int f\left(\sqrt[n]{ax+b}\right)dx = \frac{n}{a}\int u^{n-1}f(u)du \qquad \text{με } u = \sqrt[n]{ax+b}$$

$$\int f\left(\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx = n(ad-bc)\int f(u)\frac{u^{n-1}}{(cu^n-a)^2}du \qquad \text{με } u = \sqrt[n]{\frac{ax+b}{cx+d}}$$

Αν η ολοκληρωτέα συνάρτηση εξαρτάται μόνο από μία παράσταση της μορφής $(a^2 \pm x^2)^{1/2}$, τότε δοκιμάζουμε κάποια τριγωνομετρική συνάρτηση του x.

$$\int f\left(\sqrt{a^2-x^2}\right)dx = a\int f(a\cos u)\cos u\,du \qquad \text{με } x = a\sin u$$

$$\int f\left(\sqrt{x^2+a^2}\right)dx = a\int f\left(\frac{a}{\cos u}\right)\frac{1}{\cos^2 u}du \qquad \text{με } x = a\tan u$$

Αν η ολοκληρωτέα συνάρτηση εξαρτάται μόνο από μία παράσταση της μορφής $(x^2 \pm a^2)^{1/2}$, τότε δοκιμάζουμε κάποια υπερβολική συνάρτηση του x.

$$\int f\left(\sqrt{x^2-a^2}\right)dx = a\int f(a\sinh u)\sinh u\,du \qquad \text{με } x = a\cosh u$$

$$\int f\left(\sqrt{x^2+a^2}\right)dx = a\int f(a\cosh u)\cosh u\,du \qquad \text{με } x = a\sinh u$$

Αν η ολοκληρωτέα συνάρτηση περιέχει μόνο e^{ax} ή $\ln x$ ή $\sin^{-1}x$, κτλ. χρησιμοποιούμε την παράσταση αυτή ως νέα μεταβλητή ολοκλήρωσης. Έτσι

$$\int f\left(e^{ax}\right)dx = \frac{1}{a}\int \frac{f(u)}{u}du \qquad \text{με } u = e^{ax}$$

$$\int f(\ln x)dx = \int f(u)e^{u}du \qquad \text{με } u = \ln x$$

$$\int f\left(\sin^{-1}\frac{x}{a}\right)dx = \int f(u)\cos u\,du \qquad \text{με } u = \sin^{-1}\frac{x}{a}$$

(όμοια για άλλες αντίστροφες τριγωνομετρικές συναρτήσεις)

Αν η ολοκληρωτέα συνάρτηση περιέχει μόνο $\sin x$ και $\cos x$, χρησιμοποιούμε ως νέα μεταβλητή ολοκλήρωσης την $\tan(x/2)$.

$$\int f(\sin x,\, \cos x)\,dx = 2\int f\left(\frac{2u}{1+u^2},\, \frac{1-u^2}{1+u^2}\right)\frac{du}{1+u^2} \qquad \text{με } u = \tan\frac{x}{2}$$

Ομοια, αν η ολοκληρωτέα συνάρτηση περιέχει μόνο $\sinh x$ και $\cosh x$, χρησιμοποιούμε ως νέα μεταβλητή ολοκλήρωσης την $\tanh(x/2)$.

$$\int f(\sinh x,\, \cosh x)\,dx = 2\int f\left(\frac{2u}{1-u^2},\, \frac{1+u^2}{1-u^2}\right)\frac{du}{1-u^2} \qquad \text{με } u = \tanh\frac{x}{2}$$

Αν η $f(x)$ είναι ρητή συνάρτηση του x (δηλαδή λόγος δύο πολυωνύμων), μπορεί να αναλυθεί σε άθροισμα πολυωνύμου του x και όρων της μορφής

$$\frac{c}{(x-a)^k}, \qquad \frac{cx+d}{(x^2+ax+b)^m}$$

και να υπολογιστεί το αόριστο ολοκλήρωμα ως το άθροισμα των ολοκληρωμάτων αυτών των όρων.

Αν η $f(x)$ περιέχει το x σε μια παράσταση της μορφής $x^m(ax^n + b)^p$ δοκιμάζουμε τις εξής αντικαταστάσεις: Αν $(m + 1)/n$ = ακέραιος, θέτουμε $u = x^n$ και συνεχίζουμε με παραγοντική ολοκλήρωση ή θέτουμε $au + b = w$. Αν $(m + 1)/n + p$ = ακέραιος, θέτουμε $u = x^{-n}$ και συνεχίζουμε με παραγοντική ολοκλήρωση ή θέτουμε $a + bu = w$. Αν p = ακέραιος, αναπτύσσουμε τη δύναμη.

Αν η $f(x)$ περιέχει την παράσταση $(ax^2 + bx + c)^{1/2}$ δοκιμάζουμε την αντικατάσταση $u = (2ax + b)/|4ac - b^2|^{1/2}$.

7.4 Στοιχειώδη Ολοκληρώματα

$$\int a dx = ax$$

$$\int x^n dx = \begin{cases} \dfrac{x^{n+1}}{n+1}, & n \neq -1 \\ \ln x, & n = -1 \end{cases}$$

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = \sin x$$

$$\int \tan x \, dx = -\ln \cos x$$

$$\int \cot x dx = \ln \sin x$$

$$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{2}(x - \sin x \cos x)$$

$$\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}(x + \sin x \cos x)$$

$$\int \tan^2 x dx = \tan x - x$$

$$\int \cot^2 x dx = -\cot x - x$$

$$\int e^x dx = e^x$$

$$\int a^x dx = \frac{a^x}{\ln a}, \qquad a > 0, \quad a \neq 1$$

$$\int \sinh x dx = \cosh x$$

$$\int \cosh x dx = \sinh x$$

$$\int \tanh x \, dx = \ln \cosh x$$

$$\int \coth x dx = \ln \sinh x$$

$$\int \sinh^2 x dx = \frac{\sinh 2x}{4} - \frac{x}{2} = \frac{1}{2}(\sinh x \cosh x - x)$$

$$\int \cosh^2 x dx = \frac{\sinh 2x}{4} + \frac{x}{2} = \frac{1}{2}(\sinh x \cosh x + x)$$

$$\int \tanh^2 x dx = x - \tanh x$$

$$\int \coth^2 x dx = x - \coth x$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right) = -\frac{1}{a} \coth^{-1} \frac{x}{a} \qquad x^2 > a^2$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) = \frac{1}{a} \tanh^{-1} \frac{x}{a} \qquad x^2 < a^2$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) = \sinh^{-1} \frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right)$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \cos^{-1} \left|\frac{a}{x}\right| \qquad x^2 > a^2,\ a > 0$$

$$\int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a} \ln\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right)$$

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right)$$

7.5 Διάφορα Ολοκληρώματα

Με $ax+b$

$$\int (ax+b)^n\,dx = \begin{cases} \dfrac{(ax+b)^{n+1}}{a(n+1)}, & n \neq -1 \\ \dfrac{1}{a}\ln(ax+b) & n=-1 \end{cases}$$

$$\int \frac{x\,dx}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\ln(ax+b)$$

$$\int \frac{x^2\,dx}{ax+b} = \frac{(ax+b)^2}{2a^3} - \frac{2b(ax+b)}{a^3} + \frac{b^2}{a^3}\ln(ax+b)$$

$$\int \frac{x^3\,dx}{ax+b} = \frac{(ax+b)^3}{3a^4} - \frac{3b(ax+b)^2}{2a^4} + \frac{3b^2(ax+b)}{a^4} - \frac{b^3}{a^4}\ln(ax+b)$$

$$\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left(\frac{x}{ax+b}\right)$$

$$\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\ln\left(\frac{ax+b}{x}\right)$$

$$\int \frac{dx}{x^3(ax+b)} = \frac{2ax-b}{2b^2x^2} + \frac{a^2}{b^3}\ln\left(\frac{x}{ax+b}\right)$$

$$\int \frac{dx}{(ax+b)^2} = \frac{-1}{a(ax+b)}$$

$$\int \frac{x\,dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\ln(ax+b)$$

$$\int \frac{x^2\,dx}{(ax+b)^2} = \frac{ax+b}{a^3} - \frac{b^2}{a^3(ax+b)} - \frac{2b}{a^3}\ln(ax+b)$$

$$\int \frac{x^3\,dx}{(ax+b)^2} = \frac{(ax+b)^2}{2a^4} - \frac{3b(ax+b)}{a^4} + \frac{b^3}{a^4(ax+b)} + \frac{3b^2}{a^4}\ln(ax+b)$$

$$\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} + \frac{1}{b^2}\ln\left(\frac{x}{ax+b}\right)$$

$$\int \frac{dx}{x^2(ax+b)^2} = -\frac{a}{b^2(ax+b)} - \frac{1}{b^2 x} + \frac{2a}{b^3}\ln\left(\frac{ax+b}{x}\right)$$

$$\int \frac{dx}{x^3(ax+b)^2} = -\frac{(ax+b)^2}{2b^4x^2} + \frac{3a(ax+b)}{b^4x} - \frac{a^3x}{b^4(ax+b)} - \frac{3a^2}{b^4}\ln\left(\frac{ax+b}{x}\right)$$

$$\int \frac{dx}{(ax+b)^3} = -\frac{1}{2(ax+b)^2}$$

$$\int \frac{x\,dx}{(ax+b)^3} = -\frac{1}{a^2(ax+b)} + \frac{b}{2a^2(ax+b)^2}$$

$$\int \frac{x^2\,dx}{(ax+b)^3} = \frac{2b}{a^3(ax+b)} - \frac{b^2}{2a^3(ax+b)^2} + \frac{1}{a^3}\ln(ax+b)$$

$$\int \frac{x^3\,dx}{(ax+b)^3} = \frac{x}{a^3} - \frac{3b^2}{a^4(ax+b)} + \frac{b^3}{2a^4(ax+b)^2} - \frac{3b}{a^4}\ln(ax+b)$$

$$\int \frac{dx}{x(ax+b)^3} = \frac{a^2x^2}{2b^3(ax+b)^2} - \frac{2ax}{b^3(ax+b)} - \frac{1}{b^3}\ln\left(\frac{ax+b}{x}\right)$$

$$\int \frac{dx}{x^2(ax+b)^3} = -\frac{a}{2b^2(ax+b)^2} - \frac{2a}{b^3(ax+b)} - \frac{1}{b^3x} + \frac{3a}{b^4}\ln\left(\frac{ax+b}{x}\right)$$

$$\int \frac{dx}{x^3(ax+b)^3} = \frac{a^4x^2}{2b^5(ax+b)} - \frac{(ax+b)^2}{2b^5x^2} - \frac{6a^2}{b^5}\ln\left(\frac{ax+b}{x}\right)$$

$$\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n+1}}{(n+1)a^2}, \qquad n \neq -1, -2$$

$$\int \frac{x^n}{ax+b}\,dx = \frac{x^n}{na} - \frac{bx^{n-1}}{(n-1)a^2} + \frac{b^2x^{n-2}}{(n-2)a^3} - \cdots + (-1)^{n-1}\frac{b^{n-1}x}{1\cdot a^n} + \frac{(-1)^n b^n}{a^{n+1}}\ln(ax+b)$$

$$\int x^2(ax+b)^n\,dx = \frac{(ax+b)^{n+3}}{(n+3)a^3} - \frac{2b(ax+b)^{n+2}}{(n+2)a^3} + \frac{b^2(ax+b)^{n+1}}{(n+1)a^3} \qquad n \neq -1, -2, -3$$

$$\int x^m (ax+b)^n\,dx = \begin{cases} \dfrac{x^{m+1}(ax+b)^n}{m+n+1} + \dfrac{nb}{m+n+1}\displaystyle\int x^m (ax+b)^{n-1}\,dx \\ \dfrac{x^m (ax+b)^{n+1}}{(m+n+1)a} - \dfrac{mb}{(m+n+1)a}\displaystyle\int x^{m-1} (ax+b)^n\,dx \\ \dfrac{-x^{m+1}(ax+b)^{n+1}}{(n+1)b} + \dfrac{m+n+2}{(n+1)b}\displaystyle\int x^m (ax+b)^{n+1}\,dx \end{cases}$$

Με ρίζες

$$\int \sqrt{ax+b}\,dx = \frac{2}{3a}\sqrt{(ax+b)^3}$$

$$\int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$$

$$\int x^2\sqrt{ax+b}\,dx = \frac{2(15a^2x^2-12abx+8b^2)}{105a^3}\sqrt{(ax+b)^3}$$

$$\int x^3\sqrt{ax+b}\,dx = \frac{2(35a^3x^3-30a^2bx^2+24ab^2x-16b^3)}{315a^4}\sqrt{(ax+b)^3}$$

$$\int \frac{\sqrt{ax+b}}{x}dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$$

$$\int \frac{\sqrt{ax+b}}{x^2}dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}}$$

$$\int \frac{\sqrt{ax+b}}{x^3}dx = -\frac{(ax+2b)\sqrt{ax+b}}{4bx^2} - \frac{a^2}{8b}\int \frac{dx}{x\sqrt{ax+b}}$$

$$\int \frac{dx}{\sqrt{ax+b}} = \frac{2}{a}\sqrt{ax+b}$$

$$\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$$

$$\int \frac{x^2\,dx}{\sqrt{ax+b}} = \frac{2(3a^2x^2-4abx+8b^2)}{15a^3}\sqrt{ax+b}$$

$$\int \frac{x^3}{\sqrt{ax+b}}\,dx = \frac{2(5a^3x^3 - 6a^2bx^2 + 8ab^2x - 16b^3)}{35a^4}\sqrt{ax+b}$$

$$\int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{b}}\ln\left(\dfrac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right), & b>0 \\[2ex] \dfrac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\dfrac{ax+b}{-b}}, & b<0 \end{cases}$$

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$$

$$\int \frac{dx}{x^3\sqrt{ax+b}} = \frac{3ax-2b}{4b^2x^2}\sqrt{ax+b} + \frac{3a^2}{8b^2}\int \frac{dx}{x\sqrt{ax+b}}$$

$$\int x^n\sqrt{ax+b}\,dx = \frac{2x^n}{(2n+3)a}\sqrt{(ax+b)^3} - \frac{2nb}{(2n+3)a}\int x^{n-1}\sqrt{ax+b}\,dx$$

$$= \frac{2}{a^{n+1}}\int u^2(u^2-b)^n\,du, \qquad u=\sqrt{ax+b}$$

$$\int \frac{\sqrt{ax+b}}{x^n}\,dx = -\frac{\sqrt{ax+b}}{(n-1)x^{n-1}} + \frac{a}{2(n-1)}\int \frac{dx}{x^{n-1}\sqrt{ax+b}}$$

$$= -\frac{\sqrt{(ax+b)^3}}{(n-1)bx^{n-1}} - \frac{(2n-5)a}{(2n-2)b}\int \frac{\sqrt{ax+b}}{x^{n-1}}\,dx$$

$$\int \frac{x^n}{\sqrt{ax+b}}\,dx = \frac{2x^n\sqrt{ax+b}}{(2n+1)a} - \frac{2nb}{(2n+1)a}\int \frac{x^{n-1}}{\sqrt{ax+b}}\,dx$$

$$= \frac{2}{a^{n+1}}\int (u^2-b)^n\,du, \qquad u=\sqrt{ax+b}$$

$$\int \frac{dx}{x^n\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{(n-1)bx^{n-1}} - \frac{(2n-3)a}{(2n-2)b}\int \frac{dx}{x^{n-1}\sqrt{ax+b}}$$

$$\int (ax+b)^{n/2}\,dx = \frac{2(ax+b)^{(n+2)/2}}{a(n+2)}$$

$$\int x(ax+b)^{n/2}\,dx = \frac{2(ax+b)^{(n+4)/2}}{a^2(n+4)} - \frac{2b(ax+b)^{(n+2)/2}}{a^2(n+2)}$$

$$\int x^2(ax+b)^{n/2}\,dx = \frac{2(ax+b)^{(n+6)/2}}{a^3(n+6)} - \frac{4b(ax+b)^{(n+2)/2}}{a^3(n+4)} + \frac{2b^2(ax+b)^{(n+2)/2}}{a^3(n+2)}$$

$$\int \frac{(ax+b)^{n/2}}{x}\,dx = \frac{2(ax+b)^{n/2}}{n} + b\int \frac{(ax+b)^{(n-2)/2}}{x}\,dx$$

$$\int \frac{(ax+b)^{n/2}}{x^2}\,dx = -\frac{(ax+b)^{(n+2)/2}}{bx} + \frac{na}{2b}\int \frac{(ax+b)^{n/2}}{x}\,dx$$

$$\int \frac{dx}{x(ax+b)^{n/2}} = \frac{2}{(n-2)b(ax+b)^{(n-2)/2}} + \frac{1}{b}\int \frac{dx}{x(ax+b)^{(n-2)/2}}$$

$$\int \frac{dx}{x^2(ax+b)^{n/2}} = \frac{-1}{bx(ax+b)^{(n-2)/2}} - \frac{na}{2b}\int \frac{dx}{x(ax+b)^{n/2}}$$

Με $ax+b$ και $cx+d$

$$\int \frac{ax+b}{cx+d}\,dx = \frac{ax}{c} + \frac{bc-ad}{c^2}\ln(cx+d)$$

$$\int \frac{dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\ln\left(\frac{cx+d}{ax+b}\right)$$

$$\int \frac{x\,dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\left\{\frac{b}{a}\ln(ax+b) - \frac{d}{c}\ln(cx+d)\right\}$$

$$\int \frac{dx}{(ax+b)^2(cx+d)} = \frac{1}{bc-ad}\left\{\frac{1}{ax+b} + \frac{c}{bc-ad}\ln\left(\frac{cx+d}{ax+b}\right)\right\}$$

$$\int \frac{x\,dx}{(ax+b)^2(cx+d)} = \frac{1}{bc-ad}\left\{\frac{d}{bc-ad}\ln\left(\frac{ax+b}{cx+d}\right) - \frac{b}{a(ax+b)}\right\}$$

$$\int \frac{x^2\,dx}{(ax+b)^2(cx+d)} = \frac{b^2}{(bc-ad)a^2(ax+b)} + \frac{1}{(bc-ad)^2}\left\{\frac{d^2}{c}\ln(cx+d) + \frac{b(bc-2ad)}{a^2}\ln(ax+b)\right\}$$

$$\int \frac{dx}{(ax+b)^m (cx+d)^n} = -\frac{1}{(n-1)(bc-ad)} \left\{ \frac{1}{(ax+b)^{m-1}(cx+d)^{n-1}} \right.$$

$$\left. +a(m+n-2)\int \frac{dx}{(ax+b)^m (cx+d)^{n-1}} \right\}$$

$$\int \frac{(ax+b)^m}{(cx+d)^n}dx = \begin{cases} \dfrac{-1}{(n-1)(bc-ad)} \left\{ \dfrac{(ax+b)^{m+1}}{(cx+d)^{n-1}} + (n-m-2)a\displaystyle\int \frac{(ax+b)^m}{(cx+d)^{n-1}}dx \right\} \\ \dfrac{-1}{(n-m-1)c} \left\{ \dfrac{(ax+b)^m}{(cx+d)^{n-1}} + m(bc-ad)\displaystyle\int \frac{(ax+b)^{m-1}}{(cx+d)^n}dx \right\} \\ \dfrac{-1}{(n-1)c} \left\{ \dfrac{(ax+b)^m}{(cx+d)^{n-1}} - ma\displaystyle\int \frac{(ax+b)^{m-1}}{(cx+d)^{n-1}}dx \right\} \end{cases}$$

Με ρίζες

$$\int \frac{cx+d}{\sqrt{ax+b}}dx = \frac{2(acx+3ad-2bc)}{3a^2}\sqrt{ax+b}$$

$$\int (cx+d)\sqrt{ax+b}\,dx = \frac{2(3acx+5ad-2bc)}{15a^2}\sqrt{ax+b}$$

$$\int \frac{dx}{(cx+d)\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{c(bc-ad)}} \ln\left(\dfrac{c\sqrt{(ax+b)} - \sqrt{c(bc-ad)}}{c\sqrt{(ax+b)} + \sqrt{c(bc-ad)}} \right), & c(bc-ad)>0 \\ \dfrac{2}{\sqrt{c(ad-bc)}} \tan^{-1}\sqrt{\dfrac{c(ax+b)}{ad-bc}}, & c(bc-ad)<0 \end{cases}$$

$$\int \frac{\sqrt{ax+b}}{cx+d}dx = \begin{cases} \dfrac{2\sqrt{ax+b}}{c} + \dfrac{\sqrt{c(bc-ad)}}{c^2} \ln\left(\dfrac{c\sqrt{(ax+b)} - \sqrt{c(bc-ad)}}{c\sqrt{(ax+b)} + \sqrt{c(bc-ad)}} \right), & c(bc-ad)>0 \\ \dfrac{2\sqrt{ax+b}}{c} - \dfrac{2\sqrt{c(ad-bc)}}{c^2} \tan^{-1}\sqrt{\dfrac{c(ax+b)}{ad-bc}}, & c(bc-ad)<0 \end{cases}$$

$$\int \frac{dx}{\sqrt{(ax+b)(cx+d)}} = \begin{cases} \dfrac{2}{\sqrt{ac}} \ln\left(\sqrt{ac(ax+b)} + a\sqrt{cx+d} \right), & ac>0 \\ \dfrac{2}{\sqrt{-ac}} \tan^{-1}\sqrt{\dfrac{-c(ax+b)}{a(cx+d)}}, & ac<0 \end{cases}$$

$$\int \frac{x\,dx}{\sqrt{(ax+b)(cx+d)}} = \frac{\sqrt{(ax+b)(cx+d)}}{ac} - \frac{bc+ad}{2ac}\int \frac{dx}{\sqrt{(ax+b)(cx+d)}}$$

$$\int \sqrt{(ax+b)(cx+d)}\,dx = \frac{2acx+bc+ad}{4ac}\sqrt{(ax+b)(cx+d)} - \frac{(bc-ad)^2}{8ac}\int \frac{dx}{\sqrt{(ax+b)(cx+d)}}$$

$$\int \sqrt{\frac{cx+d}{ax+b}}\,dx = \frac{\sqrt{(ax+b)(cx+d)}}{a} + \frac{ad-bc}{2a}\int \frac{dx}{\sqrt{(ax+b)(cx+d)}}$$

$$\int \frac{dx}{(cx+d)\sqrt{(ax+b)(cx+d)}} = \frac{2\sqrt{ax+b}}{(ad-bc)\sqrt{cx+d}}$$

$$\int (cx+d)^n\sqrt{ax+b}\,dx = \frac{2(cx+d)^{n+1}\sqrt{ax+b}}{(2n+3)c} + \frac{bc-ad}{(2n+3)c}\int \frac{(cx+d)^n}{\sqrt{ax+b}}\,dx$$

$$\int \frac{dx}{(cx+d)^n\sqrt{ax+b}} = \frac{\sqrt{ax+b}}{(n-1)(ad-bc)(cx+d)^{n-1}} + \frac{(2n-3)a}{2(n-1)(ad-bc)}\int \frac{dx}{(cx+d)^{n-1}\sqrt{ax+b}}$$

$$\int \frac{(cx+d)^n}{\sqrt{ax+b}}\,dx = \frac{2(cx+d)^n\sqrt{ax+b}}{(2n+1)a} + \frac{2n(ad-bc)}{(2n+1)a}\int \frac{(cx+d)^{n-1}\,dx}{\sqrt{ax+b}}$$

$$\int \frac{\sqrt{ax+b}}{(cx+d)^n}\,dx = -\frac{\sqrt{ax+b}}{(n-1)c(cx+d)^{n-1}} + \frac{a}{2(n-1)c}\int \frac{dx}{(cx+d)^{n-1}\sqrt{ax+b}}$$

Με $x^2 + a^2$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2)$$

$$\int \frac{x^2\,dx}{x^2+a^2} = x - a\tan^{-1}\frac{x}{a}$$

$$\int \frac{x^3 dx}{x^2+a^2} = \frac{x^2}{2} - \frac{a^2}{2}\ln(x^2+a^2)$$

$$\int \frac{dx}{x(x^2+a^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

$$\int \frac{dx}{x^2(x^2+a^2)} = -\frac{1}{a^2 x} - \frac{1}{a^3}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x^3(x^2+a^2)} = -\frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

$$\int \frac{dx}{(cx+d)(x^2+a^2)} = \frac{1}{(a^2c^2+d^2)}\left[c\ln(cx+d) - \frac{c}{2}\ln(x^2+a^2) + \frac{d}{a}\tan^{-1}\frac{x}{a}\right]$$

$$\int \frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x\,dx}{(x^2+a^2)^2} = -\frac{1}{2(x^2+a^2)}$$

$$\int \frac{x^2\,dx}{(x^2+a^2)^2} = -\frac{x}{2(x^2+a^2)} + \frac{1}{2a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x^3\,dx}{(x^2+a^2)^2} = \frac{a^2}{2(x^2+a^2)} + \frac{1}{2}\ln(x^2+a^2)$$

$$\int \frac{dx}{x(x^2+a^2)^2} = \frac{1}{2a^2(x^2+a^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

$$\int \frac{dx}{x^2(x^2+a^2)^2} = -\frac{1}{a^4 x} - \frac{x}{2a^4(x^2+a^2)} - \frac{3}{2a^5}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x^3(x^2+a^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(x^2+a^2)} - \frac{1}{a^6}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

$$\int \frac{dx}{(x^2+a^2)^3} = \frac{x}{4a^2(x^2+a^2)^2} + \frac{3x}{8a^4(x^2+a^2)} + \frac{3}{8a^5}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x\,dx}{(x^2+a^2)^3} = \frac{-1}{4(x^2+a^2)^2}$$

$$\int \frac{x^2\,dx}{(x^2+a^2)^3} = \frac{-x}{4(x^2+a^2)^2} + \frac{x}{8a^2(x^2+a^2)} + \frac{1}{8a^3}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x^3\,dx}{(x^2+a^2)^3} = \frac{-1}{2(x^2+a^2)} + \frac{a^2}{4(x^2+a^2)^2}$$

$$\int \frac{dx}{x(x^2+a^2)^3} = \frac{1}{4a^2(x^2+a^2)^2} + \frac{1}{2a^4(x^2+a^2)} + \frac{1}{2a^6}\ln\frac{x^2}{x^2+a^2}$$

$$\int \frac{dx}{x^2(x^2+a^2)^3} = \frac{-1}{a^6 x} - \frac{x}{4a^4(x^2+a^2)^2} - \frac{7x}{8a^6(x^2+a^2)} - \frac{15}{8a^7}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x^3(x^2+a^2)^3} = \frac{-1}{2a^6x^2} - \frac{1}{a^6(x^2+a^2)} - \frac{1}{4a^4(x^2+a^2)^2} - \frac{3}{2a^8}\ln\frac{x^2}{x^2+a^2}$$

$$\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(x^2+a^2)^{n-1}}$$

$$\int \frac{x\,dx}{(x^2+a^2)^n} = -\frac{1}{2(n-1)(x^2+a^2)^{n-1}}$$

$$\int \frac{dx}{x(x^2+a^2)^n} = \frac{1}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{1}{a^2}\int \frac{dx}{x(x^2+a^2)^{n-1}}$$

$$\int \frac{x^m\,dx}{(x^2+a^2)^n} = \int \frac{x^{m-2}dx}{(x^2+a^2)^{n-1}} - a^2\int \frac{x^{m-2}dx}{(x^2+a^2)^n}$$

$$\int \frac{dx}{x^m(x^2+a^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^m(x^2+a^2)^{n-1}} - \frac{1}{a^2}\int \frac{dx}{x^{m-2}(x^2+a^2)^n}$$

Με ρίζες

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) = \sinh^{-1}\frac{x}{a}$$

$$\int \frac{x\,dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}$$

$$\int \frac{x^2\,dx}{\sqrt{x^2+a^2}} = \frac{x\sqrt{x^2+a^2}}{2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int \frac{x^3\,dx}{\sqrt{x^2+a^2}} = \frac{(x^2+a^2)^{3/2}}{3} - a^2\sqrt{x^2+a^2}$$

$$\int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

$$\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x}$$

$$\int \frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int x\sqrt{x^2+a^2}\,dx = \frac{(x^2+a^2)^{3/2}}{3}$$

$$\int x^2\sqrt{x^2+a^2}\,dx = \frac{x(x^2+a^2)^{3/2}}{4} - \frac{a^2x\sqrt{x^2+a^2}}{8} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int x^3\sqrt{x^2+a^2}\,dx = \frac{(x^2+a^2)^{5/2}}{5} - \frac{a^2(x^2+a^2)^{3/2}}{3}$$

$$\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} - a\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

$$\int \frac{\sqrt{x^2+a^2}}{x^2}\,dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int \frac{\sqrt{x^2+a^2}}{x^3}\,dx = -\frac{\sqrt{x^2+a^2}}{2x^2} - \frac{1}{2a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

$$\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$$

$$\int \frac{x^2\,dx}{(x^2+a^2)^{3/2}} = -\frac{x}{\sqrt{x^2+a^2}} + \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int \frac{x^3\,dx}{(x^2+a^2)^{3/2}} = \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}}$$

$$\int \frac{dx}{x(x^2+a^2)^{3/2}} = \frac{1}{a^2\sqrt{x^2+a^2}} - \frac{1}{a^3}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

$$\int \frac{dx}{x^2(x^2+a^2)^{3/2}} = -\frac{\sqrt{x^2+a^2}}{a^4x} - \frac{x}{a^4\sqrt{x^2+a^2}}$$

$$\int \frac{dx}{x^3(x^2+a^2)^{3/2}} = -\frac{1}{2a^2x^2\sqrt{x^2+a^2}} - \frac{3}{2a^4\sqrt{x^2+a^2}} + \frac{3}{2a^5}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

$$\int (x^2+a^2)^{3/2}\,dx = \frac{x(x^2+a^2)^{3/2}}{4} + \frac{3a^2x\sqrt{x^2+a^2}}{8} + \frac{3}{8}a^4\ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int x(x^2+a^2)^{3/2}\,dx = \frac{(x^2+a^2)^{5/2}}{5}$$

$$\int x^2(x^2+a^2)^{3/2}\,dx = \frac{x(x^2+a^2)^{5/2}}{6} - \frac{a^2x(x^2+a^2)^{3/2}}{24} - \frac{a^4x\sqrt{x^2+a^2}}{16} - \frac{a^6}{16}\ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int x^3(x^2+a^2)^{3/2}\,dx = \frac{(x^2+a^2)^{7/2}}{7} - \frac{a^2(x^2+a^2)^{5/2}}{5}$$

$$\int \frac{(x^2+a^2)^{3/2}}{x}\,dx = \frac{(x^2+a^2)^{3/2}}{3} + a^2\sqrt{x^2+a^2} - a^3\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

$$\int \frac{(x^2+a^2)^{3/2}}{x^2}\,dx = -\frac{(x^2+a^2)^{3/2}}{x} + \frac{3x\sqrt{x^2+a^2}}{2} + \frac{3}{2}a^2\ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int \frac{(x^2+a^2)^{3/2}}{x^3}\,dx = -\frac{(x^2+a^2)^{3/2}}{2x^2} + \frac{3}{2}\sqrt{x^2+a^2} - \frac{3}{2}a\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

$$\int \frac{x\,dx}{(x^2+a^2)^{(2n+1)/2}} = \frac{-1}{(2n-1)(x^2+a^2)^{(2n-1)/2}}$$

$$\int \frac{x^3\,dx}{(x^2+a^2)^{(2n+1)/2}} = \frac{-1}{(2n-3)(x^2+a^2)^{(2n-3)/2}} + \frac{a^2}{(2n-1)(x^2+a^2)^{(2n-1)/2}}$$

$$\int x(x^2+a^2)^{(2n+1)/2}\,dx = \frac{(x^2+a^2)^{(2n+3)/2}}{2n+3}$$

$$\int x^3(x^2+a^2)^{(2n+1)/2}\,dx = \frac{(x^2+a^2)^{(2n+5)/2}}{2n+5} - \frac{a^2(x^2+a^2)^{(2n+3)/2}}{2n+3}$$

Mε $x^2 - a^2$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

$$= -\frac{1}{a}\coth^{-1}\frac{x}{a} \qquad x^2 > a^2$$

$$= -\frac{1}{a}\tanh^{-1}\frac{x}{a} \qquad x^2 < a^2$$

$$\int \frac{x\,dx}{x^2-a^2} = \frac{1}{2}\ln\left|x^2-a^2\right|$$

$$\int \frac{x^2\,dx}{x^2-a^2} = x + \frac{a}{2}\ln\left|\frac{x-a}{x+a}\right|$$

$$\int \frac{x^3\,dx}{x^2-a^2} = \frac{x^2}{2} + \frac{a^2}{2}\ln\left|x^2-a^2\right|$$

$$\int \frac{dx}{x\left(x^2-a^2\right)} = \frac{1}{2a^2}\ln\left|\frac{x^2-a^2}{x^2}\right|$$

$$\int \frac{dx}{x^2(x^2-a^2)} = \frac{1}{a^2x} + \frac{1}{2a^3}\ln\left|\frac{x-a}{x+a}\right|$$

$$\int \frac{dx}{x^3(x^2-a^2)} = \frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln\left|\frac{x^2}{x^2-a^2}\right|$$

$$\int \frac{dx}{(x^2-a^2)^2} = \frac{-x}{2a^2(x^2-a^2)} - \frac{1}{4a^3}\ln\left|\frac{x-a}{x+a}\right|$$

$$\int \frac{x\,dx}{(x^2-a^2)^2} = \frac{-1}{2(x^2-a^2)}$$

$$\int \frac{x^2\,dx}{(x^2-a^2)^2} = \frac{-x}{2(x^2-a^2)} + \frac{1}{4a}\ln\left|\frac{x-a}{x+a}\right|$$

$$\int \frac{x^3\,dx}{(x^2-a^2)^2} = \frac{-a^2}{2(x^2-a^2)} + \frac{1}{2}\ln\left|x^2-a^2\right|$$

$$\int \frac{dx}{x(x^2-a^2)^2} = \frac{-1}{2a^2(x^2-a^2)} + \frac{1}{2a^4}\ln\left|\frac{x^2}{x^2-a^2}\right|$$

$$\int \frac{dx}{x^2(x^2-a^2)^2} = \frac{-1}{a^4x} - \frac{x}{2a^4(x^2-a^2)} - \frac{3}{4a^5}\ln\left|\frac{x-a}{x+a}\right|$$

$$\int \frac{dx}{x^3(x^2-a^2)^2} = \frac{-1}{2a^4x^2} - \frac{1}{2a^4(x^2-a^2)} + \frac{1}{a^6}\ln\left|\frac{x^2}{x^2-a^2}\right|$$

$$\int \frac{dx}{(x^2-a^2)^n} = \frac{-x}{2(n-1)a^2(x^2-a^2)^{n-1}} - \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(x^2-a^2)^{n-1}}$$

$$\int \frac{x\,dx}{(x^2-a^2)^n} = \frac{-1}{2(n-1)(x^2-a^2)^{n-1}}$$

$$\int \frac{dx}{x(x^2-a^2)^n} = \frac{-1}{2(n-1)(x^2-a^2)^{n-2}} - \frac{1}{a^2}\int \frac{dx}{x(x^2-a^2)^{n-1}}$$

$$\int \frac{x^m\,dx}{(x^2-a^2)^n} = \int \frac{x^{m-2}dx}{(x^2-a^2)^{n-1}} + a^2\int \frac{x^{m-2}dx}{(x^2-a^2)^n}$$

$$\int \frac{dx}{x^m (x^2-a^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^{m-2}(x^2-a^2)^n} - \frac{1}{a^2}\int \frac{dx}{x^m (x^2-a^2)^{n-1}}$$

Με ρίζες του $x^2 - a^2$ $(x^2 > a^2)$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right)$$

$$\int \frac{x\,dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}$$

$$\int \frac{x^2\,dx}{\sqrt{x^2-a^2}} = \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)$$

$$\int \frac{x^3\,dx}{\sqrt{x^2-a^2}} = \frac{(x^2-a^2)^{3/2}}{3} + a^2\sqrt{x^2-a^2}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cos^{-1}\left|\frac{a}{x}\right| \qquad x^2 > a^2$$

$$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2 x}$$

$$\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\cos^{-1}\left|\frac{a}{x}\right|$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)$$

$$\int x\sqrt{x^2-a^2}\,dx = \frac{(x^2-a^2)^{3/2}}{3}$$

$$\int x^2\sqrt{x^2-a^2}\,dx = \frac{x(x^2-a^2)^{3/2}}{4} + \frac{a^2x\sqrt{x^2-a^2}}{8} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2-a^2}\right)$$

$$\int x^3\sqrt{x^2-a^2}\,dx = \frac{(x^2-a^2)^{5/2}}{5} + \frac{a^2(x^2-a^2)^{3/2}}{3}$$

$$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\cos^{-1}\left|\frac{a}{x}\right|$$

$$\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = -\frac{\sqrt{x^2-a^2}}{x} + \ln\left(x+\sqrt{x^2-a^2}\right)$$

$$\int \frac{\sqrt{x^2-a^2}}{x^3}\,dx = -\frac{\sqrt{x^2-a^2}}{2x^2} + \frac{1}{2a}\cos^{-1}\left|\frac{a}{x}\right|$$

$$\int \frac{dx}{(x^2-a^2)^{3/2}} = -\frac{x}{a^2\sqrt{x^2-a^2}}$$

$$\int \frac{x\,dx}{(x^2-a^2)^{3/2}} = -\frac{1}{\sqrt{x^2-a^2}}$$

$$\int \frac{x^2\,dx}{(x^2-a^2)^{3/2}} = \frac{x}{\sqrt{x^2-a^2}} + \ln x + \sqrt{x^2-a^2}$$

$$\int \frac{x^3\,dx}{(x^2-a^2)^{3/2}} = \sqrt{x^2-a^2} - \frac{a^2}{\sqrt{x^2-a^2}}$$

$$\int \frac{dx}{x(x^2-a^2)^{3/2}} = -\frac{1}{a^2\sqrt{x^2-a^2}} - \frac{1}{a^3}\cos^{-1}\left|\frac{a}{x}\right|$$

$$\int \frac{dx}{x^2(x^2-a^2)^{3/2}} = -\frac{\sqrt{x^2-a^2}}{a^4 x} - \frac{x}{a^4\sqrt{x^2-a^2}}$$

$$\int \frac{dx}{x^3(x^2-a^2)^{3/2}} = \frac{1}{2a^2x^2\sqrt{x^2-a^2}} - \frac{3}{2a^4\sqrt{x^2-a^2}} - \frac{3}{2a^5}\cos^{-1}\left|\frac{a}{x}\right|$$

$$\int (x^2-a^2)^{3/2}\,dx = \frac{x(x^2-a^2)^{3/2}}{4} - \frac{3a^2x\sqrt{x^2-a^2}}{8} + \frac{3}{8}a^4\ln\left(x+\sqrt{x^2-a^2}\right)$$

$$\int x(x^2-a^2)^{3/2}\,dx = \frac{(x^2-a^2)^{5/2}}{5}$$

$$\int x^2(x^2-a^2)^{3/2}\,dx = \frac{x(x^2-a^2)^{5/2}}{6} + \frac{a^2x(x^2-a^2)^{3/2}}{24} - \frac{a^4x\sqrt{x^2-a^2}}{16} + \frac{a^6}{16}\ln\left(x+\sqrt{x^2-a^2}\right)$$

$$\int x^3(x^2-a^2)^{3/2}\,dx=\frac{(x^2-a^2)^{7/2}}{7}+\frac{a^2(x^2-a^2)^{5/2}}{5}$$

$$\int\frac{(x^2-a^2)^{3/2}}{x}\,dx=\frac{(x^2-a^2)^{3/2}}{3}-a^2\sqrt{x^2-a^2}+a^3\cos^{-1}\left|\frac{a}{x}\right|$$

$$\int\frac{(x^2-a^2)^{3/2}}{x^2}\,dx=-\frac{(x^2-a^2)^{3/2}}{x}+\frac{3x\sqrt{x^2-a^2}}{2}-\frac{3}{2}a^2\ln\left(x+\sqrt{x^2-a^2}\right)$$

$$\int\frac{(x^2-a^2)^{3/2}}{x^3}\,dx=-\frac{(x^2-a^2)^{3/2}}{2x^2}+\frac{3\sqrt{x^2-a^2}}{2}-\frac{3}{2}a\cos^{-1}\left|\frac{a}{x}\right|$$

$$\int\frac{x\,dx}{(x^2-a^2)^{(2n+1)/2}}=\frac{-1}{(2n-1)(x^2-a^2)^{(2n-1)/2}}$$

$$\int\frac{x^3\,dx}{(x^2-a^2)^{(2n+1)/2}}=\frac{-1}{(2n-3)(x^2-a^2)^{(2n-3)/2}}-\frac{a^2}{(2n-1)(x^2-a^2)^{(2n-1)/2}}$$

$$\int x(x^2-a^2)^{(2n+1)/2}\,dx=\frac{(x^2-a^2)^{(2n+3)/2}}{2n+3}$$

Με ρίζες του $a^2 - x^2$ $(a^2 > x^2)$

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$$

$$\int\frac{x\,dx}{\sqrt{a^2-x^2}}=-\sqrt{a^2-x^2}$$

$$\int\frac{x^2\,dx}{\sqrt{a^2-x^2}}=-\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$\int\frac{x^2\,dx}{\sqrt{a^2-x^2}}=-\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$\int\frac{x^3\,dx}{\sqrt{a^2-x^2}}=\frac{(a^2-x^2)^{3/2}}{3}-a^2\sqrt{a^2-x^2}$$

$$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$$

$$\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} - \frac{1}{2a^3}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$\int x\sqrt{a^2-x^2}\,dx = -\frac{(a^2-x^2)^{3/2}}{3}$$

$$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x(a^2-x^2)^{3/2}}{4} + \frac{a^2x\sqrt{a^2-x^2}}{8} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}$$

$$\int x^3\sqrt{a^2-x^2}\,dx = \frac{(a^2-x^2)^{5/2}}{5} - \frac{a^2(a^2-x^2)^{3/2}}{3}$$

$$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a}$$

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a}$$

$$\int \frac{\sqrt{a^2-x^2}}{x^3}\,dx = -\frac{\sqrt{a^2-x^2}}{2x^2} + \frac{1}{2a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}}$$

$$\int \frac{x\,dx}{(a^2-x^2)^{3/2}} = \frac{1}{\sqrt{a^2-x^2}}$$

$$\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a}$$

$$\int \frac{x^3\,dx}{(a^2-x^2)^{3/2}} = \sqrt{a^2-x^2} + \frac{a^2}{\sqrt{a^2-x^2}}$$

$$\int \frac{dx}{x(a^2-x^2)^{3/2}} = \frac{1}{a^2\sqrt{a^2-x^2}} - \frac{1}{a^3}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \frac{dx}{x^2(a^2-x^2)^{3/2}} = -\frac{\sqrt{a^2-x^2}}{a^4x} + \frac{x}{a^4\sqrt{a^2-x^2}}$$

$$\int \frac{dx}{x^3(a^2-x^2)^{3/2}} = -\frac{1}{2a^2x^2\sqrt{a^2-x^2}} + \frac{3}{2a^4\sqrt{a^2-x^2}} - \frac{3}{2a^5}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int (a^2-x^2)^{3/2}\,dx = \frac{x(a^2-x^2)^{3/2}}{4} + \frac{3a^2x\sqrt{a^2-x^2}}{8} + \frac{3}{8}a^4\sin^{-1}\frac{x}{a}$$

$$\int x(a^2-x^2)^{3/2}\,dx = -\frac{(a^2-x^2)^{5/2}}{5}$$

$$\int x^2(a^2-x^2)^{3/2}\,dx = -\frac{x(a^2-x^2)^{5/2}}{6} + \frac{a^2x(a^2-x^2)^{3/2}}{24} + \frac{a^4x\sqrt{a^2-x^2}}{16} + \frac{a^6}{16}\sin^{-1}\frac{x}{a}$$

$$\int x^3(a^2-x^2)^{3/2}\,dx = \frac{(a^2-x^2)^{7/2}}{7} - \frac{a^2(a^2-x^2)^{5/2}}{5}$$

$$\int \frac{(a^2-x^2)^{3/2}}{x}\,dx = \frac{(a^2-x^2)^{3/2}}{3} + a^2\sqrt{a^2-x^2} - a^3\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \frac{(a^2-x^2)^{3/2}}{x^2}\,dx = -\frac{(a^2-x^2)^{3/2}}{x} - \frac{3x\sqrt{a^2-x^2}}{2} - \frac{3}{2}a^2\sin^{-1}\frac{x}{a}$$

$$\int \frac{(a^2-x^2)^{3/2}}{x^3}\,dx = -\frac{(a^2-x^2)^{3/2}}{2x^2} - \frac{3\sqrt{a^2-x^2}}{2} + \frac{3a}{2}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \frac{x\,dx}{(a^2-x^2)^{(2n+1)/2}} = \frac{1}{(2n-1)(a^2-x^2)^{(2n-1)/2}}$$

$$\int \frac{x^3\,dx}{(a^2-x^2)^{(2n+1)/2}} = \frac{-1}{(2n-3)(a^2-x^2)^{(2n-3)/2}} + \frac{a^2}{(2n-1)(a^2-x^2)^{(2n-1)/2}}$$

$$\int x(a^2-x^2)^{(2n+1)/2}\,dx = -\frac{(a^2-x^2)^{(2n+3)/2}}{2n+3}$$

$$\int x^3(a^2-x^2)^{(2n+1)/2}\,dx = \frac{(a^2-x^2)^{(2n+5)/2}}{2n+5} - \frac{a^2(a^2-x^2)^{(2n+3)/2}}{2n+3}$$

Με $ax^k + b$

$$\int \frac{dx}{ax^2+b} = \frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{a}{b}}\right), \qquad ab>0$$

$$= \frac{1}{2\sqrt{-ab}}\ln\frac{b+x\sqrt{-ab}}{b-x\sqrt{-ab}} \qquad ab<0$$

$$\int \frac{dx}{ax^3+b} = \frac{\lambda}{6b}\ln\frac{(x+\lambda)^2}{x^2-\lambda x+\lambda^2} + \frac{\lambda}{\sqrt{3}b}\tan^{-1}\frac{2x-\lambda}{\lambda\sqrt{3}} \qquad \lambda=\sqrt[3]{\frac{b}{a}}$$

$$\int \frac{x\,dx}{ax^3+b} = -\frac{1}{6a\lambda}\ln\frac{(x+\lambda)^2}{x^2-\lambda x+\lambda^2} + \frac{1}{\sqrt{3}a\lambda}\tan^{-1}\frac{2x-\lambda}{\lambda\sqrt{3}} \qquad \lambda=\sqrt[3]{\frac{b}{a}}$$

$$\int \frac{x^2\,dx}{ax^3+b} = \frac{1}{3a}\ln(ax^3+b)$$

$$\int \frac{x^3\,dx}{ax^3+b} = \frac{x}{a} - \frac{b}{a}\int\frac{dx}{ax^3+b}$$

$$\int \frac{dx}{(ax^3+b)^2} = \frac{x}{3b(ax^3+b)} + \frac{2}{3b}\int\frac{dx}{ax^3+b}$$

$$\int \frac{x\,dx}{(ax^3+b)^2} = \frac{x^2}{3b(ax^3+b)} + \frac{1}{3b}\int\frac{x\,dx}{ax^3+b}$$

$$\int \frac{x^2\,dx}{(ax^3+b)^2} = -\frac{1}{3b(ax^3+b)}$$

$$\int \frac{x^3\,dx}{(ax^3+b)^2} = -\frac{x}{3b(ax^3+b)} + \frac{1}{3b}\int \frac{dx}{ax^3+b}$$

$$\int \frac{dx}{x(ax^3+b)} = \frac{1}{3b}\ln \frac{x^3}{ax^3+b}$$

$$\int \frac{dx}{x^2(ax^3+b)} = -\frac{1}{bx} - \frac{a}{b}\int \frac{x\,dx}{ax^3+b}$$

$$\int \frac{dx}{x^3(ax^3+b)} = -\frac{1}{2bx^2} - \frac{a}{b}\int \frac{dx}{ax^3+b}$$

$$\int \frac{dx}{x(ax^3+b)^2} = \frac{1}{3b(ax^3+b)} + \frac{1}{3b^2}\ln \frac{x^3}{ax^3+b}$$

$$\begin{aligned}\int x^n(ax^k+b)^m\,dx &= \frac{x^{n-k+1}(ax^k+b)^{m+1}}{a(km+n+1)} - \frac{b(n-k+1)}{a(km+n+1)}\int x^{n-k}(ax^k+b)^m\,dx \\ &= \frac{x^{n+1}(ax^k+b)^{m+1}}{b(n+1)} - \frac{a(km+k+n+1)}{b(n+1)}\int x^{n+k}(ax^k+b)^m\,dx\end{aligned}$$

Με $ax^2 + bx + c$

Αν $b^2 = 4ac$, τότε $ax^2 + bx + c = a(x + b/2a)^2$ και είναι προτιμότερο να αναχθεί το ολοκλήρωμα σε άλλο απλούστερο πριν από τον υπολογισμό. Το ίδιο ισχύει και σε άλλες ειδικές περιπτώσεις, όπου ένα από τα a, b, c είναι μηδέν.

$$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}}\tan^{-1}\dfrac{2ax+b}{\sqrt{4ac-b^2}}, & b^2<4ac \\[2ex] \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ac+b+\sqrt{b^2-4ac}}\right|, & b^2>4ac \\[2ex] \dfrac{1}{a(x_1-x_2)}\ln\left|\dfrac{x-x_1}{x-x_2}\right|, & b^2>4ac \end{cases}$$

(x_1, x_2 είναι οι ρίζες της $ax^2 + bx + c = 0$)

$$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\ln\left|ax^2+bx+c\right| - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c}$$

$$\int \frac{x^2\,dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2}\ln\left|ax^2+bx+c\right| + \frac{b^2-2ac}{2a^2}\int \frac{dx}{ax^2+bx+c}$$

$$\int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c}\ln\left|\frac{x^2}{ax^2+bx+c}\right| - \frac{b}{2c}\int \frac{dx}{ax^2+bx+c}$$

$$\int \frac{dx}{x^2(ax^2+bx+c)} = \frac{b}{2c^2}\ln\left|\frac{ax^2+bx+c}{x^2}\right| - \frac{1}{cx} + \frac{b^2-2ac}{2c^2}\int \frac{dx}{ax^2+bx+c}$$

$$\int \frac{dx}{(ax^2+bx+c)^2} = \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \frac{2a}{4ac-b^2}\int \frac{dx}{ax^2+bx+c}$$

$$\int \frac{x\,dx}{(ax^2+bx+c)^2} = \frac{bx+2c}{(b^2-4ac)(ax^2+bx+c)} + \frac{b}{b^2-4ac}\int \frac{dx}{ax^2+bx+c}$$

$$\int \frac{x^2\,dx}{(ax^2+bx+c)^2} = \frac{(b^2-2ac)x+bc}{a(4ac-b^2)(ax^2+bx+c)} + \frac{2c}{4ac-b^2}\int \frac{dx}{ax^2+bx+c}$$

$$\int \frac{dx}{x(ax^2+bx+c)^2} = \frac{1}{2c(ax^2+bx+c)} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^2} + \frac{1}{c}\int \frac{dx}{x(ax^2+bx+c)}$$

$$\int \frac{dx}{x^2(ax^2+bx+c)^2} = \frac{-1}{cx(ax^2+bx+c)} - \frac{3a}{c}\int \frac{dx}{(ax^2+bx+c)^2} - \frac{2b}{c}\int \frac{dx}{x(ax^2+bx+c)^2}$$

$$\int \frac{x^n\,dx}{ax^2+bx+c} = \frac{x^{n-1}}{(n-1)a} - \frac{c}{a}\int \frac{x^{n-2}\,dx}{ax^2+bx+c} - \frac{b}{a}\int \frac{x^{n-1}\,dx}{ax^2+bx+c}$$

$$\int \frac{dx}{x^n(ax^2+bx+c)} = \frac{-1}{(n-1)cx^{n-1}} - \frac{b}{c}\int \frac{dx}{x^{n-1}(ax^2+bx+c)} - \frac{a}{c}\int \frac{dx}{x^{n-2}(ax^2+bx+c)}$$

$$\int \frac{dx}{(ax^2+bx+c)^n} = \frac{2ax+b}{(n-1)(4ac-b^2)(ax^2+bx+c)^{n-1}}$$

$$+\frac{2(2n-3)a}{(n-1)(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^{n-1}}$$

$$\int \frac{x^m dx}{(ax^2+bx+c)^n} = \frac{-x^{m-1}}{(2n-m-1)a(ax^2+bx+c)^{n-1}}$$

$$+\frac{(m-1)c}{(2n-m-1)a}\int \frac{x^{m-2}dx}{(ax^2+bx+c)^n} - \frac{(n-m)b}{(2n-m-1)a}\int \frac{x^{m-1}dx}{(ax^2+bx+c)^n}$$

$$\int \frac{dx}{x^m(ax^2+bx+c)^n} = -\frac{1}{(m-1)cx^{m-1}(ax^2+bx+c)^{n-1}}$$

$$-\frac{(m+2n-3)a}{(m-1)c}\int \frac{dx}{x^{m-2}(ax^2+bx+c)^2} - \frac{(m+n-2)}{(m-1)c}\int \frac{dx}{x^{m-1}(ax^2+bx+c)^n}$$

Με ρίζες

Αν $b^2 = 4ac$ ή ένα από τα a, b, c είναι μηδέν, ισχύει η ίδια παρατήρηση με τα ολοκληρώματα με $ax^2 + bx + c$.

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2\sqrt{a}\sqrt{ax^2+bx+c}+2ax+b\right| & a>0 \\[2ex] \dfrac{1}{\sqrt{a}}\sinh^{-1}\left(\dfrac{2ax+b}{\sqrt{4ac-b^2}}\right) & a>0,\ 4ac>b^2 \\[2ex] \dfrac{-1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{2ax+b}{\sqrt{b^2-4ac}}\right) & \begin{cases} a<0,\ b^2>4ac \\ |2ax+b|<\sqrt{b^2-4ac} \end{cases} \end{cases}$$

$$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int \frac{x^2\,dx}{\sqrt{ax^2+bx+c}} = \frac{2ax-3b}{4a^2}\sqrt{ax^2+bx+c} + \frac{3b^2-4ac}{8a^2}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right| & c>0 \\[2ex] \dfrac{-1}{\sqrt{c}}\sinh^{-1}\left(\dfrac{bx+2c}{|x|\sqrt{4ac-b^2}}\right) & c>0,\ 4ac>b^2 \\[2ex] \dfrac{1}{\sqrt{-c}}\sin^{-1}\left(\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}\right) & c<0,\ b^2>4ac \end{cases}$$

$$\int \frac{dx}{x^2\sqrt{ax^2+bx+c}} = -\frac{\sqrt{ax^2+bx+c}}{cx} - \frac{b}{2c}\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

$$\int \sqrt{ax^2+bx+c}\,dx = \frac{(2a+b)\sqrt{ax^2+bx+c}}{4a} + \frac{4ac-b^2}{8a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int x\sqrt{ax^2+bx+c}\,dx = \frac{(ax^2+bx+c)^{3/2}}{3a} - \frac{b(2ax+b)}{8a^2}\sqrt{ax^2+bx+c}$$

$$-\frac{b(4ac-b^2)}{16a^2}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int x^2\sqrt{ax^2+bx+c}\,dx = \frac{6ax-5b}{24a^2}(ax^2+bx+c)^{3/2} + \frac{5b^2-4ac}{16a^2}\int \sqrt{ax^2+bx+c}\,dx$$

$$\int \frac{\sqrt{ax^2+bx+c}}{x}dx = \sqrt{ax^2+bx+c} + \frac{b}{2}\int \frac{dx}{\sqrt{ax^2+bx+c}} + c\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

$$\int \frac{\sqrt{ax^2+bx+c}}{x^2}dx = -\frac{\sqrt{ax^2+bx+c}}{x} + a\int \frac{dx}{\sqrt{ax^2+bx+c}} + \frac{b}{2}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int \frac{dx}{(ax^2+bx+c)^{3/2}} = \frac{4ax+2b}{(4ac-b^2)\sqrt{ax^2+bx+c}}$$

$$\int \frac{x\,dx}{(ax^2+bx+c)^{3/2}} = \frac{2(bx+2c)}{(b^2-4ac)\sqrt{ax^2+bx+c}}$$

$$\int \frac{x^2\,dx}{(ax^2+bx+c)^{3/2}} = \frac{(2b^2-4ac)x+2bc}{a(4ac-b^2)\sqrt{ax^2+bx+c}} + \frac{1}{a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int \frac{dx}{x(ax^2+bx+c)^{3/2}} = \frac{1}{c\sqrt{ax^2+bx+c}} + \frac{1}{c}\int \frac{dx}{x\sqrt{ax^2+bx+c}} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^{3/2}}$$

$$\int \frac{dx}{x^2(ax^2+bx+c)^{3/2}} = -\frac{ax^2+bx+c}{c^2x\sqrt{ax^2+bx+c}} + \frac{b^2-2ac}{2c^2}\int \frac{dx}{(ax^2+bx+c)^{3/2}}$$

$$-\frac{3b}{2c^2}\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

$$\int (ax^2+bx+c)^{n+1/2}dx = \frac{(2ax+b)(ax^2+bx+c)^{n+1/2}}{4(n+1)a}$$

$$+\frac{(2n+1)(4ac-b^2)}{8(n+1)a}\int (ax^2+bx+c)^{n-1/2}dx$$

$$\int x(ax^2+bx+c)^{n+1/2}dx = \frac{(ax^2+bx+c)^{n+3/2}}{(2n+3)a} - \frac{b}{2a}\int (ax^2+bx+c)^{n+1/2}dx$$

$$\int \frac{dx}{(ax^2+bx+c)^{n+1/2}} = \frac{2(2ax+b)}{(2n-1)(4ac-b^2)(ax^2+bx+c)^{n-1/2}}$$

$$+\frac{8a(n-1)}{(2n-1)(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^{n-1/2}}$$

$$\int \frac{x\,dx}{(ax^2+bx+c)^{n+1/2}} = \frac{-1}{(2n-1)a(ax^2+bx+c)^{n-1/2}} - \frac{b}{2a}\int \frac{dx}{(ax^2+bx+c)^{n+1/2}}$$

$$\int \frac{dx}{x(ax^2+bx+c)^{n+1/2}} = \frac{1}{(2n-1)c(ax^2+bx+c)^{n-1/2}}$$

$$+\frac{1}{c}\int \frac{dx}{x(ax^2+bx+c)^{n-1/2}} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^{n+1/2}}$$

Με x^3+a^3

$$\int \frac{dx}{x^3+a^3} = \frac{1}{6a^2}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{1}{a^2\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

$$\int \frac{xdx}{x^3+a^3} = \frac{1}{6a}\ln\frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{a\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

$$\int \frac{x^2dx}{x^3+a^3} = \frac{1}{3}\ln\left|x^3+a^3\right|$$

$$\int \frac{x^3dx}{x^3+a^3} = x - \frac{a}{6}\ln\frac{(x+a)^2}{x^2-ax+a^2} - \frac{a}{\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

$$\int \frac{dx}{x(x^3+a^3)} = \frac{1}{3a^3}\ln\left|\frac{x^3}{x^3+a^3}\right|$$

$$\int \frac{dx}{x^2(x^3+a^3)} = -\frac{1}{a^3x} - \frac{1}{6a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2} - \frac{1}{a^4\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

$$\int \frac{dx}{x^3(x^3+a^3)} = \frac{-1}{2a^3x^2} - \frac{1}{6a^5}\ln\frac{(x+a)^2}{x^2-ax+a^2} - \frac{1}{a^5\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

$$\int \frac{dx}{(x^3+a^3)^2} = \frac{x}{3a^3(x^3+a^3)} + \frac{1}{9a^5}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{2}{3a^5\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

$$\int \frac{x\,dx}{(x^3+a^3)^2} = \frac{x^2}{3a^3(x^3+a^3)} + \frac{1}{18a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{3a^4\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

$$\int \frac{x^2\,dx}{(x^3+a^3)^2} = \frac{-1}{3(x^3+a^3)}$$

$$\int \frac{x^3\,dx}{(x^3+a^3)^2} = \frac{-x}{3(x^3+a^3)} + \frac{1}{18a^2}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{1}{3\sqrt{3}a^2}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

$$\int \frac{dx}{x(x^3+a^3)^2} = \frac{1}{3a^3(x^3+a^3)} + \frac{1}{3a^6}\ln\left|\frac{x^3}{x^3+a^3}\right|$$

$$\int \frac{dx}{x^2(x^3+a^3)^2} = \frac{-1}{a^6x} - \frac{x^2}{3a^6(x^3+a^3)} - \frac{4}{3a^6}\int\frac{x\,dx}{x^3+a^3}$$

$$\int \frac{dx}{x^3(x^3+a^3)^2} = \frac{-1}{2a^6x^2} - \frac{x}{3a^6(x^3+a^3)} - \frac{5}{18a^8}\ln\frac{(x+a)^2}{x^2-ax+a^2} - \frac{5}{3\sqrt{3}a^8}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

$$\int \frac{x^n\,dx}{x^3+a^3} = \frac{x^{n-2}}{n-2} - a^3\int\frac{x^{n-3}\,dx}{x^3+a^3}$$

$$\int \frac{dx}{x^n(x^3+a^3)} = \frac{-1}{a^3(n-1)x^{n-1}} - \frac{1}{a^3}\int\frac{dx}{x^{n-3}(x^3+a^3)}$$

Με $x^4 \pm a^4$

$$\int \frac{dx}{x^4+a^4} = \frac{1}{4a^3\sqrt{2}}\ln\left(\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2}\right) - \frac{1}{2a^3\sqrt{2}}\tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

$$\int \frac{x\,dx}{x^4+a^4} = \frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2}$$

$$\int \frac{x^2\,dx}{x^4+a^4} = \frac{1}{4a\sqrt{2}}\ln\left(\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2}\right) - \frac{1}{2a\sqrt{2}}\tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

$$\int \frac{x^3\,dx}{x^4+a^4} = \frac{1}{4}\ln\left(x^4+a^4\right)$$

$$\int \frac{dx}{x(x^4+a^4)} = \frac{1}{4a^4}\ln\left(\frac{x^4}{x^4+a^4}\right)$$

$$\int \frac{dx}{x^2(x^4+a^4)} = -\frac{1}{a^4x} - \frac{1}{4a^5\sqrt{2}}\ln\left(\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2}\right) + \frac{1}{2a^5\sqrt{2}}\tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

$$\int \frac{dx}{x^3(x^4+a^4)} = -\frac{1}{2a^4x^2} - \frac{1}{2a^6}\tan^{-1}\frac{x^2}{a^2}$$

$$\int \frac{dx}{x^4-a^4} = \frac{1}{4a^3}\ln\left|\frac{x-a}{x+a}\right| - \frac{1}{2a^3}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x\,dx}{x^4-a^4} = \frac{1}{4a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right|$$

$$\int \frac{x^2\,dx}{x^4-a^4} = \frac{1}{4a}\ln\left|\frac{x-a}{x+a}\right| + \frac{1}{2a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x^3\,dx}{x^4-a^4} = \frac{1}{4}\ln\left|x^4-a^4\right|$$

$$\int \frac{dx}{x(x^4-a^4)} = \frac{1}{4a^4}\ln\left|\frac{x^4-a^4}{x^4}\right|$$

$$\int \frac{dx}{x^2(x^4-a^4)} = \frac{1}{a^4x} + \frac{1}{4a^5}\ln\left|\frac{x-a}{x+a}\right| + \frac{1}{2a^5}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x^3(x^4-a^4)} = \frac{1}{2a^4x^2} + \frac{1}{4a^6}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right|$$

Με $x^n \pm a^n$

$$\int \frac{dx}{x(x^n+a^n)} = \frac{1}{na^n}\ln\left|\frac{x^n}{x^n+a^n}\right|$$

$$\int \frac{x^{n-1}dx}{x^n+a^n} = \frac{1}{n}\ln\left|x^n+a^n\right|$$

$$\int \frac{x^m\,dx}{(x^n+a^n)^r} = \int \frac{x^{m-n}\,dx}{(x^n+a^n)^{r-1}} - a^n\int \frac{x^{m-n}\,dx}{(x^n+a^n)^r}$$

$$\int \frac{dx}{x^m(x^n+a^n)^r} = \frac{1}{a^n}\int \frac{dx}{x^m(x^n+a^n)^{r-1}} - \frac{1}{a^n}\int \frac{dx}{x^{m-n}(x^n+a^n)^r}$$

$$\int \frac{dx}{x\sqrt{x^n+a^n}} = \frac{1}{n\sqrt{a^n}}\ln\left|\frac{\sqrt{x^n+a^n}-\sqrt{a^n}}{\sqrt{x^n+a^n}+\sqrt{a^n}}\right|$$

$$\int \frac{dx}{x(x^n-a^n)} = \frac{1}{na^n}\ln\left|\frac{x^n-a^n}{x^n}\right|$$

$$\int \frac{x^{n-1}dx}{x^n-a^n} = \frac{1}{n}\ln\left|x^n-a^n\right|$$

$$\int \frac{x^m\,dx}{(x^n-a^n)^r} = a^n\int \frac{x^{m-n}\,dx}{(x^n-a^n)^r} + \int \frac{x^{m-n}\,dx}{(x^n-a^n)^{r-1}}$$

$$\int \frac{dx}{x^m(x^n-a^n)^r} = \frac{1}{a^n}\int \frac{dx}{x^{m-n}(x^n-a^n)^r} - \frac{1}{a^n}\int \frac{dx}{x^m(x^n-a^n)^{r-1}}$$

$$\int \frac{dx}{x\sqrt{x^n-a^n}} = \frac{2}{n\sqrt{a^n}}\cos^{-1}\sqrt{\frac{a^n}{x^n}}$$

$$\int \frac{x^{p-1}\,dx}{x^{2m}+a^{2m}} = \frac{1}{ma^{2m-p}}\sum_{k=1}^{m}\sin\frac{(2k-1)p\pi}{2m}\tan^{-1}\left(\frac{x+a\cos[(2k-1)\pi/2m]}{a\sin[(2k-1)\pi/2m]}\right)$$
$$-\frac{1}{2ma^{2m-p}}\sum_{k=1}^{m}\cos\frac{(2k-1)p\pi}{2m}\ln\left(x^2+2ax\cos\frac{(2k-1)\pi}{2m}+a^2\right)$$

όπου $0 < p \le 2m$.

$$\int \frac{x^{p-1}\,dx}{x^{2m}-a^{2m}} = \frac{1}{2ma^{2m-p}} \sum_{k=1}^{m-1} \cos\frac{kp\pi}{m} \ln\left(x^2 - 2ax\cos\frac{k\pi}{m} + a^2\right)$$

$$- \frac{1}{ma^{2m-p}} \sum_{k=1}^{m-1} \sin\frac{kp\pi}{m} \tan^{-1}\left(\frac{x - a\cos(k\pi/m)}{a\sin(k\pi/m)}\right)$$

$$+ \frac{1}{2ma^{2m-p}} \{\ln(x-a) + (-1)^p \ln(x+a)\}$$

όπου $0 < p \le 2m$.

$$\int \frac{x^{p-1}\,dx}{x^{2m+1}+a^{2m+1}} = \frac{2(-1)^{p-1}}{(2m+1)a^{2m-p+1}} \sum_{k=1}^{m} \sin\frac{2kp\pi}{2m+1} \tan^{-1}\left(\frac{x + a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)$$

$$- \frac{2(-1)^{p-1}}{(2m+1)a^{2m-p+1}} \sum_{k=1}^{m} \cos\frac{2kp\pi}{2m+1} \ln\left(x^2 + 2ax\cos\frac{2k\pi}{2m+1} + a^2\right)$$

$$+ \frac{(-1)^{p-1}\ln(x+a)}{(2m+1)a^{2m-p+1}}$$

όπου $0 < p \le 2m+1$.

$$\int \frac{x^{p-1}\,dx}{x^{2m+1}-a^{2m+1}} = -\frac{2}{(2m+1)a^{2m-p+1}} \sum_{k=1}^{m} \sin\frac{2kp\pi}{2m+1} \tan^{-1}\left(\frac{x - a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)$$

$$+ \frac{1}{(2m+1)a^{2m-p+1}} \sum_{k=1}^{m} \cos\frac{2kp\pi}{2m+1} \ln\left(x^2 - 2ax\cos\frac{2p\pi}{2m+1} + a^2\right)$$

$$+ \frac{\ln(x-a)}{(2m+1)a^{2m-p+1}}$$

όπου $0 < p \le 2m+1$.

Με $\sin ax$

$$\int \sin ax\,dx = -\frac{\cos ax}{a}$$

$$\int x\sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$$

$$\int x^2 \sin ax\,dx = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax$$

$$\int x^3 \sin ax dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax + \left(\frac{6x}{a^3} - \frac{x^3}{a}\right)\cos ax$$

$$\int \frac{\sin ax}{x} dx = \mathrm{Si}(ax) = ax - \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} - \frac{(ax)^7}{7\cdot 7!} + \cdots$$

$$\int \frac{\sin ax}{x^2} dx = -\frac{\sin ax}{x} + a\int \frac{\cos ax}{x} dx$$

$$\int \frac{dx}{\sin ax} = \frac{1}{a}\ln\left|\frac{1}{\sin ax} - \cot ax\right| = \frac{1}{2a}\ln\left|\frac{1-\cos ax}{1+\cos ax}\right| = \frac{1}{a}\ln\left|\tan\frac{ax}{2}\right|$$

$$\int \frac{x\,dx}{\sin ax} = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{3\cdot 3!} + \frac{7(ax)^5}{3\cdot 5\cdot 5!} + \frac{31(ax)^7}{3\cdot 7\cdot 7!} + \cdots + \frac{2(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int x\sin^2 ax dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$$

$$\int x^2 \sin^2 ax\, dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax - \frac{x\cos 2ax}{4a^2}$$

$$\int x^3 \sin^2 ax\, dx = \frac{x^4}{8} - \left(\frac{x^3}{4a} - \frac{3x}{8a^3}\right)\sin 2ax - \left(\frac{3x^2}{8a^2} - \frac{3}{16a^4}\right)\cos 2ax$$

$$\int \sin^3 ax dx = -\frac{\cos ax}{a} + \frac{\cos^3 ax}{3a}$$

$$\int \sin^4 ax dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

$$\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$$

$$\int \frac{x\,dx}{\sin^2 ax} = -\frac{x}{a}\cot ax + \frac{1}{a^2}\ln|\sin ax|$$

$$\int \frac{dx}{\sin^3 ax} = \frac{-\cos ax}{2a\sin^2 ax} + \frac{1}{2a}\ln\left|\tan\frac{ax}{2}\right|$$

$$\int \frac{x\,dx}{\sin^3 ax} = \frac{-x\cos ax}{2a\sin^2 ax} - \frac{1}{2a^2 \sin ax} + \frac{1}{2}\int \frac{x\,dx}{\sin ax}$$

$$\int \sin ax \sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \qquad a \neq \pm b$$

$$\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$$\int \frac{x dx}{1-\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + \frac{2}{a^2}\ln\left|\sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)\right|$$

$$\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

$$\int \frac{x dx}{1+\sin ax} = -\frac{x}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln\left|\sin\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right|$$

$$\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

$$\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

$$\int \frac{dx}{b+c\sin ax} = \begin{cases} \dfrac{2}{a\sqrt{b^2-c^2}}\tan^{-1}\dfrac{b\tan\dfrac{ax}{2}+c}{\sqrt{b^2-c^2}} & b^2 > c^2 \\[2ex] \dfrac{1}{a\sqrt{c^2-b^2}}\ln\left|\dfrac{b\tan\dfrac{ax}{2}+c-\sqrt{c^2-b^2}}{b\tan\dfrac{ax}{2}+c+\sqrt{c^2-b^2}}\right| & b^2 < c^2 \end{cases}$$

$$\int \frac{dx}{(b+c\sin ax)^2} = \frac{q\cos ax}{a(b^2-c^2)(b+c\sin ax)} + \frac{b}{b^2-c^2}\int \frac{dx}{b+c\sin ax} \qquad b \neq \pm c$$

$$\int \frac{dx}{b^2+c^2\sin^2 ax} = \frac{1}{ab\sqrt{b^2+c^2}}\tan^{-1}\frac{\sqrt{b^2+c^2}\,\tan ax}{b}$$

$$\int \frac{dx}{b^2 - c^2 \sin ax} = \begin{cases} \dfrac{1}{ab\sqrt{b^2-c^2}} \tan^{-1} \dfrac{\sqrt{b^2-c^2}\,\tan ax}{b} & b^2 > c^2 \\ \dfrac{1}{2ab\sqrt{c^2-b^2}} \ln\left|\dfrac{\sqrt{c^2-b^2}\,\tan ax + b}{\sqrt{c^2-b^2}\,\tan ax - b}\right| & b^2 < c^2 \end{cases}$$

$$\int \sin(\ln x)\,dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)]$$

$$\int x\sin(x^2)\,dx = -\frac{1}{2}\cos(x^2$$

$$\int x^3 \sin(x^2)\,dx = -\frac{x^2}{2}\cos(x^2) + \frac{1}{2}\sin(x^2)$$

$$\int x^n \sin ax\,dx = -\frac{x^n \cos ax}{a} + \frac{nx^{n-1}\sin ax}{a^2} - \frac{n(n-1)}{a^2}\int x^{n-2}\sin ax\,dx$$

$$\int \frac{\sin ax}{x^n}dx = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}dx$$

$$\int \sin^n ax dx = -\frac{\sin^{n-1} ax \cos ax}{an} + \frac{n-1}{n}\int \sin^{n-2} ax dx$$

$$\int \frac{dx}{\sin^n ax} = -\frac{\cos ax}{a(n-1)\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}, \qquad n > 1$$

$$\int \frac{x dx}{\sin^n ax} = -\frac{x\cos ax}{a(n-1)\sin^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x dx}{\sin^{n-2} ax}, \quad n > 2$$

Με ρίζες ($K = \sqrt{1 - k^2 \sin^2 x}$)

$$\int \frac{\sin x\,dx}{\sqrt{1+k^2\sin^2 x}} = -\frac{1}{k}\sin^{-1}\frac{k\cos x}{\sqrt{1+k^2}}$$

$$\int \frac{\sin x\,dx}{K} = -\frac{1}{k}\ln\left(k\cos x + K\right)$$

$$\int (\sin x)\sqrt{1+k^2\sin^2 x}\,dx = -\frac{\cos x}{2}\sqrt{1+k^2\sin^2 x} - \frac{1+k^2}{2k}\sin^{-1}\frac{k\cos x}{\sqrt{1+k^2}}$$

$$\int K\sin x\,dx = -\frac{1}{2}K\cos x - \frac{1-k^2}{2k}\ln\left(K + k\cos x\right)$$

$$\int \frac{dx}{(\sin x)\sqrt{1+k^2\sin^2 x}} = \frac{1}{2}\ln\frac{\sqrt{1+k^2\sin^2 x}-\cos x}{\sqrt{1+k^2\sin^2 x}+\cos x}$$

$$\int \frac{dx}{K\sin x} = -\frac{1}{2}\ln\frac{K+\cos x}{K-\cos x}$$

$$\int \frac{K}{\sin x}dx = \frac{1}{2}\ln\frac{K+\cos x}{K-\cos x} + k\ln\left(K + k\cos x\right)$$

$$\int \frac{\sin x\,dx}{K^3} = \frac{\cos x}{(k^2-1)K}$$

Με cos *ax*

$$\int \cos ax dx = \frac{\sin ax}{a}$$

$$\int x\cos ax dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$$

$$\int x^2\cos ax dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$$

$$\int x^3\cos ax dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\sin ax$$

$$\int \frac{\cos ax}{x}dx = \ln|x| - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \frac{(ax)^6}{6\cdot 6!} + \cdots$$

$$\int \frac{\cos ax}{x^2}dx = -\frac{\cos ax}{x} - a\int\frac{\sin ax}{x}dx$$

$$\int \frac{\cos ax}{x^3}dx = -\frac{\cos ax}{2x^2} + \frac{a\sin ax}{2x} - \frac{a^2}{2}\int\frac{\cos ax}{x}dx$$

$$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln\left|\frac{1}{\cos ax} + \tan ax\right| = \frac{1}{a}\ln\left|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right| = \frac{1}{2a}\ln\left|\frac{1+\sin ax}{1-\sin ax}\right|$$

$$\int \frac{x\,dx}{\cos ax} = \frac{x^2}{2} + \frac{a^2x^4}{8} + \frac{5a^4x^6}{144} + \cdots + \frac{E_n a^{2n} x^{2n+2}}{(2n+2)(2n)!} + \cdots$$

$$\int \frac{x^2\,dx}{\cos ax} = \frac{x^3}{3} + \frac{a^2x^5}{5\cdot 2!} + \frac{5a^4x^7}{7\cdot 4!} + \frac{61a^6x^9}{9\cdot 6!} + \cdots + \frac{E_{n-1} a^{2n-2} x^{2n+1}}{(2n+1)(2n-2)!}$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int x\cos^2 ax\,dx = \frac{x^2}{4} + \frac{x\sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$$

$$\int x^2\cos^2 ax\,dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax + \frac{x\cos 2ax}{4a^2}$$

$$\int x^3\cos^2 ax\,dx = \frac{x^4}{8} + \left(\frac{x^3}{4a} - \frac{3x}{8a^3}\right)\sin 2ax + \left(\frac{3x^2}{8a^2} - \frac{3}{16a^4}\right)\cos 2ax$$

$$\int \cos^3 ax\,dx = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a}$$

$$\int \cos^4 ax\,dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

$$\int \frac{dx}{\cos^2 ax} = \frac{\tan ax}{a}$$

$$\int \frac{x\,dx}{\cos^3 ax} = \frac{x\sin ax}{2a\cos^2 ax} - \frac{1}{2a^2\cos ax} + \frac{1}{2}\int \frac{x\,dx}{\cos ax}$$

$$\int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a\cos^2 ax} + \frac{1}{2a}\ln\left|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right|$$

$$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \qquad a \neq \pm b$$

$$\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2}$$

$$\int \frac{x\,dx}{1-\cos ax} = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\ln\left|\sin\frac{ax}{2}\right|$$

$$\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}$$

$$\int \frac{x\,dx}{1+\cos ax} = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\ln\left|\cos\frac{ax}{2}\right|$$

$$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$$

$$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^3\frac{ax}{2}$$

$$\int \frac{dx}{b+c\cos ax} = \begin{cases} \dfrac{2}{a\sqrt{b^2-c^2}}\tan^{-1}\left(\sqrt{\dfrac{b-c}{b+c}}\tan\dfrac{ax}{2}\right), & b^2 > c^2 \\[2ex] \dfrac{1}{a\sqrt{c^2-b^2}}\ln\left(\dfrac{\tan\dfrac{ax}{2}+\sqrt{\dfrac{c+b}{c-b}}}{\tan\dfrac{ax}{2}-\sqrt{\dfrac{c+b}{c-b}}}\right), & b^2 < c^2 \end{cases}$$

$$\int \frac{dx}{(c+d\cos ax)^2} = -\frac{d\sin ax}{a(c^2-d^2)(c+d\cos ax)} + \frac{c}{c^2-d^2}\int\frac{dx}{c+d\cos ax}, \qquad c \neq \pm d$$

$$\int \frac{dx}{c^2+d^2\cos^2 ax} = \frac{1}{ac\sqrt{c^2+d^2}}\tan^{-1}\frac{c\tan ax}{\sqrt{c^2+d^2}}, \qquad c > 0$$

$$\int \frac{dx}{c^2-d^2\cos^2 ax} = \begin{cases} \dfrac{1}{ac\sqrt{c^2-d^2}}\tan^{-1}\dfrac{c\tan ax}{\sqrt{c^2-d^2}} & c^2 > d^2 \\[2ex] \dfrac{1}{2ac\sqrt{d^2-c^2}}\ln\left|\dfrac{c\tan ax-\sqrt{d^2-c^2}}{c\tan ax+\sqrt{d^2-c^2}}\right| & c^2 < d^2 \end{cases}$$

$$\int \cos(\ln x)\,dx = \frac{x}{2}[\sin(\ln x)+\cos(\ln x)]$$

$$\int x\cos(x^2)\,dx = -\frac{1}{2}\sin(x^2)$$

$$\int x^3\cos(x^2)\,dx = -\frac{x^2}{2}\sin(x^2) + \frac{1}{2}\cos(x^2)$$

$$\int x^m \cos ax dx = \frac{x^m \sin ax}{a} + \frac{mx^{m-1}}{a^2}\cos ax - \frac{m(m-1)}{a^2}\int x^{m-2}\cos ax dx$$

$$\int \frac{\cos ax}{x^n}dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int \frac{\sin ax}{x^{n-1}}dx$$

$$\int \cos^n ax dx = \frac{\sin ax \cos^{n-1} ax}{an} + \frac{n-1}{n}\int \cos^{n-2} ax dx$$

$$\int \frac{dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} ax}$$

$$\int \frac{x dx}{\cos^n ax} = \frac{x\sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\cos^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x dx}{\cos^{n-2} ax}$$

Με sin *ax* και cos *ax*

$$\int \sin ax \cos ax dx = \frac{\sin^2 ax}{2a}$$

$$\int \sin ax \cos bx\, dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}, \qquad a \neq \pm b$$

$$\int \sin^n ax \cos ax\, dx = \frac{\sin^{n+1} ax}{(n+1)a}, \qquad n \neq -1$$

$$\int \sin ax \cos^n ax\, dx = -\frac{\cos^{n+1} ax}{(n+1)a}, \qquad n \neq -1$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

$$\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a}\ln|\tan ax|$$

$$\int \frac{dx}{\sin^2 ax \cos ax} = \frac{1}{a}\ln\left|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right| - \frac{1}{a\sin ax}$$

$$\int \frac{dx}{\sin ax \cos^2 ax} = \frac{1}{a}\ln\left|\tan\frac{ax}{2}\right| + \frac{1}{a\cos ax}$$

$$\int \frac{dx}{\sin^2 ax \cos^2 ax} = -\frac{2\cot 2ax}{a}$$

$$\int \frac{\sin^2 ax}{\cos ax} dx = -\frac{\sin ax}{a} + \frac{1}{a} \ln\left|\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right|$$

$$\int \frac{\cos^2 ax}{\sin ax} dx = \frac{\cos ax}{a} + \frac{1}{a} \ln\left|\tan\frac{ax}{2}\right|$$

$$\int \frac{\sin ax}{\cos^2 ax} = \frac{1}{a\cos ax}$$

$$\int \frac{\cos ax}{\sin^2 ax} = -\frac{1}{a\sin ax}$$

$$\int \frac{dx}{\cos ax(1 \pm \sin ax)} = \mp \frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a} \ln\left|\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right|$$

$$\int \frac{dx}{\sin ax(1 \pm \cos ax)} = \pm \frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a} \ln\left|\tan\frac{ax}{2}\right|$$

$$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}} \ln\left|\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)\right|$$

$$\int \frac{dx}{1 + \sin ax + \cos ax} = \frac{1}{a} \ln\left|1 + \tan\frac{ax}{2}\right|$$

$$\int \frac{dx}{1 + \sin ax - \cos ax} = \frac{1}{a} \ln\left|\frac{\tan\frac{ax}{2}}{1 + \tan\frac{ax}{2}}\right|$$

$$\int \frac{\sin ax\,dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \frac{1}{2a} \ln|\sin ax \pm \cos ax|$$

$$\int \frac{\cos ax\,dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a} \ln|\sin ax \pm \cos ax|$$

$$\int \frac{dx}{(\sin ax \pm \cos ax)^2} = \frac{1}{2a} \tan\left(ax \mp \frac{\pi}{4}\right)$$

$$\int \frac{\sin ax dx}{p+q\cos ax} = -\frac{1}{aq}\ln|p+q\cos ax|$$

$$\int \frac{\cos ax\, dx}{b+c\sin ax} = \frac{1}{ac}\ln|b+c\sin ax|$$

$$\int \frac{\sin ax\, dx}{(b+c\cos ax)^n} = \frac{1}{ac(n-1)(b+c\cos ax)^{n-1}}$$

$$\int \frac{\cos ax dx}{(b+c\sin ax)^n} = -\frac{1}{ac(n-1)(b+c\sin ax)^{n-1}}$$

$$\int \frac{dx}{b\sin ax + c\cos ax} = \frac{1}{a\sqrt{b^2+c^2}}\ln\left|\tan\left(\frac{ax+\tan^{-1}(c/b)}{2}\right)\right|$$

$$\int \frac{\sin x\, dx}{a\sin x + b\cos x} = \frac{1}{a^2+b^2}\left[ax - b\ln\left|\sin\left(x+\tan^{-1}\frac{b}{a}\right)\right|\right]$$

$$\int \frac{\cos x\, dx}{a\sin x + b\cos x} = \frac{1}{a^2+b^2}\left[bx + a\ln\left|\sin\left(x+\tan^{-1}\frac{b}{a}\right)\right|\right]$$

$$\int \frac{dx}{b\sin ax + c\cos ax + d} = \begin{cases} \dfrac{2}{a\sqrt{D}}\tan^{-1}\left(\dfrac{b+(d-c)\tan(ax/2)}{\sqrt{D}}\right), & c \neq d,\ D>0 \\ \dfrac{1}{a\sqrt{-D}}\ln\left|\dfrac{b-\sqrt{-D}+(d-c)\tan(ax/2)}{b+\sqrt{-D}+(d-c)\tan(ax/2)}\right|, & c\neq d,\ D<0 \\ \dfrac{1}{ab}\ln\left|c+b\tan\dfrac{ax}{2}\right|, & c=d,\ b\neq 0 \\ \dfrac{-2}{a}\left[b+(d-c)\tan\dfrac{ax}{2}\right]^{-1}, & D=0 \end{cases}$$

$$\left[D = d^2 - b^2 - c^2\right]$$

$$\int \frac{dx}{b^2\sin^2 ax + c^2\cos^2 ax} = \frac{1}{abc}\tan^{-1}\left(\frac{b\tan ax}{c}\right)$$

$$\int \frac{dx}{b^2\sin^2 ax - c^2\cos^2 ax} = \frac{1}{2abc}\ln\left|\frac{b\tan ax - c}{b\tan ax + c}\right|$$

$$\int \sin^m ax \cos^n ax\,dx = \begin{cases} -\dfrac{\sin^{m-1} ax \cos^{n+1} ax}{a(m+n)} + \dfrac{m-1}{m+n}\displaystyle\int \sin^{m-2} ax \cos^n ax\,dx \\ \dfrac{\sin^{m+1} ax \cos^{n-1} ax}{a(m+n)} + \dfrac{n-1}{m+n}\displaystyle\int \sin^{m} ax \cos^{n-2} ax\,dx \end{cases}$$

$$\int \frac{\sin^m ax}{\cos^n ax}dx = \begin{cases} \dfrac{\sin^{m-1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^{n-2} ax}dx, & n \neq 1 \\ \dfrac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\sin^{m} ax}{\cos^{n-2} ax}dx, & n \neq 1 \\ \dfrac{-\sin^{m-1} ax}{a(m-n)\cos^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^{n} ax}dx, & m \neq n \end{cases}$$

$$\int \frac{\cos^m ax}{\sin^n ax}dx = \begin{cases} -\dfrac{\cos^{m-1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^{n-2} ax}dx, & n \neq 1 \\ -\dfrac{\cos^{m+1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\cos^{m} ax}{\sin^{n-2} ax}dx, & n \neq 1 \\ \dfrac{\cos^{m-1} ax}{a(m-n)\sin^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^{n} ax}dx, & m \neq n \end{cases}$$

$$\int \frac{dx}{\sin^m ax \cos^n ax} = \begin{cases} \dfrac{1}{a(n-1)\sin^{m-1} ax \cos^{n-1} ax} + \dfrac{m+n-2}{n-1}\displaystyle\int \frac{dx}{\sin^m ax \cos^{n-2} ax}, & n > 1 \\ \dfrac{-1}{a(m-1)\sin^{m-1} ax \cos^{n-1} ax} + \dfrac{m+n-2}{m-1}\displaystyle\int \frac{dx}{\sin^{m-2} ax \cos^{n} ax}, & m > 1 \end{cases}$$

Με ρίζες ($K = \sqrt{1-k^2\sin^2 x}$)

$$\int K\cos x\,dx = \frac{\sin x}{2}K + \frac{1}{2k}\sin^{-1}(k\sin x)$$

$$\int \frac{K}{\cos x}dx = \frac{\sqrt{1-k^2}}{2}\ln\frac{K+\sqrt{1-k^2}\sin x}{K-\sqrt{1-k^2}\sin x} + k\sin^{-1}(k\sin x)$$

$$\int K\sin x\cos x\,dx = -\frac{1}{3k^2}K^3$$

$$\int \frac{K}{\sin x \cos x} dx = \frac{1}{2} \ln \frac{1-K}{1+K} + \frac{\sqrt{1-k^2}}{2} \ln \frac{K+\sqrt{1-k^2}}{K-\sqrt{1-k^2}}$$

$$\int \frac{\cos x}{K} dx = \frac{1}{k} \tan^{-1} \frac{k \sin x}{K} = \frac{1}{k} \sin^{-1}(k \sin x)$$

$$\int \frac{dx}{K \cos x} = \frac{-1}{2\sqrt{1-k^2}} \ln \frac{K-\sqrt{1-k^2}\,\sin x}{K+\sqrt{1-k^2}\,\sin x}$$

$$\int \frac{\sin x \cos x}{K} dx = -\frac{1}{k^2} K$$

$$\int \frac{dx}{K \sin x \cos x} = \frac{1}{2} \ln \frac{1-K}{1+K} + \frac{1}{2\sqrt{1-k^2}} \ln \frac{K+\sqrt{1-k^2}}{K-\sqrt{1-k^2}}$$

$$\int \frac{\cos x\, dx}{K^3} = \frac{\sin x}{K}$$

$$\int \frac{\sin x \cos x}{K^3} dx = \frac{1}{k^2} K$$

$$\int (\cos x)\sqrt{1+k^2 \sin^2 x}\, dx = \frac{\sin x}{2} \sqrt{1+k^2 \sin^2 x} + \frac{1}{2k} \ln\left(k \sin x + \sqrt{1+k^2 \sin^2 x}\right)$$

$$\int \frac{\cos x\, dx}{\sqrt{1+k^2 \sin^2 x}} = \frac{1}{k} \ln\left(k \sin x + \sqrt{1+k^2 \sin^2 x}\right)$$

$$\int \frac{dx}{(\cos x)\sqrt{1+k^2 \sin^2 x}} = \frac{1}{2\sqrt{1+k^2}} \ln \frac{\sqrt{1+k^2 \sin^2 x} + \sqrt{1+k^2}\,\sin x}{\sqrt{1+k^2 \sin^2 x} - \sqrt{1+k^2}\,\sin x}$$

Με $\tan ax$

$$\int \tan ax\, dx = -\frac{1}{a} \ln|\cos ax|$$

$$\int \tan^2 ax\, dx = \frac{\tan ax}{a} - x$$

$$\int \tan^3 ax\, dx = \frac{\tan^2 ax}{2a} + \frac{1}{a} \ln|\cos ax|$$

$$\int \frac{dx}{\tan ax \cos^2 ax} = \frac{1}{a}\ln|\tan ax|$$

$$\int \frac{dx}{\tan ax} = \frac{1}{a}\ln|\sin ax|$$

$$\int x \tan ax\, dx = \frac{ax^3}{3} + \frac{a^3x^5}{15} + \frac{2a^5x^7}{105} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n a^{2n-1}x^{2n+1}}{(2n+1)!} + \cdots, \qquad |x| < \frac{\pi}{2}$$

$$\int \frac{\tan ax}{x}\, dx = ax + \frac{a^3x^3}{3} + \frac{2a^5x^5}{75} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n a^{2n-1}x^{2n-1}}{(2n-1)(2n)!} + \cdots, \qquad |x| < \frac{\pi}{2}$$

$$\int x \tan^2 ax dx = \frac{x \tan ax}{a} + \frac{1}{a^2}\ln|\cos ax| - \frac{x^2}{2}$$

$$\int \frac{dx}{b + c\tan ax} = \int \frac{\cos ax\, dx}{b\cos ax + c\sin ax} = \frac{bx}{b^2+c^2} + \frac{c}{a(b^2+c^2)}\ln|c\sin ax + b\cos ax|$$

$$\int \tan^n ax\, dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \tan^{n-2} ax\, dx, \qquad n \neq 1$$

$$\int \frac{\tan^n ax}{\cos^2 ax} dx = \frac{\tan^{n+1} ax}{(n+1)a}$$

$$\int \frac{\tan x\, dx}{\sqrt{1 \pm k^2 \sin^2 x}} = \frac{1}{2\sqrt{1 \pm k^2}} \ln \frac{\sqrt{1 \pm k^2 \sin^2 x} + \sqrt{1 \pm k^2}}{\sqrt{1 \pm k^2 \sin^2 x} - \sqrt{1 \pm k^2}}$$

Mε cot *ax*

$$\int \cot ax dx = \frac{1}{a}\ln|\sin ax|$$

$$\int \cot^2 ax dx = -\frac{\cot ax}{a} - x$$

$$\int \cot^3 ax\, dx = -\frac{\cot^2 ax}{2a} - \frac{1}{a}\ln|\sin ax|$$

$$\int \frac{dx}{\sin^2 ax \cot ax} = -\frac{1}{a}\ln|\cot ax$$

$$\int \frac{dx}{\cot ax} = -\frac{1}{a}\ln|\cos ax|$$

$$\int x\cot ax\,dx = \frac{x}{a} - \frac{ax^3}{9} - \frac{a^3x^5}{225} - \cdots - \frac{2^{2n}B_n a^{2n-1}x^{2n+1}}{(2n+1)!} - \cdots$$

$$\int \frac{\cot ax}{x}dx = -\frac{1}{ax} - \frac{ax}{3} - \frac{(ax)^3}{135} - \cdots - \frac{2^{2n}B_n(ax)^{2n-1}}{(2n-1)(2n)!} - \cdots$$

$$\int x\cot^2 ax\,dx = -\frac{x\cot ax}{a} + \frac{1}{a^2}\ln|\sin ax| - \frac{x^2}{2}$$

$$\int \frac{dx}{b+c\cot ax} = \int \frac{\sin x\,dx}{b\sin x + c\cos x} = \frac{bx}{b^2+c^2} - \frac{c}{a(b^2+c^2)}\ln|b\sin ax + c\cos ax|$$

$$\int \cot^n ax dx = -\frac{\cot^{n-1} ax}{(n-1)a} - \int \cot^{n-2} ax dx$$

$$\int \frac{\cot^n ax}{\sin^2 ax}dx = -\frac{\cot^{n+1} ax}{(n+1)a}$$

$$\int \frac{\cot x\,dx}{\sqrt{1\pm k^2\sin^2 x}} = \frac{1}{2}\ln\frac{1-\sqrt{1\pm k^2\sin^2 x}}{1+\sqrt{1\pm k^2\sin^2 x}}$$

Με αντίστροφες τριγωνομετρικές συναρτήσεις

$$\int \sin^{-1}\frac{x}{a}dx = x\sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2}$$

$$\int x\sin^{-1}\frac{x}{a}dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{4}$$

$$\int x^2\sin^{-1}\frac{x}{a}dx = \frac{x^3}{3}\sin^{-1}\frac{x}{a} + \frac{(x^2+2a^2)\sqrt{a^2-x^2}}{9}$$

$$\int x^3\sin^{-1}\frac{x}{a}dx = \left(\frac{x^4}{4} - \frac{3a^4}{32}\right)\sin^{-1}\frac{x}{a} + \frac{(2x^3+3xa^2)\sqrt{a^2-x^2}}{32}$$

$$\int \frac{\sin^{-1}(x/a)}{x}dx = \frac{x}{a} + \frac{(x/a)^3}{2\cdot3\cdot3} + \frac{1\cdot3(x/a)^5}{2\cdot4\cdot5\cdot5} + \frac{1\cdot3\cdot5(x/a)^7}{2\cdot4\cdot6\cdot7\cdot7} + \cdots, \qquad |x|<a$$

$$\int \frac{\sin^{-1}(x/a)}{x^2}\,dx = -\frac{\sin^{-1}(x/a)}{x} - \frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \frac{\sin^{-1}(x/a)}{x^3}\,dx = \frac{-1}{2x^2}\sin^{-1}\frac{x}{a} - \frac{\sqrt{a^2-x^2}}{2a^2x}$$

$$\int \left(\sin^{-1}\frac{x}{a}\right)^2 dx = x\left(\sin^{-1}\frac{x}{a}\right)^2 - 2x + 2\sqrt{a^2-x^2} - 2x + 2\sqrt{a^2-x^2}\,\sin^{-1}\frac{x}{a}$$

$$\int \cos^{-1}\frac{x}{a}\,dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2}$$

$$\int x\cos^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\cos^{-1}\frac{x}{a} - \frac{x\sqrt{a^2-x^2}}{4}$$

$$\int x^2\cos^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cos^{-1}\frac{x}{a} - \frac{(x^2+a^2)\sqrt{a^2-x^2}}{9}$$

$$\int x^3\cos^{-1}\frac{x}{a}\,dx = \left(\frac{x^4}{4} - \frac{3a^4}{32}\right)\cos^{-1}\frac{x}{a} - \frac{(2x^3+3xa^2)\sqrt{a^2-x^2}}{32}$$

$$\int \frac{\cos^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln|x| - \int \frac{\sin^{-1}(x/a)}{x}\,dx$$

$$\int \frac{\cos^{-1}(x/a)}{x^2}\,dx = -\frac{\cos^{-1}(x/a)}{x} + \frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \frac{\cos^{-1}(x/a)}{x^3}\,dx = \frac{-1}{2x^2}\cos^{-1}\frac{x}{a} + \frac{\sqrt{a^2-x^2}}{2a^2x}$$

$$\int \left(\cos^{-1}\frac{x}{a}\right)^2 dx = x\left(\cos^{-1}\frac{x}{a}\right)^2 - 2x - 2\sqrt{a^2-x^2}\,\cos^{-1}\frac{x}{a}$$

$$\int \tan^{-1}\frac{x}{a}\,dx = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln(x^2+a^2)$$

$$\int x\tan^{-1}\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\tan^{-1}\frac{x}{a} - \frac{ax}{2}$$

$$\int x^2\tan^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\tan^{-1}\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(x^2+a^2)$$

$$\int x^3 \tan^{-1}\frac{x}{a}\,dx = \frac{x^4 - a^4}{4}\tan^{-1}\frac{x}{a} - \frac{ax^3}{12} + \frac{a^3x}{4}$$

$$\int \frac{\tan^{-1}(x/a)}{x}\,dx = \begin{cases} \dfrac{x}{a} - \dfrac{(x/a)^3}{3^2} + \dfrac{(x/a)^5}{5^2} - \dfrac{(x/a)^7}{7^2} + \cdots & \left|\dfrac{x}{a}\right| \le 1 \\ \dfrac{\pi(a/x)}{2|a/x|}\ln|x| + \dfrac{a}{x} - \dfrac{(a/x)^3}{3^2} + \dfrac{(a/x)^5}{5^2} - \dfrac{(a/x)^7}{7^2} + \cdots & \left|\dfrac{x}{a}\right| > 1 \end{cases}$$

$$\int \frac{\tan^{-1}(x/a)}{x^2}\,dx = -\frac{1}{x}\tan^{-1}\frac{x}{a} - \frac{1}{2a}\ln\left(\frac{x^2 + a^2}{x^2}\right)$$

$$\int \frac{\tan^{-1}(x/a)}{x^3}\,dx = -\left(\frac{1}{2x^2} + \frac{1}{2a^2}\right)\tan^{-1}\frac{x}{a} - \frac{1}{2ax}$$

$$\int \cot^{-1}\frac{x}{a}\,dx = x\cot^{-1}\frac{x}{a} + \frac{a}{2}\ln(x^2 + a^2)$$

$$\int x\cot^{-1}\frac{x}{a}\,dx = \frac{1}{2}(x^2 + a^2)\cot^{-1}\frac{x}{a} + \frac{ax}{2}$$

$$\int x^2\cot^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cot^{-1}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(x^2 + a^2)$$

$$\int x^3\cot^{-1}\frac{x}{a}\,dx = \frac{x^4 - a^4}{4}\cot^{-1}\frac{x}{a} + \frac{ax^3}{12} - \frac{a^3x}{4}$$

$$\int \frac{\cot^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln|x| - \int \frac{\tan^{-1}(x/a)}{x}\,dx$$

$$\int \frac{\cot^{-1}(x/a)}{x^2}\,dx = -\frac{\cot^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2 + a^2}{x^2}\right)$$

$$\int \frac{\cot^{-1}(x/a)}{x^3}\,dx = \frac{-1}{2x^2}\cot^{-1}\frac{x}{a} + \frac{1}{2ax} + \frac{1}{2a^2}\tan^{-1}\frac{x}{a}$$

$$\int x^n\sin^{-1}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\sin^{-1}\frac{x}{a} - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{a^2 - x^2}}\,dx \qquad n \ne -1$$

$$\int x^n\cos^{-1}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\cos^{-1}\frac{x}{a} + \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{a^2 - x^2}}\,dx \qquad n \ne -1$$

$$\int x^n \tan^{-1}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\tan^{-1}\frac{x}{a} - \frac{a}{n+1}\int \frac{x^{n+1}}{x^2+a^2}\,dx \qquad n \neq -1$$

$$\int x^n \cot^{-1}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\cot^{-1}\frac{x}{a} + \frac{a}{n+1}\int \frac{x^{n+1}}{x^2+a^2}\,dx \qquad n \neq -1$$

Mε e^{ax}

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

$$\int a^x\,dx = \int e^{x\ln a}\,dx = \frac{e^{x\ln a}}{\ln a} = \frac{a^x}{\ln a}$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$

$$\int x^2 e^{ax}\,dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right)$$

$$\int x^3 e^{ax}\,dx = \frac{e^{ax}}{a}\left(x^3 - \frac{3x^2}{a} + \frac{6x}{a^2} - \frac{6}{a^3}\right)$$

$$\int \frac{e^{ax}}{x}\,dx = \ln|x| + \frac{ax}{1\cdot 1!} + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^3}{3\cdot 3!} + \cdots$$

$$\int \frac{dx}{b+ce^{ax}} = \frac{x}{b} - \frac{1}{ab}\ln|b+ce^{ax}|$$

$$\int \frac{dx}{(b+ce^{ax})^2} = \frac{x}{b^2} + \frac{1}{ab(b+ce^{ax})} - \frac{1}{ab^2}\ln(b+ce^{ax})$$

$$\int \frac{dx}{be^{ax}+ce^{-ax}} = \begin{cases} \dfrac{1}{a\sqrt{bc}}\tan^{-1}\left(\sqrt{\dfrac{b}{c}}e^{ax}\right) & bc > 0 \\[2ex] \dfrac{1}{2a\sqrt{-bc}}\ln\left(\dfrac{e^{ax}-\sqrt{-c/b}}{e^{ax}+\sqrt{-c/b}}\right) & bc < 0 \end{cases}$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

$$\int e^{ax}\sin^2 x\,dx = \frac{e^{ax}}{a^4+4}\left(a\sin^2 x - 2\sin x\cos x + \frac{2}{a}\right)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$\int e^{ax}\cos^2 x\,dx = \frac{e^{ax}}{a^4+4}\left(a\cos^2 x + 2\sin x\cos x + \frac{2}{a}\right)$$

$$\int xe^{ax}\sin bx\,dx = \frac{xe^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} - \frac{e^{ax}\{(a^2-b^2)\sin bx - 2ab\cos bx\}}{(a^2+b^2)^2}$$

$$\int xe^{ax}\cos bx\,dx = \frac{xe^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} - \frac{e^{ax}\{(a^2-b^2)\cos bx - 2ab\sin bx\}}{(a^2+b^2)^2}$$

$$\int e^{ax}\ln x\,dx = \frac{e^{ax}\ln x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x}dx$$

$$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}dx$$

$$= \frac{e^{ax}}{a}\left(x^n - \frac{nx^{n-1}}{a} + \frac{n(n-1)x^{n-2}}{a^2} - \cdots \frac{(-1)^n n!}{a^n}\right) \qquad [n = \text{θετικός ακέραιος}]$$

$$\int \frac{e^{ax}}{x^n}dx = -\frac{e^{ax}}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{e^{ax}}{x^{n-1}}dx$$

$$\int e^{ax}\sin^n bx\,dx = \frac{e^{ax}\sin^{n-1} bx}{a^2+n^2b^2}(a\sin bx - nb\cos bx) + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\sin^{n-2} bx\,dx$$

$$\int e^{ax}\cos^n bx\,dx = \frac{e^{ax}\cos^{n-1} bx}{a^2+n^2b^2}(a\cos bx + nb\sin bx) + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\cos^{n-2} bx\,dx$$

$$\int \frac{dx}{\sqrt{b+ce^{ax}}} = \begin{cases} \dfrac{1}{a\sqrt{b}}\ln\dfrac{\sqrt{b+ce^{ax}}-\sqrt{b}}{\sqrt{b+ce^{ax}}+\sqrt{b}}, & b>0 \\[2ex] \dfrac{2}{a\sqrt{-b}}\tan^{-1}\dfrac{\sqrt{b+ce^{ax}}}{\sqrt{-b}}, & b<0 \end{cases}$$

Με ln$x$ $(x > 0)$

$$\int \ln x\,dx = x\ln x - x$$

$$\int x\ln x\,dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$$

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right)$$

$$\int \frac{\ln x}{x}dx = \frac{1}{2}\ln^2 x$$

$$\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{2}$$

$$\int \ln^2 x\,dx = x\ln^2 x - 2x\ln x + 2x$$

$$\int \ln(x^2 + a^2)dx = x\ln(x^2 + a^2) - 2x + 2a\tan^{-1}\frac{x}{a}$$

$$\int \ln\left|x^2 - a^2\right|dx = x\ln\left|x^2 - a^2\right| - 2x + a\ln\left|\frac{x+a}{x-a}\right|$$

$$\int x\ln\left|x^2 \pm a^2\right|dx = \frac{1}{2}[(x^2 \pm a^2)\ln\left|x^2 \pm a^2\right| - x^2]$$

$$\int \frac{dx}{\ln x} = \ln(\ln x) + \ln x + \frac{\ln^2 x}{2\cdot 2!} + \frac{\ln^3 x}{3\cdot 3!} + \cdots$$

$$\int \frac{dx}{x\ln x} = \ln\left|\ln x\right|$$

$$\int \sin(\ln x)\,dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x)$$

$$\int \cos(\ln x)\,dx = \frac{1}{2}x\sin(\ln x) + \frac{1}{2}x\cos(\ln x)$$

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right), \qquad n \neq -1$$

$$\int x^n \ln^2 x = \frac{x^{n+1}}{n+1}\ln^2 x - \frac{2x^{n+1}}{(n+1)^2}\ln x + \frac{2x^{n+1}}{(n+1)^3}, \qquad n \neq -1$$

$$\int \frac{x^n\,dx}{\ln x} = \ln|\ln x| + (n+1)\ln x + \frac{(n+1)^2 \ln^2 x}{2\cdot 2!} + \frac{(n+1)^3 \ln^3 x}{3\cdot 3!} + \cdots$$

$$\int \ln^n x\,dx = x\ln^n x - n\int \ln^{n-1} x\,dx$$

$$\int \frac{\ln^n x\,dx}{x} = \frac{\ln^{n+1} x}{n+1}, \qquad n \neq -1$$

$$\int x^m \ln^n x\,dx = \frac{x^{m+1}\ln^n x}{m+1} - \frac{n}{m+1}\int x^m \ln^{n-1} x\,dx, \qquad m \neq -1,\ n \neq -1$$

$$\int x^n \ln(x^2 \pm a^2)\,dx = \frac{x^{n+1}\ln(x^2 \pm a^2)}{n+1}\ \frac{2}{n+1}\int \frac{x^{n+2}}{x^2 \pm a^2}\,dx, \qquad n \neq -1$$

$$\int \ln\left(x + \sqrt{x^2 \pm a^2}\right)dx = x\ln\left(x + \sqrt{x^2 \pm a^2}\right) - \sqrt{x^2 \pm a^2}$$

$$\int x\ln\left(x + \sqrt{x^2 \pm a^2}\right)dx = \left(\frac{x^2}{2} \pm \frac{a^2}{4}\right)\ln\left(x + \sqrt{x^2 \pm a^2}\right) - \frac{x}{4}\sqrt{x^2 \pm a^2}$$

Με sinh *ax*

$$\int \sinh ax\,dx = \frac{\cosh ax}{a}$$

$$\int x\sinh ax\,dx = \frac{x\cosh ax}{a} - \frac{\sinh ax}{a^2}$$

$$\int x^2 \sinh ax\,dx = \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\cosh ax - \frac{2x}{a^2}\sinh ax$$

$$\int \frac{\sinh ax}{x}\,dx = ax + \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} + \frac{(ax)^7}{7\cdot 7!} + \cdots$$

$$\int \frac{\sinh ax}{x^2}\,dx = -\frac{\sinh ax}{x} + a\int \frac{\cosh ax}{x}\,dx$$

$$\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\left|\tanh\frac{ax}{2}\right|$$

$$\int \frac{x\,dx}{\sinh ax} = \frac{x}{a} - \frac{ax^3}{3\cdot 3!} + \frac{7a^3x^5}{3\cdot 5\cdot 5!} - \cdots + \frac{2(-1)^n(2^{2n}-1)B_n a^{2n-1}x^{2n+1}}{(2n+1)!} + \cdots$$

$$\int \sinh^2 ax\,dx = \frac{\sinh 2ax}{4a} - \frac{x}{2}$$

$$\int x\sinh^2 ax\,dx = \frac{x\sinh 2ax}{4a} - \frac{\cos 2ax}{8a^2} - \frac{x^2}{4}$$

$$\int \frac{dx}{\sinh^2 ax} = -\frac{\coth ax}{a}$$

$$\int \frac{x\,dx}{\sinh^2 ax} = -\frac{x}{a}\coth ax + \frac{1}{a^2}\ln|\sinh ax|$$

$$\int \sinh ax \sinh bx\,dx = \frac{\sinh(a+b)x}{2(a+b)} - \frac{\sinh(a-b)x}{2(a-b)}, \qquad a \neq \pm b$$

$$\int \sinh ax \sin bx\,dx = \frac{1}{a^2+b^2}(a\cosh ax \sin bx - b\sinh ax \cos bx)$$

$$\int \sinh ax \cos bx\,dx = \frac{1}{a^2+b^2}(a\cosh ax \cos bx + b\sinh ax \sin bx)$$

$$\int \frac{dx}{b+c\sinh ax} = \frac{1}{a\sqrt{b^2+c^2}}\ln\left|\frac{ce^{ax}+b-\sqrt{b^2+c^2}}{ce^{ax}+b+\sqrt{b^2+c^2}}\right|$$

$$\int \frac{dx}{(b+c\sinh ax)^2} = -\frac{c\cosh ax}{a(b^2+c^2)(b+c\sinh ax)} + \frac{b}{b^2+c^2}\int \frac{dx}{b+c\sinh ax}$$

$$\int \frac{dx}{b^2+c^2\sinh^2 ax} = \begin{cases} \dfrac{1}{ab\sqrt{c^2-b^2}}\tan^{-1}\dfrac{\sqrt{c^2-b^2}\tanh ax}{b} & b^2 < c^2 \\[2ex] \dfrac{1}{ab\sqrt{b^2-c^2}}\ln\left(\dfrac{b+\sqrt{b^2-c^2}\tanh ax}{b-\sqrt{b^2-c^2}\tanh ax}\right) & b^2 > c^2 \end{cases}$$

$$\int \frac{dx}{b^2 - c^2 \sinh^2 ax} = \frac{1}{2ab\sqrt{b^2+c^2}} \ln \left| \frac{b + \sqrt{b^2+c^2} \tanh ax}{b - \sqrt{b^2+c^2} \tanh ax} \right|$$

$$\int x^n \sinh ax\, dx = \frac{x^n \cosh ax}{a} - \frac{n}{a} \int x^{n-1} \cosh ax\, dx$$

$$\int \sinh^n ax\, dx = \frac{\sinh^{n-1} ax \cosh ax}{an} - \frac{n-1}{n} \int \sinh^{n-2} ax\, dx$$

$$\int \frac{\sinh ax}{x^n} dx = -\frac{\sinh ax}{(n-1)x^{n-1}} + \frac{a}{n-1} \int \frac{\cosh ax}{x^{n-1}} dx$$

$$\int \frac{dx}{\sinh^n ax} = -\frac{\cosh ax}{a(n-1)\sinh^{n-1} ax} - \frac{n-2}{n-1} \int \frac{dx}{\sinh^{n-2} ax}, \qquad n > 1$$

$$\int \frac{x dx}{\sinh^n ax} = -\frac{x \cosh ax}{a(n-1)\sinh^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sinh^{n-2} ax} - \frac{n-2}{n-1} \int \frac{x dx}{\sinh^{n-2} ax}$$

Mε cosh *ax*

$$\int \cosh ax dx = \frac{\sinh ax}{x}$$

$$\int x \cosh ax dx = \frac{x \sinh ax}{a} - \frac{\cosh ax}{a^2}$$

$$\int x^2 \cosh ax dx = -\frac{2x \cosh ax}{a^2} + \left(\frac{x^2}{a} + \frac{2}{a^3} \right) \sinh ax$$

$$\int \frac{\cosh ax}{x} dx = \ln x + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^4}{4 \cdot 4!} + \frac{(ax)^6}{6 \cdot 6!} + \cdots$$

$$\int \frac{\cosh ax}{x^2} dx = -\frac{\cosh ax}{x} + a \int \frac{\sinh ax}{x} dx$$

$$\int \frac{dx}{\cosh ax} = \frac{2}{a} \tan^{-1} e^{ax}$$

$$\int \frac{x\, dx}{\cosh ax} = \frac{x^2}{2} - \frac{a^2 x^4}{4 \cdot 2!} + \frac{5a^4 x^6}{6 \cdot 4!} - \frac{61 a^4 x^6}{8 \cdot 6!} + \cdots + \frac{(-1)^n E_n a^{2n} x^{2n+2}}{(2n+2)(2n)!} + \cdots \qquad |x| < \frac{\pi}{2}$$

$$\int \cosh^2 ax\,dx = \frac{x}{2} + \frac{\sinh 2ax}{4}$$

$$\int x\cosh^2 ax\,dx = \frac{x\sinh 2ax}{4a} - \frac{\cosh 2ax}{8a^2} + \frac{x^2}{4}$$

$$\int \frac{dx}{\cosh^2 ax} = \frac{\tanh ax}{a}$$

$$\int \frac{x\,dx}{\cosh^2 ax} = \frac{x}{a}\tanh ax - \frac{1}{a^2}\ln(\cosh ax)$$

$$\int \cosh ax \cosh bx\,dx = \frac{\sinh(a-b)x}{2(a-b)} + \frac{\sinh(a+b)x}{2(a+b)}, \qquad a \neq \pm b$$

$$\int \cosh ax \sin bx\,dx = \frac{a\sinh ax \sin bx - b\cosh ax \cos bx}{a^2+b^2}$$

$$\int \cosh ax \cos bx\,dx = \frac{a\sinh ax \cos bx + b\cosh ax \sin bx}{a^2+b^2}$$

$$\int \frac{dx}{\cosh ax + 1} = \frac{1}{a}\tanh\frac{ax}{2}$$

$$\int \frac{dx}{\cosh ax - 1} = -\frac{1}{a}\coth\frac{ax}{2}$$

$$\int \frac{x\,dx}{\cosh ax + 1} = \frac{x}{a}\tanh\frac{ax}{2} - \frac{2}{a^2}\ln\left(\cosh\frac{ax}{2}\right)$$

$$\int \frac{x\,dx}{\cosh ax - 1} = -\frac{x}{a}\coth\frac{ax}{2} + \frac{2}{a^2}\ln\left(\sinh\frac{ax}{2}\right)$$

$$\int \frac{dx}{(\cosh ax + 1)^2} = \frac{1}{2a}\tanh\frac{ax}{2} - \frac{1}{6a}\tanh^3\frac{ax}{2}$$

$$\int \frac{dx}{(\cosh ax - 1)^2} = \frac{1}{2a}\coth\frac{ax}{2} - \frac{1}{6a}\coth^3\frac{ax}{2}$$

$$\int \frac{dx}{b+c\cosh ax} = \begin{cases} \dfrac{1}{a\sqrt{b^2-c^2}} \ln\left(\dfrac{ce^{ax}+b-\sqrt{b^2-c^2}}{ce^{ax}+b+\sqrt{b^2-c^2}} \right) & b^2 > c^2 \\ \dfrac{2}{a\sqrt{c^2-b^2}} \tan^{-1} \dfrac{ce^{ax}+b}{\sqrt{c^2-b^2}} & b^2 < c^2 \end{cases}$$

$$\int \frac{dx}{b^2+c^2\cosh^2 ax} = \begin{cases} \dfrac{1}{2ab\sqrt{b^2+c^2}} \ln\left| \dfrac{b\tanh ax+\sqrt{b^2+c^2}}{b\tanh ax-\sqrt{b^2+c^2}} \right| \\ \dfrac{1}{ab\sqrt{b^2+c^2}} \tanh^{-1} \dfrac{b\tanh ax}{\sqrt{b^2+c^2}} \end{cases}$$

$$\int \frac{dx}{b^2-c^2\cosh^2 ax} = \begin{cases} \dfrac{1}{2ab\sqrt{b^2-c^2}} \ln\left| \dfrac{b\tanh ax+\sqrt{b^2-c^2}}{b\tanh ax-\sqrt{b^2-c^2}} \right| & b^2 > c^2 \\ \dfrac{-1}{ab\sqrt{c^2-b^2}} \tan^{-1} \dfrac{b\tanh ax}{\sqrt{c^2-b^2}} & b^2 < c^2 \end{cases}$$

$$\int x^n \cosh ax\, dx = \frac{x^n \sinh ax}{a} - \frac{n}{a} \int x^{n-1} \sinh ax\, dx$$

$$\int \cosh^n ax\, dx = \frac{\cosh^{n-1} ax \sinh ax}{an} + \frac{n-1}{n} \int \cosh^{n-2} ax\, dx$$

$$\int \frac{\cosh ax}{x^n} dx = -\frac{\cosh ax}{(n-1)x^{n-1}} + \frac{a}{n-1} \int \frac{\sinh ax}{x^{n-1}} dx$$

$$\int \frac{dx}{\cosh^n ax} = \frac{\sinh ax}{a(n-1)\cosh^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cosh^{n-2} ax},$$

$$\int \frac{x\,dx}{\cosh^n ax} = \frac{x \sinh ax}{a(n-1)\cosh^n ax} + \frac{1}{(n-1)(n-2)a^2 \cosh^{n-2} ax} + \frac{n-2}{n-1} \int \frac{x\,dx}{\cosh^{n-2} ax}$$

Με sinh *ax* και cosh *ax*

$$\int \sinh ax \cosh ax\, dx = \frac{\sinh^2 ax}{2a}$$

$$\int \sinh ax \cosh bx\, dx = \frac{\cosh(a+b)x}{2(a+b)} + \frac{\cosh(a-b)x}{2(a-b)}, \qquad a \neq \pm b$$

$$\int \sinh^2 ax \cosh^2 ax\, dx = \frac{\sinh 4ax}{32a} - \frac{x}{8}$$

$$\int \frac{dx}{\sinh ax \cosh ax} = \frac{1}{a} \ln|\tanh ax|$$

$$\int \frac{dx}{\sinh^2 ax \cosh ax} = -\frac{1}{a} \tan^{-1}(\sinh ax) - \frac{1}{a \sinh a}$$

$$\int \frac{dx}{\sinh ax \cosh^2 ax} = \frac{1}{a \cosh ax} + \frac{1}{a} \ln\left|\tanh \frac{ax}{2}\right|$$

$$\int \frac{dx}{\sinh^2 ax \cosh^2 ax} = -\frac{2 \coth 2ax}{a}$$

$$\int \frac{\sinh^2 ax}{\cosh ax} dx = \frac{\sinh ax}{a} - \frac{1}{a} \tanh^{-1}(\sinh ax)$$

$$\int \frac{\cosh^2 ax}{\sinh ax} dx = \frac{\cosh ax}{a} + \frac{1}{a} \ln\left|\tanh \frac{ax}{2}\right|$$

$$\int \frac{\sinh ax}{\cosh^2 ax} dx = \frac{-1}{a \cosh ax}$$

$$\int \frac{\cosh ax}{\sinh^2 ax} dx = \frac{-1}{a \sinh ax}$$

$$\int \frac{dx}{\sinh ax(\cos ax + 1)} = \frac{1}{2a} \ln\left|\tanh \frac{ax}{2}\right| + \frac{1}{2a(\cosh ax + 1)}$$

$$\int \frac{dx}{\sinh ax(\cos ax - 1)} = -\frac{1}{2a} \ln\left|\tanh \frac{ax}{2}\right| - \frac{1}{2a(\cosh ax - 1)}$$

$$\int \frac{dx}{\cosh ax(1 + \sinh ax)} = \frac{1}{2a} \ln\left|\frac{1 + \sinh ax}{\cosh ax}\right| + \frac{1}{a} \tan^{-1} e^{ax}$$

$$\int \sinh^n ax \cosh ax\, dx = \frac{\sinh^{n+1} ax}{(n+1)a}, \qquad n \neq -1$$

$$\int \cosh^n ax \sinh ax\, dx = \frac{\cosh^{n+1} ax}{(n+1)a}, \qquad n \neq -1$$

$$\int \frac{dx}{\sinh ax \cosh^n ax} = \frac{1}{a(n-1)\cosh^{n-1} ax} + \int \frac{dx}{\sinh ax \cosh^{n-2} ax}, \qquad n \neq 1$$

$$\int \frac{dx}{\sinh^n ax \cosh ax} = \frac{1}{a(n-1)\sinh^{n-1} ax} - \int \frac{dx}{\sinh^{n-2} ax \cosh ax}, \qquad n \neq 1$$

Με tanh *ax*

$$\int \tanh ax \, dx = \frac{1}{a}\ln(\cosh ax)$$

$$\int \tanh^2 ax \, dx = x - \frac{\tanh ax}{a}$$

$$\int \tanh^3 ax \, dx = \frac{1}{a}\ln(\cosh ax) - \frac{\tanh^2 a}{2a}$$

$$\int \frac{1}{\cosh^2 ax \tanh ax} dx = \frac{1}{a}\ln|\tanh ax|$$

$$\int \frac{dx}{\tanh ax} = \frac{1}{a}\ln|\sinh ax|$$

$$\int x \tanh ax \, dx = \frac{ax^3}{3} - \frac{a^3x^5}{15} + \frac{2a^5x^7}{105} - \cdots + \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n a^{2n-1}x^{2n+1}}{(2n+1)!} + \cdots$$

$$\int x \tanh^2 ax \, dx = \frac{x^2}{2} - \frac{x \tanh ax}{a} + \frac{1}{a^2}\ln(\cosh ax)$$

$$\int \frac{\tanh ax}{x} dx = ax - \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} - \cdots + \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n (ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

$$\int \frac{dx}{b + c\tanh ax} = \frac{bx}{b^2 - c^2} - \frac{c}{a(b^2 - c^2)}\ln(c\sinh ax + b\cosh ax)$$

$$\int \tanh^n ax \, dx = -\frac{\tanh^{n-1} ax}{a(n-1)} + \int \tanh^{n-2} ax \, dx$$

$$\int \frac{\tanh^n ax}{\cosh^2 ax} dx = \frac{\tanh^{n+1} ax}{(n+1)a}$$

Με coth *ax*

$$\int \coth ax\,dx = \frac{1}{a}\ln|\sinh ax|$$

$$\int \coth^2 ax\,dx = x - \frac{\coth ax}{a}$$

$$\int \coth^3 ax\,dx = \frac{1}{a}\ln|\sinh ax| - \frac{\coth^2 ax}{2a}$$

$$\int \frac{1}{\sinh^2 ax \coth ax}\,dx = -\frac{1}{a}\ln(\coth ax$$

$$\int \frac{dx}{\coth ax} = \frac{1}{a}\ln(\cosh ax)$$

$$\int x\coth ax\,dx = \frac{x}{a} + \frac{ax^3}{9} - \frac{a^3x^5}{225} + \cdots + \frac{(-1)^{n-1}2^{2n}B_n(ax)^{2n+1}}{(2n+1)!} + \cdots$$

$$\int x\coth^2 ax\,dx = \frac{x^2}{2} - \frac{x\coth ax}{a} + \frac{1}{a^2}\ln|\sinh ax$$

$$\int \frac{\coth ax}{x}\,dx = -\frac{1}{ax} + \frac{ax}{3} - \frac{(ax)^3}{135} + \cdots + \frac{(-1)^n 2^{2n} B_n (ax)^{2n-1}}{(2n-1)(2n)!} + \cdot$$

$$\int \frac{dx}{b + c\coth ax} = \frac{bx}{b^2 - c^2} - \frac{c}{a(b^2-c^2)}\ln|b\sinh ax + c\cosh ax|$$

$$\int \coth^n ax\,dx = -\frac{\coth^{n-1} ax}{a(n-1)} + \int \coth^{n-1} ax\,dx$$

$$\int \frac{\coth^n ax}{\sinh^2 ax}\,dx = -\frac{\coth^{n+1} ax}{(n+1)a}$$

Με αντίστροφες υπερβολικές συναρτήσεις

$$\int \sinh^{-1}\frac{x}{a}\,dx = x\sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2}$$

$$\int x\sinh^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2} + \frac{a^2}{4}\right)\sinh^{-1}\frac{x}{a} - \frac{x\sqrt{x^2+a^2}}{4}$$

$$\int x^2 \sinh^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\sinh^{-1}\frac{x}{a} + \frac{(2a^2 - x^2)\sqrt{x^2+a^2}}{9}$$

$$\int \frac{\sinh^{-1}(x/a)}{x}\,dx = \begin{cases} \dfrac{x}{a} - \dfrac{(x/a)^3}{2\cdot 3\cdot 3} + \dfrac{1\cdot 3(x/a)^5}{2\cdot 4\cdot 5^2} - \dfrac{1\cdot 3\cdot 5(x/a)^7}{2\cdot 4\cdot 6\cdot 7^2} + \cdots & \left|\dfrac{x}{a}\right| < 1 \\ \dfrac{\ln^2(2x/a)}{2} - \dfrac{(a/x)^2}{2^3} + \dfrac{1\cdot 3(a/x)^4}{2\cdot 4^3} - \dfrac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6^3} + \cdots & \dfrac{x}{a} > 1 \\ -\dfrac{\ln^2(-2x/a)}{2} + \dfrac{(a/x)^2}{2^3} - \dfrac{1\cdot 3\cdot 5\cdot (a/x)^4}{2\cdot 4^3} + \dfrac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6^3} - \cdots & \dfrac{x}{a} < -1 \end{cases}$$

$$\int \frac{\sinh^{-1}(x/a)}{x^2}\,dx = -\frac{\sinh^{-1}(x/a)}{x} - \frac{1}{a}\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right|$$

$$\int \cosh^{-1}\frac{x}{a}\,dx = \begin{cases} x\cosh^{-1}(x/a) - \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ x\cosh^{-1}(x/a) + \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

$$\int x\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{1}{4}(2x^2-a^2)\cosh^{-1}(x/a) - \dfrac{1}{4}x\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ \dfrac{1}{4}(2x^2-a^2)\cosh^{-1}(x/a) + \dfrac{1}{4}x\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

$$\int x^2\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{1}{3}x^3\cosh^{-1}(x/a) - \dfrac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ \dfrac{1}{3}x^3\cosh^{-1}(x/a) + \dfrac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

$$\int \frac{\cosh^{-1}(x/a)}{x}\,dx = \pm\left[\frac{1}{2}\ln^2(2x/a) + \frac{(a/x)^2}{2^3} + \frac{1\cdot 3(a/x)^4}{2\cdot 4^3} + \frac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6^3} + \cdots\right]$$

[+ αν $\cosh^{-1}(x/a) > 0$, – αν $\cosh^{-1}(x/a) < 0$]

$$\int \frac{\cosh^{-1}(x/a)}{x^2}\,dx = -\frac{\cosh^{-1}(x/a)}{x} \mp a\tan^{-1}\left(\frac{1}{\sqrt{(x/a)^2-1}}\right)$$

[– αν $\cosh^{-1}(x/a) > 0$, + αν $\cosh^{-1}(x/a) < 0$]

$$\int \tanh^{-1}\frac{x}{a}\,dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\ln(a^2 - x^2)$$

$$\int x\tanh^{-1}\frac{x}{a}\,dx=\frac{ax}{2}+\frac{1}{2}(x^2-a^2)\tanh^{-1}\frac{x}{a}$$

$$\int x^2\tanh^{-1}\frac{x}{a}\,dx=\frac{ax^2}{6}+\frac{x^3}{3}+\frac{x^3}{3}\tanh^{-1}\frac{x}{a}+\frac{a^3}{6}\ln(a^2-x^2)$$

$$\int\frac{\tanh^{-1}(x/a)}{x}\,dx=\frac{x}{a}+\frac{(x/a)^3}{3^2}+\frac{(x/a)^5}{5^2}+\frac{(x/a)^7}{7^2}+\cdots$$

$$\int\frac{\tanh^{-1}(x/a)}{x^2}\,dx=-\frac{\tanh^{-1}(x/a)}{x}+\frac{1}{2a}\ln\left(\frac{x^2}{a^2-x^2}\right)$$

$$\int\coth^{-1}\frac{x}{a}\,dx=x\coth^{-1}x+\frac{a}{2}\ln(x^2-a^2)$$

$$\int x\coth^{-1}\frac{x}{a}\,dx=\frac{ax}{2}+\frac{1}{2}(x^2-a^2)\coth^{-1}\frac{x}{a}$$

$$\int x^2\coth^{-1}\frac{x}{a}\,dx=\frac{ax^2}{6}+\frac{x^3}{3}\coth^{-1}\frac{x}{a}+\frac{a^3}{6}\ln(x^2-a^2)$$

$$\int\frac{\coth^{-1}(x/a)}{x}\,dx=-\frac{a}{x}-\frac{(a/x)^3}{3^2}-\frac{(a/x)^5}{5^2}-\cdots$$

$$\int\frac{\coth^{-1}(x/a)}{x^2}\,dx=-\frac{\coth^{-1}(x/a)}{x}+\frac{1}{2a}\ln\left(\frac{x^2}{x^2-a^2}\right)$$

$$\int x^n\sinh^{-1}\frac{x}{a}\,dx=\frac{x^{n+1}}{n+1}\sinh^{-1}\frac{x}{a}-\frac{1}{n+1}\int\frac{x^{n+1}}{\sqrt{x^2+a^2}}\,dx,\qquad n\neq-1$$

$$\int x^n\cosh^{-1}\frac{x}{a}\,dx=\begin{cases}\dfrac{x^{n+1}}{n+1}\cosh^{-1}\dfrac{x}{a}-\dfrac{1}{n+1}\displaystyle\int\frac{x^{n+1}}{\sqrt{x^2-a^2}}\,dx, & \cosh^{-1}(x/a)>0,\ n\neq-1\\[2ex] \dfrac{x^{n+1}}{n+1}\cosh^{-1}\dfrac{x}{a}+\dfrac{1}{n+1}\displaystyle\int\frac{x^{n+1}}{\sqrt{x^2-a^2}}\,dx, & \cosh^{-1}(x/a)<0,\ n\neq-1\end{cases}$$

$$\int x^n\tanh^{-1}\frac{x}{a}\,dx=\frac{x^{n+1}}{n+1}\tanh^{-1}\frac{x}{a}-\frac{a}{n+1}\int\frac{x^{n+1}}{a^2-x^2}\,dx$$

$$\int x^n\coth^{-1}\frac{x}{a}\,dx=\frac{x^{n+1}}{n+1}\coth^{-1}\frac{x}{a}-\frac{a}{n+1}\int\frac{x^{n+1}}{a^2-x^2}\,dx,\qquad n\neq-1$$