7. DIFFERENTIATION

Objectives

To understand that the gradient of a slope is the rate of change and that the gradient of a curve constantly changes.

To be able to perform basic differentiation to find a formula for the gradient of a curve.

To be able to find the stationary points of a function and to investigate their nature.

To become familiar with different techniques of differentiation.

7.1 Gradients

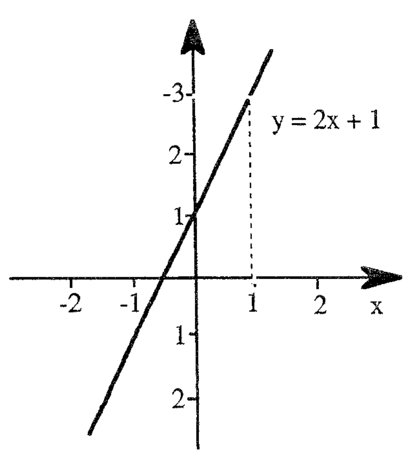
The **gradient** of a function is the rate of change of y with respect to x. Consider the straight line equation.

$$y = mx + c$$

m is the gradient, or slope of the line. For a straight line this is constant at every point along the line.

eg y = 2x + 1 has gradient 2. This is also given by the formula $\frac{y_2 - y_1}{x_2 - x_1}$

for any two points (x_1, y_1) and (x_2, y_2) on the line. Check it to see.



The Graph of the Equation y = 2x + 1

A positive gradient means the line slopes up to the right.
A negative gradient means the line slopes up to the left.
A zero gradient means the line is horizontal.

If the function is a continuous curve, the gradient is not constant. In fact, each point on the cuve has its own gradient, so we need a formula which will give us the gradient of a particular curve at any point.

The formula for the gradient is the **derivative** of the function, d/dx. This is found in a general sense as shown below.

$$y = x^n \qquad \text{then} \qquad \frac{dy}{dx} = nx^{n-1}$$

or equivalently

$$f(x) = x^n \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

Differentiate the following equations with regard to x.

$y = x^2 - 2 + x^{-2}$ $\quad$ $f(x) = 3x^2 - 2x + 1$

....

....

[Solutions: $2x - 2/x^3$, $6x - 2$]

By substituting the co-ordinates of any point into the derivative we can find the gradient at that point.

eg $\quad y = x^2 + 1$

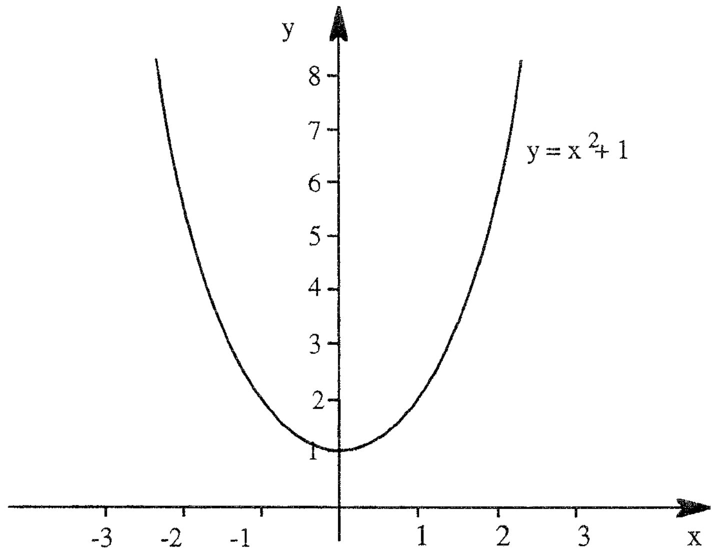
$$\frac{dy}{dx} = 2x$$

then at $\quad x = 0 \quad \frac{dy}{dx} = 0$

$x = 1 \quad \frac{dy}{dx} = 2$

$x = -1 \quad \frac{dy}{dx} = -2$

These gradients can be seen on the sketch below.



The Gradient changes as the curve changes

7.2 Limits of a Function

In calculus our main aim is to investigate the behaviour of a function around a particular point.

eg Without drawing the graph, how does the function $f(x) = \frac{x^2 - 25}{x - 5}$ behave when x = 5?

Try evaluating f(x) at x = 5. It does not work because we obtain a zero denominator. This often signifies an **assymtote** which is a limit to which the curve approaches but never touches.

We can see how the curve behaves around x = 5 below.

x	5.5	5.1	5.01	5.001	4.999	4.9	4.5
f(x)	10.5	10.1	10.01	10.001	9.999	9.9	9.5

Clearly the curve approaches or *tends towards* the value 10.

ie $f(x) \rightarrow 10$ as $x \rightarrow 5$ or $\lim_{x \rightarrow 5} f(x) = 10$

We can see this quite easily by re-writing the function as

$$f(x) = \frac{(x + 5)(x - 5)}{(x - 5)}$$

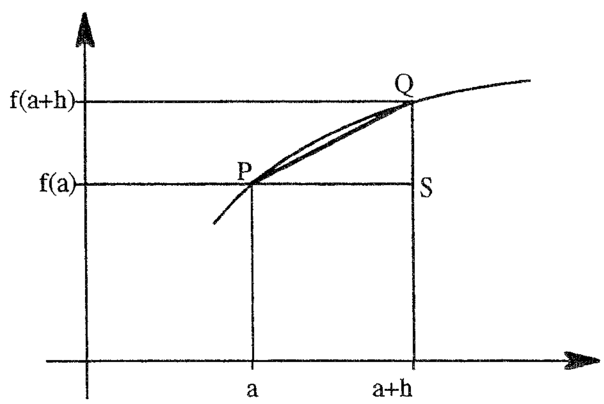
$$f(5) = 10$$

Since the denominator vanishes at $x = 5$ we say that the function is **discontinuous** at $x = 5$ and continuous everywhere else.

Differentiability

Consider a function $y = f(x)$ as represented by the diagram below, and let P be a typical point on the curve with co-ordinates (a, f(a)). The co-ordinates of a neighbouring point Q can be written (a+h, f(a+h)).

The Trigonmetric formula for differentiability



Basic trigonometry tells us that

$$\frac{f(a+h) - f(a)}{h} = \tan QPS$$

This is the slope of the line PQ and it may be regarded as the mean value of the gradient of the curve f(x).

As Q becomes closer to P and h decreases the curve PQ becomes more like the hypoteneuse PQ and so, the expression above approaches a limiting value, l say.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = l$$

If this limit exists, then we say that the function f(x) is **differentiable** and we represent the expression above by the symbol $\frac{dy}{dx}$, or $f'(x)$.

The symbol dy/dx is derived from the expression above where dx comes from δx meaning 'a little bit of x' and is used in place of h above.

eg Given that $f(x) = 2x^2 - 4$ find the limit $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\frac{f(x+h) - f(x)}{h} = \frac{2(x+h)^2 - 4 - 2x^2 + 4}{h}$$

$$= \frac{2x^2 + 4xh + 2h^2 - 2x^2}{h}$$

$$= 4x + 2h$$

$$\frac{f(x+h) - f(x)}{h} \to 4x \qquad \text{as} \qquad h \to 0$$

7.3 Stationary Points

We have seen how to find the gradient at a particular point on a curve, but what about finding the point where the gradient is zero? This is the point wher the curve changes from having a positive gradient to a negative one, or vice versa. The point where this happens is called a **stationary point**.

eg Find the stationary point of $f(x) = 2x^3 - x^2 - 4x + 2$.

$$\frac{d}{dx} = 6x^2 - 2x - 4$$

Put $\frac{d}{dx} = 0$ so that

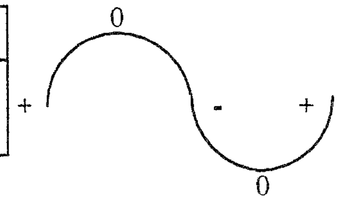
$$6x^2 - 2x - 4 = 0$$
$$(6x + 4)(x - 1) = 0$$
$$x = -2/3 \text{ or } x = 1$$

At $x = -2/3$, $f(x) = -14/27$

At $x = 1$, $f(x) = 7$.

The curve has stationary points at (-2/3, -14/27) and (1, 7). Below we can see how the gradient changes.

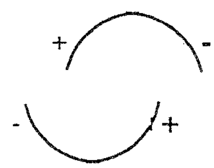
x	-1	-2/3	0	1	2
$\frac{d}{dx}$	+ve	0	-ve	0	+ve



The diagram above shows how the curve behaves. By looking at the gradients of the points either side of the stationary values, the ones with zero gradient, we find out the nature of the point. These stationary values indicate **turning points** on the curve.

The first turning point, (-2/3, -14/27) is a MAXIMUM.

The second turning point, (1, 7) is a MINIMUM.



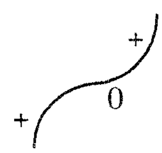
eg Find the stationary points for $y = x^3 + 1$

$\frac{dy}{dx} = 3x^2$ for stationary points $3x^2 = 0$

$x = 0$ twice.

Consider the nature of this point.

x	-1	0	1
$\frac{dy}{dx}$	+ve	0	+ve



This time the gradient goes from positive to zero and back to positive again. It is called a point of **inflexion**, and it could have been simple to just regard it as a turning point. This is why we need to investigate the nature of the stationary points.

Another way of investigating the nature of stationary points is to look at the second differential, ie the differentiated function differentiated again.

In the first example

$\frac{d}{dx} = 6x^2 - 2x - 4$

$\frac{d^2f}{dx^2} = 12x - 2$

Substituting in the values of x where we have stationary points,

$x = -2/3$ $f''(x) = -10$ NEGATIVE means it is a maximum.
$x = 1$ $f''(x) = 10$ POSITIVE means it is a minimum.

For the second example,

$$\frac{d}{dx} = 3x^2 \qquad \frac{d^2f}{dx^2} = 6x$$

Substituting the value $x = 0$ we have $f''(x) = 0$, zero means a point of inflexion.

7.4 Standard Derivatives

So far we have considered differentiation using the general formula

$$y = x^n \qquad \frac{dy}{dx} = nx^{n-1}$$

There are a number of standard formulae which help us to differentiate functions which are not of the simple form above. Indeed there are so many standard derivatives that you will never use them all and only the most important ones are listed below.

f(x)	f'(x)
sin x	cos x
cos x	- sin x
tan x	$\sec^2 x$
sin nx	n cos nx
cos nx	- n sin nx
e^x	e^x
e^{ax}	ae^{ax}
ln x	1/x

7.5 Differentiation Techniques

There are a number of different techniques used in differentiation to make the calculations simpler. An important idea is that of treating the function as if it really two functions.

Product Rule
If f and g are two functions of x then

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

Quotient Rule
Given the same two functions as above

$$\frac{d}{dx}(f/g) = \frac{g\, df/dx - f\, dg/dx}{g^2}$$

These two rules are very important tools of differentiation and you will find that they are frequently needed.

eg Differentiate with respect to x $y = (x + 1)(2x - 1)$

Let $f = x + 1$ $\quad g = 2x - 1$

$\frac{df}{dx} = 1$ $\quad \frac{dg}{dx} = 2$

So

$$\frac{d}{dx}(x + 1)(2x - 1) = 1(2x - 1) + 2(x + 1)$$
$$= 2x - 1 + 2x + 2$$
$$= 4x + 1$$

eg Differentiate with respect to x $y = \frac{(x^2 + 1)}{x - 2}$

Let $f = x^2 + 1$ $\quad g = x - 2$

$\frac{df}{dx} = 2x$ $\quad \frac{dg}{dx} = 1$

So

$$\frac{d}{dx}\frac{(x^2 + 1)}{x - 2} = \frac{2x(x - 2) - (x^2 + 1)}{(x - 2)^2}$$
$$= \frac{2x^2 - 4x - x^2 - 1}{(x - 2)^2}$$
$$= \frac{x^2 - 4x - 1}{(x - 2)^2}$$

Function of a Function

This technique is sometimes called the chain-rule. Sometimes we may be given a single function which would prove difficult to differentiate straight away. So we split the function into two as follows.

eg Differentiate with regard to x $y = (3x + 4)^2$

This can be re-written as $u = 3x + 4$ with $y = u^2$.

Now $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

So that $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = 3$

Then $\frac{dy}{dx} = 3.2u = 6(3x + 4)$

You can check this particular example by multplying out the brackets and differentiating as usual.

eg Differentiate $y = \frac{5}{(3 + 4x^2)^3}$

Let $u = 3 + 4x^2$ so $y = 5/u^3$

$\frac{du}{dx} = 8x$ $\frac{dy}{du} = -15\,u^{-4}$

Then, $\frac{dy}{dx} = 8x(-15u^{-4}) = \frac{-120x}{(3 + 4x^2)^3}$

eg Differentiate $y = \sin(x^3 + 3)$

Let $u = x^3 + 3$ so $y = \sin u$

$\frac{du}{dx} = 3x^2$ $\frac{dy}{du} = \cos u$

So $\frac{dy}{dx} = 3x^2 \cos u = 3x^2 \cos(x^3 + 3)$

This chain-rule can be extended to help us cope with more complicated functions. The following example could be called a 'function of a function of a function'.

eg Differentiate $y = \ln(\sin(3x^2 - 5))^3$

Let $u = 3x^2 - 5$ $\quad v = \sin u$ $\quad$ so $\quad y = 3 \ln v$

$\frac{du}{dx} = 6x \qquad \frac{dv}{du} = \cos u \qquad \frac{dy}{dv} = 3/v$

Then,
$$\frac{dy}{du} = \frac{dy}{dv}\frac{dv}{du} = \frac{3}{v}\cos u$$
$$= 3 \tan u$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 6x.3 \tan u$$
$$= 18x \tan(3x^2 - 5)$$

It is possible to simplify the most difficult looking functions using this method. It is possible to use as many variables as you need, as long as you appraoch it logically an in a progressive manner. For instance, say your function y had substitutions of u, v and w, the neccessary derivatives would be

$$\frac{dy}{dv} = \frac{dy}{dw}\frac{dw}{dv} \qquad \frac{dy}{du} = \frac{dy}{dv}\frac{dv}{du} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

We can use the chain-rule another way to make functions with fractional indices easier.

eg Differentiate $y = x^{1/2}$

Let $u = x^{1/2}$ and $\quad y = u^2 \quad$ so that $\quad y = x$

$\frac{dy}{du} = 2u \qquad \frac{dy}{dx} = 1$

Then
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
$$1 = 2u\frac{du}{dx}$$
$$\frac{du}{dx} = \frac{1}{2u} = \frac{1}{2\sqrt{x}}$$

This is the solution since we changed the variable at the beginning of the calculation. So we can see from this particular example that the general formula for differentiating works for fractional indices as well. The following example is more suited to the method.

eg Differentiate $y = \frac{1}{(x^3 + 2x^5)^3}$

Let $u = x^3 + 2x^5$ so $y = u^{-3}$

$\frac{du}{dx} = 3x^2 + 10x^4$ $\frac{dy}{du} = -3u^{-4}$

$$\text{Then } \frac{dy}{dx} = -3u^{-4}(3x^2 + 10x^4)$$
$$= -3(x^3 + 2x^5)^{-4}(3x^2 + 10x^4)$$
$$= \frac{-3(3x^2 + 10x^4)}{(x^3 + 2x^5)^4}$$

Implicit Differentiation.

Say that you had to differentiate a function in x and y and that the function was not of the usual form $y = f(x)$. The idea is to differentiate the x component as usual, to differentiate the y component with respect to y and then multiply it by dy/dx.

eg Differentiate $2x^2 + 3y^2 = 6$

$$4x + 6y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-2x}{3y}$$

If there are any terms containing both x and y then the product or quotient rules need to be used as appropriate.

eg Differentiate $x^3y + y^3x = a^4$

$$3x^2y + x^3\frac{dy}{dx} + 3y^2\frac{dy}{dx} + y^3 = 0$$

$$\frac{dy}{dx}(3y^2 + x^3) = -3x^2y - y^3$$

$$\frac{dy}{dx} = \frac{-y(3x^2 + y^2)}{3y^2 + x^2}$$

Parametric Differentiation

Sometimes an equation is given parametrically, that is with x and y defined in terms of a parameter t. We have to approach this type of differentiation slightly differently.

Say $x = x(t)$ and $y = y(t)$ then we can differentiate $y = f(x)$ with resect to t using the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

eg Differentiate the following with respect to x.

$$y = 2t^3 \qquad x = t^2$$

$$\frac{dy}{dt} = 6t^2 \qquad \frac{dx}{dt} = 2t$$

Then, $\frac{dy}{dx} = \frac{6t^2}{2t} = 3t$

eg Differentiate $y = a \sin \vartheta$ $x = a \cos \vartheta$

$$\frac{dy}{dt} = a \cos \vartheta \qquad \frac{dx}{dt} = -a \sin \vartheta$$

Then $\frac{dy}{dx} = \frac{a \cos \vartheta}{-a \sin \vartheta} = -\cot \vartheta$

Summary

We have looked generally at differentiation as a way of finding the gradient of the curve at any point. Also we are able to get an idea of what the graph looks like by investigating the stationary points. These tell us the nature of turning points, ie maxima or minima, as well as indicating the presence of any points of inflexion.

Differentiation is a very important technique which is used widely in many fields, so it is important to understand the basics before trying to use as a tool. The development of differentiation from first principles is touching more on the topic of mathematical analysis and a basic realisation that limits are a way of investigating a function without having to plot it, is all that is really required.

The various methods of differentiation are ways in which we can overcome complicated looking functions. Product and quotient rules, substitution, implicit and parametric differentiation should be known, so that the correct method can be employed in various situations.

Activities

1. Differentiate the following with respect to x.

   (a) x^5 (b) x^{-4} (c) $2x^7$ (d) $3/x$ (e) $2/x^2$
   (f) $x^2 + 2x$ (g) $x^6 + 3$ (h) $3x + 2x^2$ (i) $2 - 3x - 4x^3$

2. (a) Given that $v = 2 + t - 3t^2$ find dv/dt.
   (b) " " $v = 4/3\pi r^3$ " dv/dr.
   (c) " " $s = ut + 1/2at^2$ " ds/dt u, a constants.
   (d) " " $PV = 2V$ " dP/dV.

3. Differentiate using any appropriate method.

   (a) $(2x - 3)^2$ (b) $(x + 1)(x - 2)$ (c) $x^2(2x + 3)$
   (d) $(x^2 + 1)/x$ (e) $(x^2 + 1)(x^2 - 1)$ (f) $(x + 1/x)^2$

4. Find the limits in the following.

   (a) $\lim_{x \to 0} \dfrac{x^2 + 6x}{3x}$ (b) $\lim_{x \to 0} \dfrac{x + 1/x}{x - 1/x}$

5. Differentiate the following using the chain-rule.

   (a) $y = (3x - 2)^3$ (b) $z = 3v^2$ where $v = 2t - t^3$.

6. Differentiate the following with regard to x.

   (a) $(3x + 4)^2$ (b) $(2x - 1)^3$ (c) $(x^2 - 3x)^4$

7. Find the gradient of the curve at the given point.

   (a) $x^3 + 2y^3 = 3$ at (1, 1) (b) $xy + \cos xy = \pi/2$ at $(1, \pi/2)$

8. Differentiate the following.

   (a) $\sin^2 x + \sin x + 1$ (b) $3 \cos^2 x$
   (c) $\cos 4x$ (d) $\ln x + \sin x$
   (e) $\tan x + 3(\ln x)^2$ (f) $3e^{2x} - xe^{-x}$

9. Find the derivatives of the following functions.

   (a) $\dfrac{x^2 + 1}{x + 1}$ (b) $\dfrac{2x^3 + 3x}{3x^2 + 6}$ (c) $(x^2 + 4)\sqrt{(x + 1)}$

   (d) $\sqrt{(x + 2)^3}\sqrt{(x + 1)^{-1}}$ (e) $(3x^2 - 2x)^{1/3}(2x^2 + 5)^{1/2}$

10. Differentiate the curve given parametrically as

$y = t\cos t - \sin t \qquad x = t\sin t + \cos t.$

11. Find the stationary points and investigate the nature of them for the following equations.

(a) $f(x) = x^3 - 3x^2 - 24x + 20$ (b) $y = x^4$
(c) $f(x) = 3x^4 + 4x^3 - 6x^2 - 12x + 5$ (d) $f(x) = x - 1/x^2$
(e) $y = \cos 3x$ (f) $y = (x + 1)^3(x + 2)$
(g) $y = (x - 1)^3$

[Solutions: **1** (a) $5x^4$, (b) $-4x^{-5}$, (c) $14x^6$, (d) $-3x^{-3}$, (e) $-4x^{-3}$, (f) $2x + 2$, (g) $6x^5$, (h) $3 + 4x$, (i) $-3 - 12x^2$;
2 (a) $1 - 6t$, (b) $4\pi r^2$, (c) $u + at$, (d) $-25/V^2$;
3 (a) $8x - 6$, (b) $2x - 1$, (c) $6x^2 + 6x$, (d) $1 - 1/x^2$, (e) $4x^3$, (f) $2x - 2/x^3$;
4 (a) 2, (b) -1;
5 (a) $9(3x - 2)^2$, (b) $6(2t - t^3)(2 - 3t^2)$;
6 (a) $6(3x + 4)$, (b) $6(2x - 1)^2$, (c) $4(x^2 - 3x)^3(2x - 3)$;
7 (a) $dy/dx = -x^2/2y^2$, gradient = 1/2, (b) $dy/dx = -y(1 - y\sin xy)/(1 - x\sin xy)$, gradient = $-\pi/2$;
8 (a) $2\cos x\sin x + \cos x$, (b) $-6\cos x\sin x$, (c) $-4\sin 4x$, (d) $1/x + \cos x$, (e) $\sec^2 x + 6\ln x/x$,
(f) $6e^{2x} - e^{-x} + xe^{-x}$;
9 (a) $(x^2 + 2x - 1)/(x + 1)^2$, (b) $(2x^4 + 9x^2 + 6)/3(x^2 + 2)^2$, (c) $(5x^2 + 4x + 4)/2\sqrt{(x + 1)}$,
(d) $(x + 7/2)\sqrt{(x + 2)}/\sqrt{(x + 3)^3}$, (e) $1/3(3x^2 - 2x)^{-2/3}(6x - 2)(2x^2 + 5)^{1/2} + (3x^2 - 2x)^{1/3}2x(2x^2 + 5)^{-1/2}$;
10 $(2\cos t + t\sin t)/(2\sin t - t\cos t)$;
11 (a) (4 -60) min, (-2, 60) max, (b) (0, 0) min, (c) (-1, 10) inflexion, (1, -6), min, (1/3, 16/27) inflexion,
(d) $(2^{1/3}, 1/2(2)^{1/3})$ min, (e) $(2n\pi/3, 1)$ max, $((2n + 1)\pi/3, -1)$ min, (f) (-7/3, 64/81) min, (g) (1, 0) inflexion.]