# 4. TRIGONOMETRY

## Objectives

To be able to use the basic trigonometric functions sine, cosine and tangent.

To understand the relationship between the functions and to use this knowledge to solve trigonometric equations.

To be able to use trigonometric identities to rearrange an equation to a suitable form.

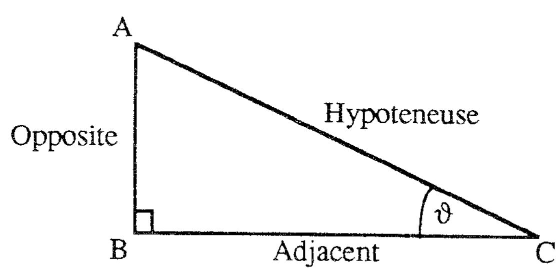
To know what radians are and why we use them.

To be able to solve any triangles given three of its measurements.

### 4.1 The Angles in a Triangle

The basis of trigonometry is the right-angled triangle.

*The Standard Right -angled Triangle*




Pythagoras' theorem gives us the result $AB^2 + BC^2 = AC^2$

ie "The sum of the squares on the opposite and adjacent sides equals the square on the hypoteneuse."

From this we formulate the trigonometric functions sine, cosine and tangent which enable us to find the angles and sides of a triangle.

$$\tan \vartheta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{BC}{AB}$$ Two Old Angels

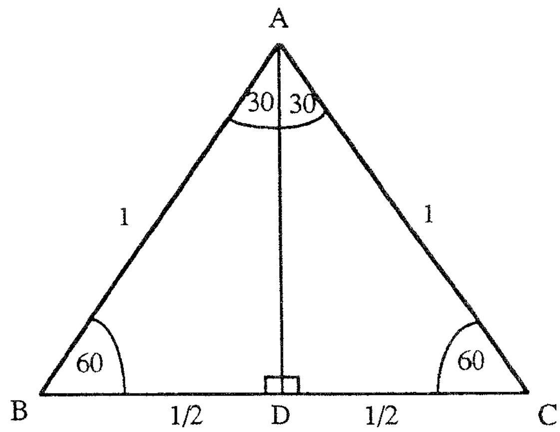
$$\sin \vartheta = \frac{\text{Opposite}}{\text{Hypoteneuse}} = \frac{BC}{AC}$$ Sitting On High

$$\tan \vartheta = \frac{\text{Adjacent}}{\text{Hypoteneuse}} = \frac{AB}{AC}$$ Chatting About Heaven

This little rhyme is an alternative to SOHCAHTOA as a way of remembering the formulae.

The angle $\vartheta$ can be either of the two angles which are not right-angles, but the opposite and adjacent sides change according to which angle is being looked at.

Consider the equilateral triangle in which all sides and angles are equal. This triangle can be divided into two right-angled triangles of equal dimensions and we are able to formulate some very important results from these.

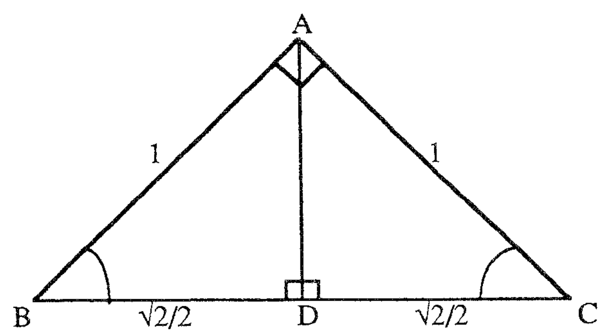



*The basic Equilateral Triangle used to find useful trigonometric values*

From Pythagoras, AD = $\sqrt{3}/2$. Using the formulae for sine, cosine and tangent we can find the following useful results.

$\sin 30^\circ = 1/2\cos 30^\circ = \sqrt{3}/2\tan 30^\circ = 1/\sqrt{3}$
$\sin 60^\circ = \sqrt{3}/2\cos 60^\circ = 1/2\tan 60^\circ = \sqrt{3}$

See if you can work out similar results for the triangle below.



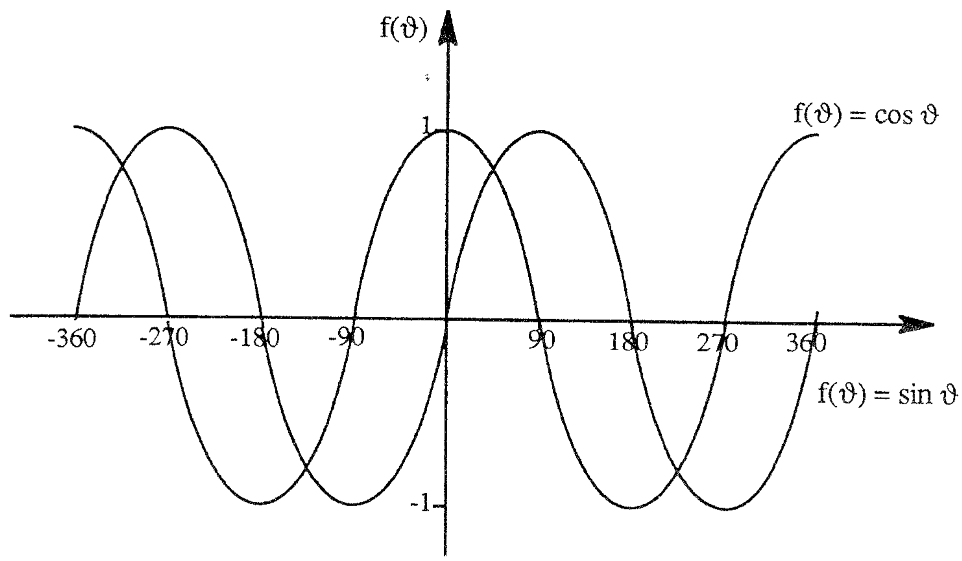

*Another useful triangle for working out trigonometric values*

| | sin | cos | tan |
|---|---|---|---|
| 0° | 0 | 1 | 0 |
| 30° | ...... | ...... | ...... |
| 45° | ...... | ...... | ...... |
| 60° | ...... | ...... | ...... |
| 90° | ...... | ...... | ...... |

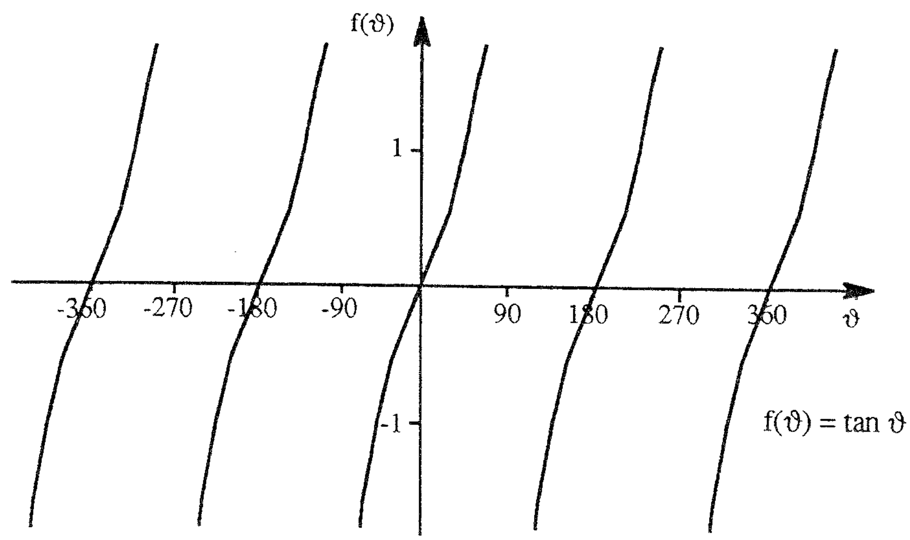
You should see some similarity between the results for 30° and 60°, after all they are complementary angles, ie they sum to 90°.

So far we have been concerned with acute angles but we can also use these formulae to find the trigonometric values of obtuse angles. An obtuse angle is one which is over 180°. Since there are 360° in a circle there are repetitions of results for $\vartheta$ outside of the range $0° \leq \vartheta \leq 360°$. By drawing a graph of the functions we will see the relationship between them more clearly.

*The Sine and Cosine curves*




*The Tangent Curve.*




You should notice that the following are true.

$\cos(-\vartheta) = \cos\vartheta$
$\sin(-\vartheta) = -\sin\vartheta$
$\tan(-\vartheta) = -\tan\vartheta$

The cosine curve is symmetric about the y axis and each curve is repeated every 360°.

## 4.2 Trigonometric Equations

Before we try to solve trigonometric equations it is important to know the following standard identities.

$\tan \vartheta = \dfrac{\sin \vartheta}{\cos \vartheta}$ $\quad$ $\sin^2 + \cos^2\vartheta = 1$

$\sec \vartheta = \dfrac{1}{\cos \vartheta}$ $\quad$ $\operatorname{cosec} \vartheta = \dfrac{1}{\sin \vartheta}$ $\quad$ $\cot \vartheta = \dfrac{1}{\tan \vartheta}$

$1 + \tan^2 \vartheta = \sec^2\vartheta$ $\quad$ $1 + \cot^2 \vartheta = \operatorname{cosec}^2 \vartheta$

The trigonometric functions are **periodic** so we are unable to find just one solution to equations. Usually we take the **principal value** which is the only solution in the region $-90° \leq \vartheta \leq 90°$ for $\sin \vartheta$ and $\tan \vartheta$.

$0° \leq \vartheta \leq 180°$ for $\cos \vartheta$

eg Solve $2 \sin \vartheta + 1 = 0$

$\sin \vartheta = -1/2$

$\vartheta = \sin^{-1}(-1/2)$ (or arcsin 1/2 = -30° principal value)

$\vartheta = -150°, -30°, 210°, 330°.$

These values are all solutions to the equation, but the principal value is -30° since it is the only solution in the range $-90° \leq \vartheta \leq 90°$ nearest the origin. This is the solution your calculator would give, ssince calculators **only** return values for $\vartheta$ in the respective ranges of principal values s as indicated above.

eg Find values for x between 0° and 360° where x is given by

$\cos x = -\sqrt{3}/2$

$x = \cos^{-1}(-\sqrt{3}/2)$ (or $x = \arccos -\sqrt{3}/2$)

$x = 150°$

This is the solution given by a calculator, but it is clear from the graph that there is another solution within this region. The solution is given by $(360 - 150)° = 210°$

eg Solve $\sin \vartheta + 2 \cos \vartheta \sin \vartheta = 0$ for $0 \leq \vartheta \leq 360$

$\sin \vartheta (1 + 2 \cos \vartheta) = 0$

$\sin \vartheta = 0$ or $1 + 2 \cos \vartheta = 0$

$\vartheta = 0°$ $\quad$ $\cos \vartheta = -1/2$

$\vartheta = 120°$

These are the principal values , so looking at the graphs we find the other solutions in the range.

$\vartheta = 0°, 180, 360.$ or $\vartheta = 120°, (360 - 120)°, = 240°.$

By now you should be aware of a pattern emerging. In general we find other solutions from the principal solution using the formulae below. This can be validated by inspection of the graphs of sine, cosine and tangent illustrated previously.

| | |
|---|---|
| $\vartheta = 180n° + (-1)^n p.v.°$ | for sine |
| $\vartheta = 360\,n° \pm p.v.°$ | for cosine |
| $\vartheta = 180n° + p.v.°$ | for tangent |

where p.v. stands for the principal value.

eg Solve $\cos\vartheta - \sin\vartheta = 1 0 \le \vartheta \le 360$

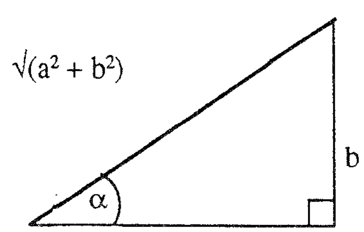
This equation is in terms of cos and sin, so we need to change it so that it has only one trigonometric function. It is often useful to reduce equations like $a\cos\vartheta + b\sin\vartheta = c$ to a single term such as $r\cos(\vartheta - \alpha) = c$, where c is a constant. This is possible if we can find values of r and α so that,

$$r(\cos\vartheta\cos\alpha + \sin\vartheta\sin\alpha) \equiv a\cos\vartheta + b\sin\vartheta$$

Comparing coefficients of $\cos\vartheta$ and $\sin\vartheta$ we have,

$$r\cos\alpha = a \quad (1)$$
$$r\sin\alpha = b \quad (2)$$

On dividing, we obtain the ratio $\tan\alpha = b/a$ and hence the triangle below.




Then $\cos a = \dfrac{a}{\sqrt{(a^2+b^2)}}$ so $r = \sqrt{(a^2 + b^2)}$ from (1)

Now $a\cos\vartheta + b\sin\vartheta \equiv r\cos(\vartheta - \alpha).$

eg. Solve $\cos\vartheta - \sin\vartheta = 1$ $0° \le \vartheta \le 360°$

So $a = 1, b = -1$ then, $\tan\alpha = -1$ $r = \sqrt{(1^2 + 1^2)}$

$a = -45°$ $r = \sqrt{2}$

So $\cos\vartheta - \sin\vartheta \equiv \sqrt{2}\cos(\vartheta - \alpha) = 1$

$$\cos(\vartheta - \alpha) = 1/\sqrt{2}$$

$$\vartheta - \alpha = 360n^\circ \pm 45^\circ$$

$$\vartheta = 360n^\circ \pm 45^\circ - 45^\circ$$

If $n = 0, 1$ $\quad \vartheta = 0^\circ, -90^\circ, 360^\circ, 270^\circ$.

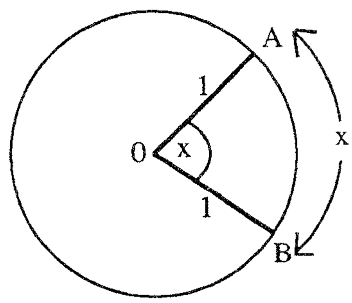
The specified range of solutions is $0^\circ \le \vartheta \le 360^\circ$, so our solution is $\vartheta = 0^\circ, 270^\circ, 360^\circ$

Further variations on this method use $r\cos(\vartheta + \alpha)$ and $r\sin(\vartheta \pm \alpha)$ which are derived using the identities for $\cos(A \pm B)$ and $\sin(A \pm B)$, (section 4.3).

## 4.3 Radian Measure

A far more mathematical way of measuring angles is to use radians instead of degrees. This is because on a circle of radius 1, the angle AOB is x radians if the length of the arc AB is x.

*The measurement of radians*




The relationship between degrees and radians is given below.

| Degrees | 360 | 180 | 90 | $\vartheta$ |
|---|---|---|---|---|
| Radians | $2\pi$ | $\pi$ | $\pi/2$ | $\vartheta\pi/180$ |

As with degrees, there are a number of key trig values which are useful to know. See if you can complete the chart.

| x | sin x | cos x | tan x |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| $\pi/6$ | ...... | ...... | ...... |
| $\pi/4$ | ...... | ...... | ...... |
| $\pi/3$ | $\sqrt{3}/2$ | 1/2 | $\sqrt{3}$ |
| $\pi/2$ | 1 | 0 | -- |
| $2\pi/3$ | ...... | ...... | ...... |
| $3\pi/4$ | ...... | ...... | ...... |
| $5\pi/6$ | ...... | ...... | ...... |
| $\pi$ | ...... | ...... | ...... |

Using your calculator, you need to change the mode from degrees (deg) to radians (rad) and use the pre-programmed value of $\pi$. Most scientific calculators allow you to convert an angle in radians to one in degrees and vice versa, so read your calculator manual!.

eg Solve $\cos x + \cos 5x = 0$ for general values of x

$$\cos 5x = -\cos x$$
$$= \cos(\pi - x) = \cos(\pi + x)$$

since the cosine curve is symmetrical

$5x = 2n\pi - (\pi - x)$ or $5x = 2n\pi - (\pi + x)$

since we require general values we can convert the general formula for degrees into radians.

| | |
|---|---|
| $4x = \pi(2n - 1)$ | $6x = \pi(2n - 1)$ |
| $x = (\pi/4)(2n - 1)$ | $x = (\pi/6)(2n - 1)$ |

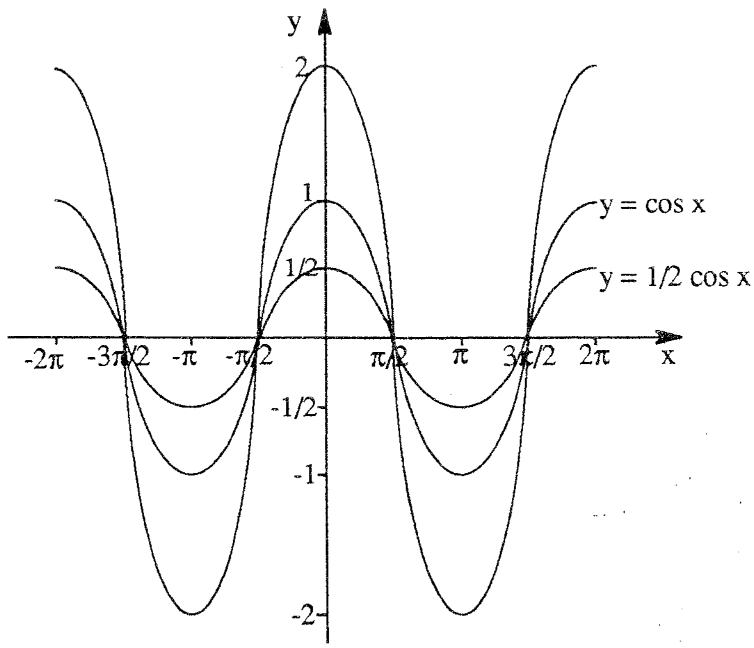
These solutions are found using the same formula for general solutions as we use for degrees. Note that n varies and that the solution for n = 0 is the one given by your calculator.

| | | |
|---|---|---|
| $\sin \vartheta = 0$ | gives | $\vartheta = n\pi + (-1)^n$ p.v. |
| $\cos \vartheta = 0$ | gives | $\vartheta = 2n\pi \pm$ p.v. |
| $\tan \vartheta = 0$ | gives | $\vartheta = n\pi +$ p.v. |

## 4.4 Change of Scale

We have already seen the graphs for $y = \cos x$ and $y = \sin x$, but what happens with $y = a \cos x$ and $y = a \sin x$ ?

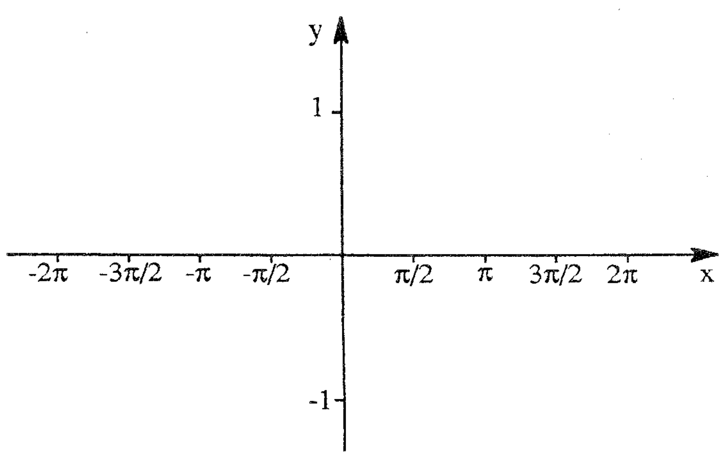
These new curves have the same period $2\pi$ and have peaks and troughs at the same values of x. However the height of these peaks and troughs is multiplied by a factor of a, a is called the **ampilitude**, such that if $a > 1$ the amplitude is increased and if $0 < a < 1$ the amplitude is decreased (when compared with the case $a = 1$). Below are the graphs of $y = a \cos x$ for $a = 2$, 1 and 1/2.




*The cosine curve with varying amplitudes*

What do you think happens when $y = \cos 2x$ ? Tabulate the values below and sketch the curve.

| x | -2π | -3π/2 | -π | 0 | π/2 | π3 | π/2 | 2π |
|---|---|---|---|---|---|---|---|---|
| cos 2x | | | | | | | | |



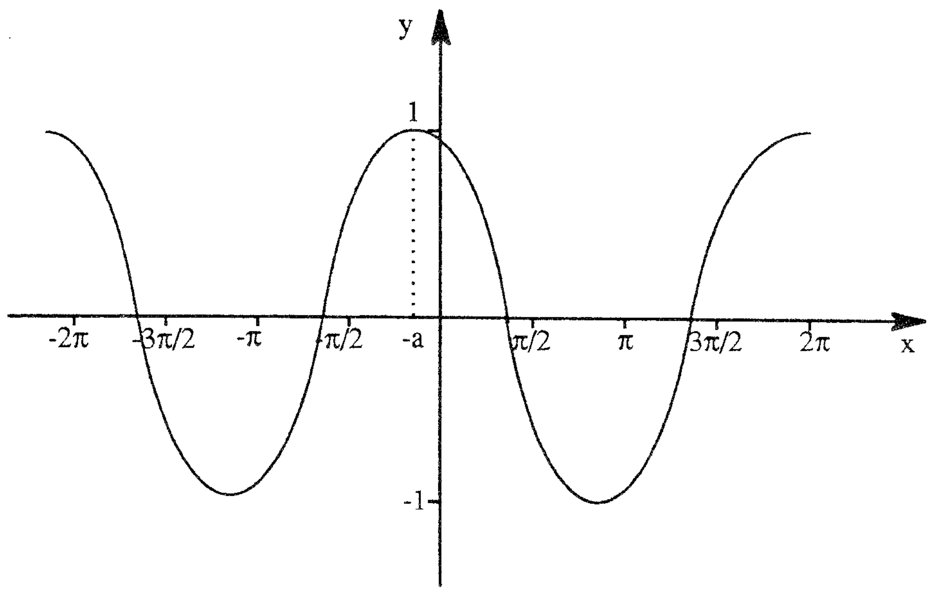

*Another variation of the cosine curve*

Note that this curve is 'twice as frequent' compared with the basic curve y = cos x. In fact y =cos 3x is 'three times as frequent', etc. showing that in cos ax or sin ax, a is closely related to the **frequency** of the sine or cosine wave.

So y = a cos x is stretched or squashed along the y axis and y = cos a x is stretched or squashed along the x axis. It is a similar case for y = a sin x and y = sin a x.

What happens to a function such as y = cos (x + a)? This is shifted along the x axis by a factor of -a.

*The Cosine curve shifted along the x axis by a factor of -a*




## 4.5 More Trigonometric Identities

A lot of trigonometric equations are expressed in a way which cannot be solved immediately so we use various identities which allow us to express the same equation in a way which can be solved.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

If B is written as A in the first and third identities above , we obtain:

$$\cos 2A = \cos^2 A - \sin^2 A \quad \text{and} \quad \sin 2A = 2 \sin A \cos A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A \qquad (\text{using } \sin^2 A + \cos^2 A = 1)$$

eg Express sin 3A in terms of sin A.

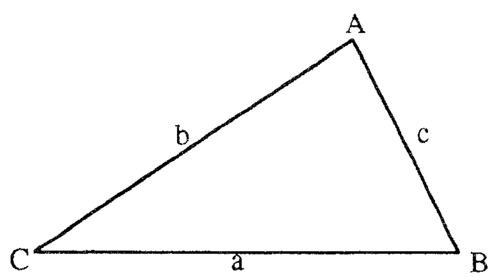
$$\begin{aligned}\sin 3A &= \sin(2A + A)\\ &= \sin 2A \cos A + \cos 2A \sin A\\ &= 2\sin A\cos A\cos A + (2\cos^2 A - 1)\sin A\\ &= 2\sin A\cos^2 A + (\cos^2 A - \sin^2 A)\sin A\\ &= 2\sin A(1 - \sin^2 A) + (1 - \sin^2 A - \sin^2 A)\sin A\\ &= 2\sin A - 2\sin^3 A + \sin A - \sin^3 A - \sin^3 A\\ &= 3\sin A - 4\sin^3 A\end{aligned}$$

## 4.6 Solution of Triangles

We have seen how to calculate angles and sides of a right-angled triangle and you can probably remember the area of a triangle to be 1/2(base x height).

Now suppose that you did not have enough information about a triangle to solve it and that the triangle was not right-angled. There are three trig formulae which allow us to solve any triangle providing we know three things about it.

*The General Triangle*




The conventional way of labelling a triangle is as above, with angles having the same letter as the side opposite. The angles and sides can all be irregular.

The trigonometric formula for the area of a triangle is

$$\begin{aligned}\text{Area} &= 1/2\ a.b.\sin C\\ &= 1/2\ a.c.\sin B\\ &= 1/2\ b.c\ \sin A\end{aligned}$$

ie 1/2 x (2 sides x sine of the angle between)

By dividing these equations by 1/2(abc) we arrive at the formula below which is called the **Sine Formula.**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly we can derive another formula called the **Cosine Formula.**

$a^2 = b^2 + c^2 - 2bc \cos A$

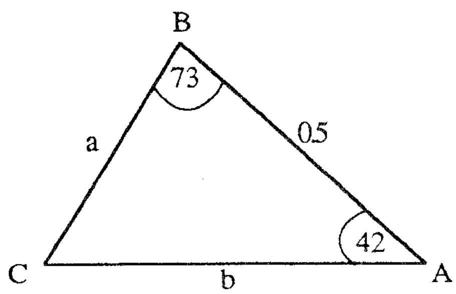
$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

When do we use each formula? That depends on what you already know about the triangle.

| | |
|---|---|
| 2 angle & any side | sine formula |
| 2 sides & any angle | sine formula |
| 2 sides & angle between | cosine formula |
| 3 sides | cosine formula |

eg Solve the triangle below.




area $= 1/2 \times 0.37 \times 0.53 \times \sin 65°$

$= 0.09$

Angle C is $180° - (42 + 27)° = 65°$

$a = \dfrac{c \sin A}{\sin C}$ from the sine formula

$a = \dfrac{0.5 \times \sin 42°}{\sin 65°} = 0.37$

$b^2 = (0.37)2 + (0.5)2 - (2 \times 0.37 \times \cos 73)$

$b^2 = 0.28$

$b = 0.53$

### Summary

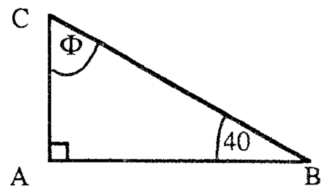
Basic trigonometry originates from the right-angled triangle, but you will have seen now how the same trigonometry can be applied to any triangle. The standard angles in both degrees and radians are very useful to know because they will occur again and again in many different contexts. Apart from solving triangles, we use the trig functions to solve equations in a general or specific case. Trigonometric equations may look daunting at first but they are really quite simple as long as you remember the relevant graphs. The trigonometric identities are especially important because they allow us to manipulate equations to a simpler or better-looking form but they really come into their own in calculus

## Activities

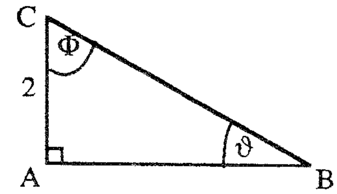
1. If $\sin \vartheta = 4/5$ find possible values for $\cos \vartheta$ and $\tan \vartheta$.
   If $\tan \vartheta = -5/12$ find possible values for $\cos \vartheta$ and $\sin \vartheta$.

2. Find the missing sides and angles if possible.

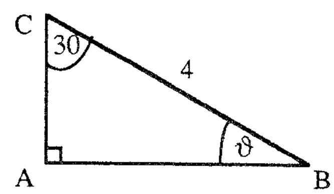
(a)




(b)




(c)




3. Solve the following equations for $0 \leq \vartheta \leq 360$.

   (a) $2 \sin \vartheta + 1 = 0$
   (b) $\sin \vartheta \cos 2\vartheta = 0$
   (c) $(1 + 3 \sin \vartheta)(4 - \cos \vartheta) = 0$

4. Find the principal values of $\vartheta$ for the following.

   (a) $\cos^2 \vartheta + \cos \vartheta - 2 = 0$
   (b) $\cos \vartheta \sin \vartheta - \cos \vartheta - \sin \vartheta + 1 = 0$
   (c) $2 \cos \vartheta \sin \vartheta - \sin \vartheta + 2 \cos \vartheta = 1$

5. Solve the following.

   (a) $\sin 2x - \sin x = 0$ $\quad 0 \leq x \leq \pi$
   (b) $\sin x = \cos 2x$ $\quad 0 \leq x \leq 2\pi$
   (c) $\sin 2x - \cos x = 0$ $\quad$ general solutions
   (d) $\sin 5x + \sin 3x = 0$ $\quad 0 < x < \pi$

6. Eliminate $\vartheta$ from the sinultaneous equations to obtain a single solution.

$$x \cos \vartheta + y \sin \vartheta = 1$$
$$y \cos \vartheta + x \sin \vartheta = 2$$

7. Plot the function $y = \sin(2x + 1)$

8. Simplify (a) $\dfrac{\sin 2A\,(2\cos 2A + 1)}{2\sin 3A}$

(b) $\sin(A + B) + \sin(A - B)$

(c) $\cos(A + B) + \cos(A - B)$

9. Solve in the range $0 < \vartheta \leq 180$

(a) $\sin 2\vartheta = \cos\vartheta$ (b) $2\cos 2\vartheta = 1 + 4\cos\vartheta$

10. Solve the triangle with angles 50°, 57.8° and the side inbetween of length 8 units.

11. Solve the triangle ABC given

(a) $A = 15°$ $a = 2$ $b = 3$
(b) $B = 82°$ $b = 5$ $c = 4$
(c) $B = 38°$ $a = 12$ $b = 9$
(d) $C = 150°$ $a = 6$ $c = 10$

12. Solve the triangle ABC given

(a) $A = 75°$ $B = 55°$ $c = 9$
(b) $B = 68°$ $C = 82°$ $c = 15$
(c) $A = 121°$ $C = 13°$ $a = 3$
(d) $A = 32°$ $C = 87°$ $b = 22$
(e) $a = 13$ $b = 5$ $c = 12$
(f) $a = 3$ $b = 7$ $c = 5$
(g) $a = 4$ $b = 5$ $c = 6$

[Solutions: **1** 3/5, 4/3 or -3/5, -4/3; 12/13, -5/13 or -12/13, 5/13.
**2** (a) f = 50°; (b) $\vartheta$ = 33°41', f = 26°19', BC = $\sqrt{13}$; (c) $\vartheta$ = 60°, AB = 2, AC = $2\sqrt{3}$.
**3** (a) 210°, 330°; (b) 0°, 45°, 135°, 180°, 225°, 315°; (c) 199.5°, 350.5°.
**4** (a) 0°; (b) 0°, 90°; (c) -90°, -60°, 60°.
**5** (a) 0, $\pi/3$, $\pi$; (b) $\pi/6$, $5\pi/6$, $3\pi/2$; (c) $(n + 1)\pi$, $(2n + 1/6)\pi$, $(2n + 5/6)\pi$; (d) $\pi/4$, $\pi/2$, $3\pi/4$.
**6** $(x^2 - y^2)2 = (x - 2y)2 + (y - 2x)2$.
**8** (a) cos A; (b) 2 sin A cos B; (c) 2 cos A sin B.
**9** (a) 30°, 90°, 150°; (b) 120°. 10 A = 50°, B = 72.25°, C = 57.8°, a = 6.436, b = 8, c = 7.1
**10** area = 27.57 units2.
**11** (a) B = 22.8°, C = 142.2°, c = 4.74; or B = 157.2°, C = 78°, c = 1.05; (b) A = 45.6°, C = 52.4°, a=3.61; (c) A = 124.8°, C = 27.2°, c = 4.32; or A = 55.2°, C = 86.8°, c = 14.6; (d) A = 17.5°, B = 12.5°, b = 4.33.
**12** (a) C = 50°, a = 11.35, b = 9.62; (b) A = 30°, a = 7.57, b = 14.04; (c) B = 46°, b = 2.52, c = 0.79; (d) B = 61°, a = 13.33, c = 21.52; (e) A = 90°, C = 67.4°, B = 22.6°; (f) A = 21.8°, B = 120°, C = 38.2°; (g) A = 41.4°, B = 55.8°, C = 82.8°.]