3. INDICES AND LOGARITHMS

Objectives

To understand the role of indices, positive, negative and fractional and how they behave.

To be able to use surds when writing irrational solutions and to understand the reason for their use.

To be able to manipulate indices and surds algebraically.

To realise the value of logarithms in mathematics and be able to use them in calculations.

3.1 Indices

Powers

x^2 is the variable x raised to the power 2, ie x times x. Similarly x^{10} is x multiplied by itself ten times.

Whenever a positive number is raised to a power the answer is always positive.

Whenever a negative number is raised to a power the answer is

- positive if the power is even$(-2)^6 = 64$
- negative if the power is odd$(-2)^5 = -32$

If the number to be raised to a power is a fraction, then raise both the numerator and the denominator to that power, ie $(3/4)^2 = 9/16$.

Sometimes the power is called the **exponent** but do not confuse this with the exponential function e^x.

Negative Powers

The negative power really means inverse.

ie $x^{-1} = 1/x$ $x^3y\text{-}4 = x^3/y^4$

Fractional Powers

When the power is a fraction it really means that the number is to the root of the denominator.

ie $2^{1/2} = \sqrt{2}$ $3^{1/4} = \sqrt[4]{3}$ $5^{-1/2} = 1/\sqrt{5}$

Note that the square root of a real negative number does not exist.

Using a calculator, find the following values.

$3^4 =$ .... $3^5 =$ .... $(-3)^4 =$ .... $(-3)^5 =$ ....

$3^{-4} =$ .... $3^{-5} =$ .... $(-3)^{-4} =$ .... $(-3)^{-5} =$ ....

$3^{1/4} =$ .... $3^{1/5} =$ .... $3^{-1/4} =$ .... $3^{-1/5} =$ ....

Algebraic Operations

Variables can be added and subtracted only if they have the same index.

eg $x^2 + 2x + 3 - 4x^2 + 1 + 4x = -3x^2 + 6x + 4$

$x^3 - y^2 + 2x^2 + 3y^2 = x^3 + 2y^2 + 2x^2$

We can perform multiplication and division on them by simply adding the indices.

eg $a^4.a^2 = a^6$ $a^3b.c = a^3bc$

$a^7.a^{-3} = a^4$

If the variable is already raised to a power and it is bracketed and raised to another power, then the powers are multiplied as below.

$x^2 + 3(x^3)^2 = x^2 + 3x^6$ $(x + y^2)^2 = x^2 + 2xy^2 + y^4$

	$a^o = 1$	$a^1 = a$
Summarising	$a^m \times a^n = a^{m+n}$	$a^m \div a^n = a^{m-n}$
	$(a \times b)^n = a^n \times b^n$	(Note $(a + b)^n \neq a^n + b^n$)
	$(a^m)^n = a^{m \times n}$	

3.2 Surd Form

When the solution to a problem is an irrational number (ie. one that cannot be expresed as a fraction), it is mathematically better manners to leave it in **surd form**, rather than writing down the value given by your calculator. Surds are written using the √ sign and it is a shorter, more accurate way of writing solutions which are irrational. This is especially useful if someone else needs your solutions to carry out calculations of their own.

eg $\sqrt{2}$ can be more easily recognised than 1.41 (3 d.p) and in fact $\sqrt{2}$ is as accurate as anyone wants it to be, whereas 1.414 can only ever be accurate to, at most 3 d.p. However, to write down something like $\sqrt{8}$ is incorrect. It would be leaving things unfinished like writing 6/3 instead of 2.

Notice that
$$\begin{aligned}\sqrt{8} &= \sqrt{(2 \times 4)} \\ &= \sqrt{(2 \times 2^2)} \\ &= \sqrt{2^2} \times \sqrt{2} \\ &= 2\sqrt{2}\end{aligned}$$

This is the correct way of writing $\sqrt{8}$. The easiest way to reduce irrational numbers to surd form is to factorise them into prime numbers and place the squared terms outside the √ sign. Remember that this method only works if the numbers within the √ sign are *multiplied* .

eg $\sqrt{75} = \sqrt{(3 \times 5 \times 5)} = 5\sqrt{3}$

Simplify the following.

$\sqrt{12}$ = ...	$5\sqrt{18}$ = ...	$\sqrt{(3/4)}$ = ...
$\sqrt{48}$ = ...	$10\sqrt{96}$ = ...	$\sqrt{(1/2)}$ = ...
$\sqrt{20}$ = ...	$\sqrt{(4/5)}$ = ...	$\sqrt{(3/2)}$ = ...

Addition and Subtraction

This is very similar to algebraic addition and subtraction, only those terms with the same square root part can be added or subtracted.

eg $2\sqrt{5} + 3\sqrt{5} = 5\sqrt{5}$ $3\sqrt{5} - 2\sqrt{5} = \sqrt{5}$

$2\sqrt{3} + 3\sqrt{5}$ cannot be added together, but this looks perfectly acceptable as a solution.

Add $10\sqrt{(1/5)} + 4\sqrt{18} + 3\sqrt{45} - 8\sqrt{(1/2)}$

....

....

....

....

[Solution : $11\sqrt{5} + 8\sqrt{2}$]

Multiplication and Division

Again, this can be compared to algebraic methods but remember to simplify your solution if possible.

eg $\sqrt{4} \times \sqrt{5} = \sqrt{(4 \times 5)} = \sqrt{20} = 2\sqrt{5}$
$2\sqrt{3} \times 7\sqrt{5} = (2 \times 7)\sqrt{(3 \times 5)} = 14\sqrt{15}$

When it comes to division the same rules apply, but if a solution is left as a fraction, the denominator cannot remain irrational. The method to use is called **rationalisation** which is to multiply the numerator and denominator by the denominator, thus removing the square root in the denominator.

eg $\dfrac{8\sqrt{30}}{2\sqrt{5}} = 8/2\sqrt{30/5} = 4\sqrt{6}$

$$\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5}\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{\sqrt{5}\sqrt{3}}{3} = \frac{\sqrt{15}}{3}$$

Simplify the following.

$\sqrt{3} \times \sqrt{6} =$ .... $3\sqrt{5} \times \sqrt{7} =$ ....

$\dfrac{\sqrt{27}}{\sqrt{3}} =$ .... $\dfrac{\sqrt{60}}{\sqrt{5}} =$ ....

[Solutions: $2\sqrt{3}$, $3\sqrt{35}$, 3, $2\sqrt{3}$]

Whatever the form of the denominator, if it contains an irrational number, the whole fraction must be rationalised. Notice that in the example below, the denominator ($\sqrt{5} + 1$) is multiplied by ($\sqrt{5} - 1$). This is because it makes the arithmetic easier since we have a *difference of two squares*.

eg $$\frac{\sqrt{3} + 2}{\sqrt{5} + 1} = \frac{(\sqrt{3} + 2)(\sqrt{5} - 1)}{(\sqrt{5} + 1)(\sqrt{5} - 1)} = \frac{\sqrt{3}\sqrt{5} - \sqrt{3} + 2\sqrt{5} - 1}{\sqrt{5}\sqrt{5} - 1} = \frac{\sqrt{15} - \sqrt{3} + 2\sqrt{5} - 1}{4}$$

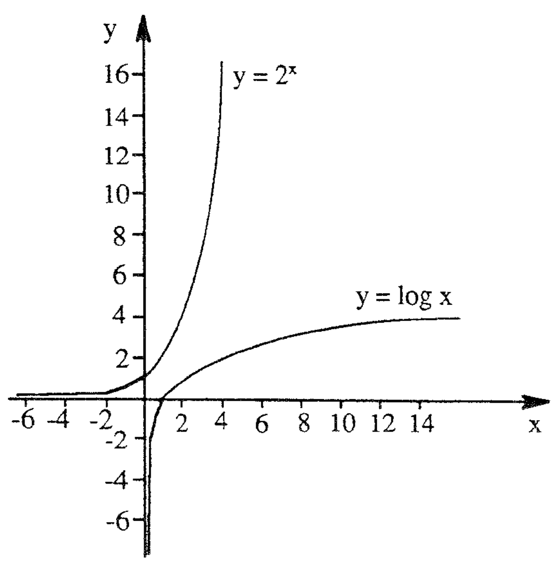
This one looks better with the numerator factorised.

ie $$\frac{(\sqrt{3} + 2)(\sqrt{5} - 1)}{4}$$

3.3 Logarithms

Below is the graph of the function y = 2x. As you can see, it stays above the x axis at all times. As x tends towards -∞ the function approaches zero, this is called an *asymtote.* As x tends towards +∞ the function becomes very large, very quickly.

This function has an inverse, $y = \log_2 x$ and this is a reflection of the function y = 2x in the line y = x.



An example of the graph of a logarithmic function and its inverse

In general we have the function $x = a^y$ so that $y = \log_a x$.

Some Rules of Logarithms

These rules may be likened to the rules for indices.

$$\log_a b + \log_a c = \log_a (bc)$$
$$\log_a b - \log_a c = \log_a (b/c)$$
$$\log_a b^d = d \log_a b$$
$$\log_a a = 1$$
$$\log_a (a^b) = a^{\log b}$$

Note that all manipulations are carried out in the same base. The base of the logarithm is a and it is important to notice because the solutions differ for different bases.

The reason for logarithms is simple, they save us a lot of very complicated arithmetic.

eg to solve $3 \times 4^x = 7$

Take logs to the base four on both sides.

$$\log_4 (3 \times 4^x) = \log_4 7$$
$$\log_4 3 + x \log_4 4 = \log_4 7$$
$$x \log_4 4 = \log_4 7 - \log_4 3$$
$$x = \log_4 (7/3)$$

Solve the equations below.

$10^x = 4$	$22x + 6 = 45 - x$
....	....
....	....
....	....
....	....

[Solutions: $x = \log_{10} 4$, $x = 1$]

Changing the Base

The formula below enables us to change the base of a logarithm. This is sometimes neccessary when the equation cannot be written as a logarithmic equation in a single base.

$$\log_a b \times \log_b c = \log_a c$$

When we carry out logarithmic calculations we use the base of ten because this is the base of our number system and apart from the base e, it is the only base on most calculators. Logarithms to the base of ten are written log x with the figure 10 generally omitted because the base is understood to be ten.

Natural Logarithms

In calculus we use the natural logarithm which has the base e. It is written as ln x and is present on most calculators. The inverse of ln x is e^x, sometimes written exp x, and $y = \ln x$ if $x = e^y$. This is called **the Exponential Function**, and has applications throughout science, technology, business, etc. eg. Exponential growth of capital under compound interest, exponential decay of a radioactive isotope etc.

Summary

The topics covered in this unit are more tools for using in mathematics.

The general rules of indices state that

$$a^n \, . \, a^m = a^{n+m}$$
$$(a^n)m = a^{nm}$$
$$(a^n) \, / \, (a^m) = a^{n-m}$$
$$(ab)^n = a^n b^n$$

Note that $ab^3 = a.b.b.b$

$(ab)^3 = ab.ab.ab$

Generally, surds can be expressed in index form, so these rules also apply to surds. The most imortant thing to remember about surds is that they give the precise value and any subsequent calculations using the surd will be accurate.

Logarithms are an important tool, especially when dealing with calculus. However, they are very useful in algebraic situations when they can really save time by simplifying complicated calculations.

Activities

1. Simplify the following.
   (a) $a^2 a^{-4}$ (b) a^{-4} / a^{-7} (c) $(a^{-3}b^{-2})^{-4}$
   (d) $(10^4 \times 10^3)$ (e) $a^b \div a^2$

2. Divide and simplify $\dfrac{\sqrt{(x^3y^3)} + \sqrt{(x^2y^2)} - \sqrt{(xy)}}{\sqrt{(xy)}}$

3. Solve (a) $4x^2 - 27 = x^2$ (b) $x^2 = 24$

4. Find the value of x given that $\dfrac{4^{3+x}}{8^{10x}} = \dfrac{2^{10-2x}}{64^{3x}}$

5. Given that log 2 = 0.301, log 3 = 0.477 and log 5 = 0.699, find
   (a) $\log_2 3$ (b) $\log_3 2$ (c) $\log_2 5$ (d) $\log_2 6$ (e) log (2.4)

7. Solve the simultaneous equations given that $\log_y x = \dfrac{1}{\log_x y}$

$$\log_x y + 2 \log_y x = 3$$
$$\log_9 y + \log_9 x = 3$$

8. Without using a calculator find
   (a) log 125 / log 5 (b) $\log_7 7 \,.\, \log_7 49$

9. If $\log(x^3y^2) = 9$ and $\log(x/y) = 2$, find without using a calculator,
   (a) log x (b) log y

10. Solve $3^x.3^{2x+3} = 10$ correct to two decimal places.

11. Given that ln y = 1-3ln x write an equation in x and y without using logarithms.

[Solutions:

1 (a) a^{-2}; (b) a^3; (c) a^4b^3; (d) 10^7; (e) $a^{b/2}$

2. $xy + \sqrt{(xy)} - 1$.

3 (a) $x = \pm 3$; (b) $x = \pm 2\sqrt{6}$.

4 $x = -1/2$.

5 $e(1 - e^x)$.

6 (a) 1.58; (b) 0.63; (c) 2.32; (d) 2.58; (e) 0.38.

7 (27, 27) or (9, 81).

8 (a) 3; (b) 2.

9 (a) 13/5; (b) 3/5.

10 -0.030.

11 $y = ex^{-3}$.]