University of the Peloponnese
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DIGITAL SIGNAL PROCESSING
Solved Examples
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SET #8 - Discrete Fourier Transform

- Discrete Fourier Transform (DFT)

- Relationship between DFT and other Transforms

- Calculation DFT with Linear Algebra

- Periodic sequence expansion - Periodic convolution

- Circular sequence shift - Circular convolution

1. Discrete Fourier Transform (DFT)

&Example 1

Compute the N-point DFT of the exponential sequence $x[n] = \alpha^n$, όπου $0 \le n \le N - 1$.

Answer: The DFT can be calculated from the definition as follows:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk} = \sum_{n=0}^{N-1} \alpha^n\, W_N^{nk} = \sum_{n=0}^{N-1} \left(\alpha W_N^k\right)^n = \frac{1 - \left(\alpha W_N^k\right)^N}{1 - \alpha W_N^k}$$
$$= \frac{1 - \alpha^N}{1 - \alpha W_N^k}, k = 0,1, \dots, N - 1$$

&Example 2

Calculate the N-point $x[n] = e^{j2\pi k_0 n/N}$, όπου $0 \le n \le N - 1$ DFT of the signal.

Answer: The DFT can be calculated from its definition as follows:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} e^{j2\pi k_0 n/N} e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} e^{-j2\pi(k-k_0)n/N}$$
$$= \frac{1 - e^{-j2\pi(k-k_0)}}{1 - e^{-j2\pi(k-k_0)/N}}, k = 0,1, \dots, N - 1$$

For $k \ne k_0$ the numerator takes a value of zero, so $X[k] = 0$. For $k = k_0$ the numerator and the denominator take a value of zero, so by the Del'Hospital rule we find:

$$X[k] = \lim_{k \to k_0} \frac{d}{dk}\left[\frac{1 - e^{-j2\pi(k-k_0)}}{1 - e^{-j2\pi(k-k_0)/N}}\right] = \left.\frac{2\pi j\, e^{-j2\pi(k-k_0)}}{\frac{2\pi j}{N} e^{\frac{j2\pi(k-k_0)}{N}}}\right|_{k=k_0} = N$$

Therefore:

$$X[k] = \begin{cases} 0, k \neq k_0 \\ N, k = k_0 \end{cases} = N\,\delta[k - k_0]$$

2. Relationship between DFT and other Transforms

> 📖**Example 3**
>
> (a) For the pulse $x[n] = u[n] - u[n-4]$ calculate the 4-point DFT and compare it with the DTFT (see Example 9.7.b).
>
> (b) Repeat the solution for 8-points DFT, after applying the procedure of zero-padding in sequence $x[n]$.

Answer: (a) The sequence is written: $x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$. We compute the 4-point DFT from the definition of DFT:

$$X[k] = \sum_{n=0}^{3} x[n]\, W_4^{nk} = W_4^0 + W_4^{1k} + W_4^{2k} + W_4^{3k},\, 0 \leq k \leq 3$$

We will calculate the points $X[k]$ for $0 \leq k \leq 3$. Taking into account the values of the phase factors we calculated in section 12.4.3, we have:

$$X[0] = W_4^0 + W_4^0 + W_4^0 + W_4^0 = 1 + 1 + 1 + 1 = 4$$
$$X[1] = W_4^0 + W_4^1 + W_4^2 + W_4^3 = 1 - j - 1 + j = 0$$
$$X[2] = W_4^0 + W_4^2 + W_4^4 + W_4^6 = 1 - 1 + 1 - 1 = 0$$
$$X[3] = W_4^0 + W_4^3 + W_4^6 + W_4^9 = 1 + j - 1 - j = 0$$

Hence the 4-point DFT of the pulse $x[n] = u[n] - u[n-4]$ given by:

$$X[k] = \begin{cases} 4, & n = 0 \\ 0, n = 1,2,\dots \end{cases}$$

From Example 10.8 it follows that the DTFT of the pulse $x[n] = u[n] - u[n-4]$ is:

$$X(e^{j\omega}) = e^{-\frac{j3\omega}{2}}\frac{\sin(2\omega)}{\sin(\omega/2)}$$

DTFT magnitude spectrum plot is shown in red in next figure, while the DFT points are shown in blue. We notice that the points of the DFT are obtained by sampling from the DTFT according to the equation $X[k] = X(e^{j\omega})\big|_{\omega = \pi k/2}, k = 0,1,2,3$.
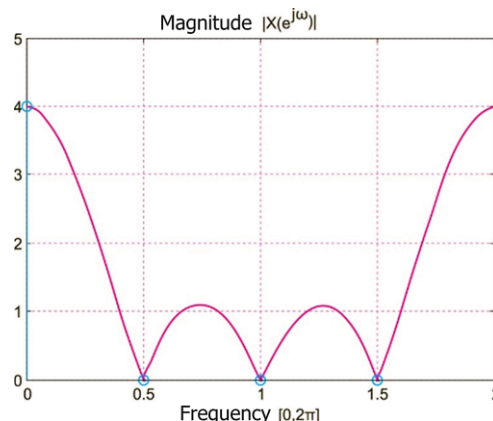
DTFT (red color) and 4-point DFT (blue color) magnitude and phase spectra of the pulse $x[n] = u[n] - u[n-4]$ in the frequency domain$[0,2\pi)$

(b) The 8-point DFT of the pulse $x[n] = u[n] - u[n-4]$ is:

$$X[k] = \sum_{n=0}^{7} x[n]\, W_8^{nk}, 0 \le k \le 7$$

We calculate the points $X[k]$ for $0 \le k \le 7$:

$X[0] = 1W_8^0 + 1W_8^0 + 1W_8^0 + 1W_8^0 + 0W_8^0 + 0W_8^0 + 0W_8^0 + 0W_8^0 = 4$

$X[1] = 1W_8^0 + 1W_8^1 + 1W_8^2 + 1W_8^3 + 0W_8^4 + 0W_8^5 + 0W_8^6 + 0W_8^7 = 1 - 2.41j$

$X[2] = 1W_8^0 + 1W_8^2 + 1W_8^4 + 1W_8^6 + 0W_8^8 + 0W_8^{10} + 0W_8^{12} + 0W_8^{14} = 0$

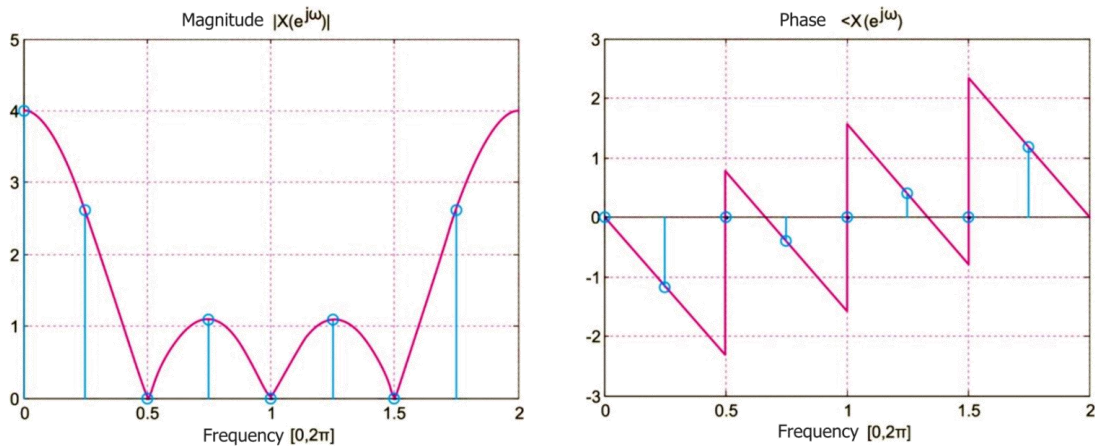$X[3] = 1W_8^0 + 1W_8^3 + 1W_8^6 + 1W_8^9 + 0W_8^{12} + 0W_8^{15} + 0W_8^{18} + 0W_8^{21} = 1 - 0.41j$

$X[4] = 1W_8^0 + 1W_8^4 + 1W_8^8 + 1W_8^{12} + 0W_8^{16} + 0W_8^{20} + 0W_8^{24} + 0W_8^{28} = 0$

$X[5] = 1W_8^0 + 1W_8^5 + 1W_8^{10} + 1W_8^{15} + 0W_8^{20} + 0W_8^{25} + 0W_8^{30} + 0W_8^{35} = 1 + 0.41j$

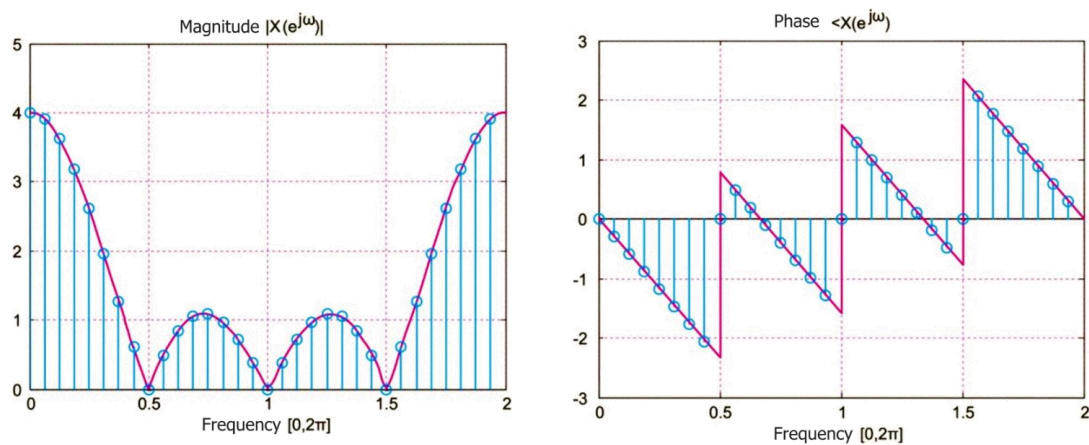$X[6] = 1W_8^0 + 1W_8^6 + 1W_8^{12} + 1W_8^{18} + 0W_8^{24} + 0W_8^{30} + 0W_8^{36} + 0W_8^{42} = 0$

$X[7] = 1W_8^0 + 1W_8^7 + 1W_8^{14} + 1W_8^{21} + 0W_8^{28} + 0W_8^{35} + 0W_8^{42} + 0W_8^{49} = 1 + 2.41j$

and we get the figure:

DTFT (red color) and 8-point DFT (blue color) magnitude and phase spectra of the pulse $x[n] = u[n] - u[n-4]$ in the frequency domain $[0,2\pi)$

We repeat for 32-point DFT and obtain the form:

DTFT (red color) and 32-point DFT (blue color) magnitude and phase spectra of the pulse $x[n] = u[n] - u[n-4]$ in the frequency domain $[0,2\pi)$

- From the above figures we find that the magnitude and phase of the DFT have even and odd symmetry, respectively. Therefore, we can keep only the frequency part $[0, \pi)$, i.e. the values $X[k], k = 0,1, \dots, (N/2) - 1$, because the information of the part $[-\pi, 0)$ is identical.

- As we mentioned above, if for a given aperiodic sequence $x[n]$ we wish to increase the density of DFT samples, then we must **increase the number** of samples of $x[n]$. This is done by **adding zeros** (zero-padding), by applying the following procedure:

  - We create a periodic extension $\tilde{x}[n]$ of $x[n]$ length $L \geq N$, adding $L - N$ zeros to its end.

  - We compute the L-point DFT $\tilde{X}[k]$ of the sequence $\tilde{x}[n]$.

  - DFT $X[k]$ of sequence $x[n]$ is: $X[k] = \tilde{X}[k]$, για $0 \leq k \leq L - 1$.

- Adding the zeros does not change anything in the sum of equation (12.9), so it does not improve the **sharpness** of the DFT, it just reduces the distance between successive samples of $X[k]$. To increase the sharpness of the DTF, **more data must be obtained from the signal,** that is, the number of samples of the signal must be increased.

- If the sequence is **periodic,** then to increase the sharpness of the DFT, we do not add zeros but include in the DFT calculation more than one period of the sequence.

📖**Example 4**

Calculate the N-points DFT of the sequence $x[n] = \delta[n - n_0]$, where $0 < n_0 < N$, by sampling in the Z-transform.

<u>Answer:</u> From Table 9.1 we know that for the sequence $\delta[n - n_0]$:

$$\delta[n - n_0] \overset{Z}{\leftrightarrow} X(z) = z^{-n_0}$$

The region of convergence is the entire field z, except 0 when $n_0 > 0$. So, the unit circle is inside the region of convergence of the Z-transform, so by sampling the function $X(z)$ at the points $z = W_N^{-k}$ for $k = 0,1, \dots, N - 1$, we find:

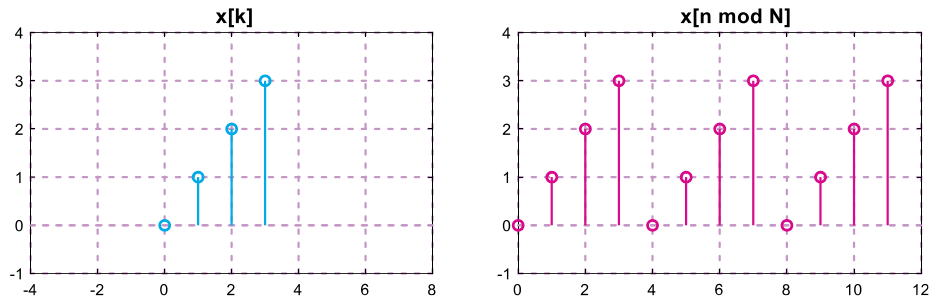$$X[k] = W_N^{n_0 k}, k = 0,1, \dots, N - 1$$

Alternative spelling:

$$X[k] = \left[1, W_N^{n_0}, W_N^{2n_0}, \dots, W_N^{(N-1)n_0}\right]$$

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3.   Periodic sequence expansion - Periodic convolution
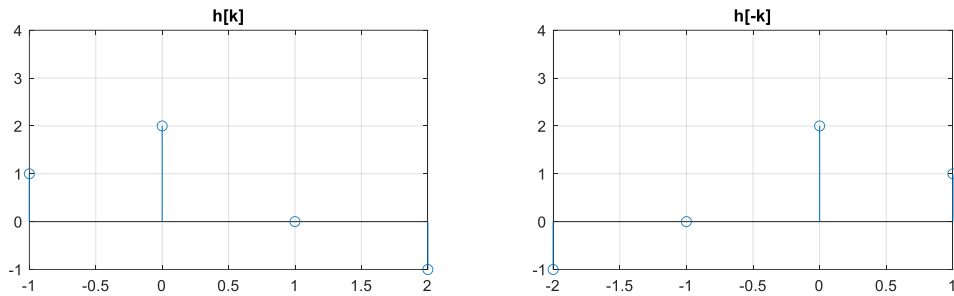
📖 **Example 5**

Calculate the periodic convolution between the discrete-time signals $x[n] = \{\hat{0}, 1, 2, 3\}$ and $h[n] = \{1, \hat{2}, 0, -1\}$.

<u>Answer:</u> The graph of $x[n]$ and its periodic expansion $\tilde{x}[n]$ for $N = 4$ is shown in the figure:

Sequence of finite length $x[n]$ and its periodic expansion $\tilde{x}[n] = x[n \bmod 4] = x\big[[n]\big]_4$


Sequence $h[k]$ and its reflection $h[-k]$

We use the graphical way of calculating the convolutional sum:

$$\tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\, \tilde{h}[n-k]$$

The graph of $h[n]$, its reflection $h[-k]$ and its periodic expansion $\tilde{h}[-k]$ for $N = 4$, shown in the next figure. The $\tilde{y}[0]$ is founded by summing the products $\tilde{x}[k]\,\tilde{h}[-k]$ for $k = 0$ up to 3. Is: $\tilde{y}[0] = 0x2 + 1x1 + 2x(-1) + 3x0 = -1$.

Then, $\tilde{h}[-k]$ is shifted to the right by 1, resulting in $\tilde{h}[1-k]$. The $\tilde{y}[1]$ is found by summing the products $\tilde{x}[k]\,\tilde{h}[1-k]$ for $k = 0$ up to 3. Is: $\tilde{y}[1] = 0x0 + 1x2 + 2x1 + 3x(-1) = 1$.
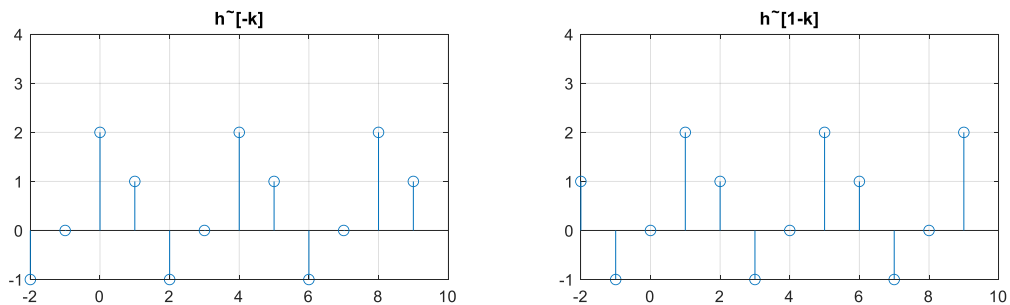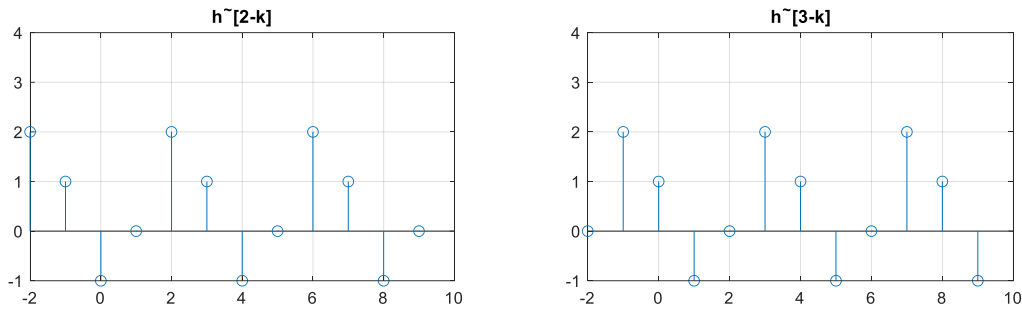

Periodic extension $h\big[[-k]\big]_4$ and shifted periodic extension $\tilde{h}[1-k] = h\big[[1-k]\big]_4$

Shifting $\tilde{h}[-k]$ to the right by 2 yields $\tilde{h}[2-k]$, figure 12.13 (a). The $\tilde{y}[2]$ is found by summing the products $\tilde{x}[k]\,\tilde{h}[2-k]$ for $k = 0$ up to 3. Is: $\tilde{y}[2] = 0x(-1) + 1x0 + 2x2 + 3x1 = 7$.
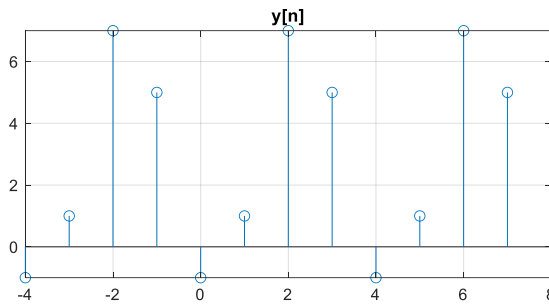
Finally, shifted $\tilde{h}[-k]$ to the right by 3, resulting in $\tilde{h}[3-k]$. The $\tilde{y}[3]$ is founded by

summing the products $\tilde{x}[k]\,\tilde{h}[3-k]$ for $k=0$ up to 3. Is: $\tilde{y}[3]=0x1+1x(-1)+2x0+3x2=5$.

Shifted periodic extension $\tilde{h}[2-k]=h\big[[2-k]\big]_4$
and shifted periodic extension $\tilde{h}[3-k]=h\big[[3-k]\big]_4$

Therefore, the periodic convolution is: $\tilde{y}[n]=\{-1,1,7,5,-\hat{1},1,7,5,-1,1,7,5\}$. The result is shown in the figure:

Periodic convolution $\tilde{y}[n]$

4. Shifted Circular sequence - Circular convolution

📖 Example 6

To calculate the circular convolution of 4 points between the discrete time signals $x[n]=\{\hat{0},1,2,3\}$ and $h[n]=\{1,\hat{2},0,-1\}$.

Answer: We calculate the circular convolution of four points from the equation:

$$y[n]=\left[\sum_{k=0}^{3}x[k]\,\tilde{h}[n-k]\right]R_4[n]$$

For $n=0$:

$$y[0]=\left[\sum_{k=0}^{3}x[k]\,\tilde{h}[-k]\right]R_4[n]=\sum_{k=0}^{3}\{\hat{0},1,2,3\}\{\hat{2},1,-1,0\}$$
$$=\sum_{k=0}^{3}\{\hat{0},1,-2,0\}\Rightarrow y[0]=-1$$

For $n=1$:

$$y[1] = \left[\sum\nolimits_{k=0}^{3} x[k]\,\tilde{h}[1-k]\right]R_4[n] = \sum\nolimits_{k=0}^{3} \{\hat{0}, 1, 2, 3\}\{\hat{0}, 2, 1, -1\}$$

$$= \sum\nolimits_{k=0}^{3} \{\hat{0}, 2, 2, -3\} \Rightarrow y[1] = 1$$

For $n = 2$:

$$y[2] = \left[\sum\nolimits_{k=0}^{3} x[k]\,\tilde{h}[2-k]\right]R_4[n] = \sum\nolimits_{k=0}^{3} \{\hat{0}, 1, 2, 3\}\{-\hat{1}, 0, 2, 1\}$$

$$= \sum\nolimits_{k=0}^{3} \{\hat{0}, 0, 4, 3\} \Rightarrow y[2] = 7$$

For $n = 3$:

$$y[3] = \left[\sum\nolimits_{k=0}^{3} x[k]\,\tilde{h}[3-k]\right]R_4[n] = \sum\nolimits_{k=0}^{3} \{\hat{0}, 1, 2, 3\}\{\hat{1}, -1, 0, 2\}$$

$$= \sum\nolimits_{k=0}^{3} \{\hat{0}, -1, 0, 6\} \Rightarrow y[3] = 5$$

Therefore it is:

$$y[n] = h[n]\,④\,x[n] = \{-\hat{1}, 1, 7, 5\}$$

We observe that it verifies the equation $y[n] = x_1[n]\,Ⓝ\,x_2[n] = [\tilde{x}_2[n] \circledast \tilde{x}_1[n]]R_N[n]$. The linear convolution between $h[n]$ and $x[n]$, is the following sequence of six points:

$$h[n] * x[n] = \{1, \hat{4}, 7, 5, -2, -3\}$$

We notice that linear convolution and circular convolution of the same sequences give different results.