University of the Peloponnese
Electrical and Computer
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DIGITAL SIGNAL PROCESSING
Solved Examples
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SET #6 – Inverse Z-Transform – Studying Discrete-Time Systems with the Z- Transform

- Inverse Z-Transform
- Relationship between Transfer Function and Difference Equation
- Solving Differential Equations
- Solving Differential Equations

1. Inverse Z-Transform

> 📖 **Example 1**
>
> Calculate the inverse Z-transform of the function:
>
> $$X(z) = \frac{3}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

<u>Answer:</u> The given Z-Transform is the sum of two exponential functions of the first degree, that is, it is already in the form of a sum of simple fractions. The poles of the transformations are $z_1 = 1/2$ and $z_2 = 1/3$. Because the region of convergence has not been determined, there are three possible cases of regions of convergence, as shown in the figure.
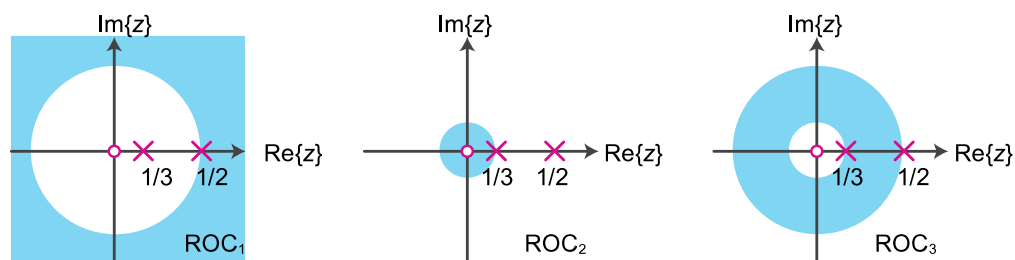
Regions of convergence (ROC)

(a) Region of Convergence $ROC_1: 1/2 < |z| < \infty$: Since the region of convergence of the function $X(z)$ is the outer surface of a circle and its poles lie on the inner side of the circle, it follows that the sequence $x[n]$ is right-sided (causal sign). Using the corresponding pair for right-hand exponential sequences from Table 9.1 or equation (9.41), we find:

$$x[n] = 3\left(\frac{1}{2}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n] = \left\{3\left(\frac{1}{2}\right)^n + 2\left(\frac{1}{3}\right)^n\right\} u[n]$$

(b) Region of Convergence ROC_2: $0 < |z| < 1/3$: Since the region of convergence of the $X(z)$function is the inner surface of a circle and its poles are on the outer side of the circle, the sequence $x[n]$is left-sided (anticausal sign). Using the corresponding pair for left-hand exponential sequences from Table 9.1 or equation (9.41), we find:

$$x[n] = -3\left(\frac{1}{2}\right)^n u[-n-1] - 2\left(\frac{1}{3}\right)^n u[-n-1] = -\left\{3\left(\frac{1}{2}\right)^n + 2\left(\frac{1}{3}\right)^n\right\} u[-n-1]$$

(c) Region of Convergence ROC_3: $1/3 < |z| < 1/2$: Since the region of convergence of the function $X(z)$is the inner surface of a circular ring, the pole $z_1$is on the outer side of the great circle while the pole $z_2$is on the inner side of the small circle, the sequence $x[n]$is double sided formed by the sum a left side sequence and a right side sequence. Similarly to above, we find:

$$x[n] = -3\left(\frac{1}{2}\right)^n u[-n-1] + 2\left(\frac{1}{3}\right)^n u[n]$$

📖**Example 2**

To calculate the convolution of the signals:

$$x[n] = 3^n u[-n] \text{ και } h[n] = \left(\frac{1}{2}\right)^n u[n]$$

<u>Answer:</u> The sequence $h[n]$is right-sided (causative) and has a Z-transformation:

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad ROC(H): |z| > \frac{1}{2}$$

The sequence is left-handed (anti-causal) and the $x[n]$Z- transform can be found using the time-shift and time-reversal properties:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{0} 3^n z^{-n} = \sum_{n=0}^{+\infty} \left(\frac{1}{3}z\right)^n$$

$$= \frac{1}{1 - \frac{1}{3}z} = -\frac{3z^{-1}}{1 - 3z^{-1}}, ROC(X): |z| < 3$$

So, the Z- transform of the convolution $y[n] = h[n] * x[n]$, is:

$$Y(z) = -\frac{1}{1 - \frac{1}{2}z^{-1}} \cdot \frac{3z^{-1}}{1 - 3z^{-1}}$$

The region of convergence is $ROC(Y) = ROC(X) \cap ROC(H)$, for which $|z| > 1/2$ and $|z| < 3$. Therefore, the region of convergence of the sequence $y[n]$ is $ROC(Y): 1/2 < |z| < 3$.

Because of the particular shape of the region of convergence we expect the sequence $y[n]$to be the sum of a right-hand side sequence and a left-hand side sequence.

Its $Y(z)$ inverse Z- transform is obtained by expanding the function $Y(z)$ into some fractions:

$$Y(z) = \frac{R_1}{1 - \frac{1}{2}z^{-1}} + \frac{R_2}{1 - 3z^{-1}} \quad (1)$$

where the rest R_1 and R_2 of the polynomial division for the corresponding poles are given by relations (9.38) and are:

$$R_1 = \left[\left(1 - \frac{1}{2}z^{-1}\right)Y(z)\right]_{z=\frac{1}{2}} = \left[\left(1 - \frac{1}{2}z^{-1}\right)\frac{1}{1 - \frac{1}{2}z^{-1}} \cdot \frac{3z^{-1}}{1 - 3z^{-1}}\right]_{z=\frac{1}{2}}$$

$$= \left[\frac{3z^{-1}}{1 - 3z^{-1}}\right]_{z=\frac{1}{2}} = \frac{6}{5}$$

$$R_2 = [(1 - 3z^{-1})\,Y(z)]_{z=3} = \left[(1 - 3z^{-1})\frac{1}{1 - \frac{1}{2}z^{-1}} \cdot \frac{3z^{-1}}{1 - 3z^{-1}}\right]_{z=3}$$

$$= \left[\frac{3z^{-1}}{1 - \frac{1}{2}z^{-1}}\right]_{z=3} = -\frac{6}{5}$$

Considering that the first fraction of $Y(z)$ (equation 1) is left-hand side, while the second is right-hand side and using Table 9.1 or equation (9.31), we obtain:

$$y[n] = \left(\frac{6}{5}\right)\left(\frac{1}{2}\right)^n u[n] + \left(\frac{6}{5}\right)3^n u[-n-1]$$

2. Relationship between Transfer Function and Difference Equation

📖Example 3

An initially relaxed LSI system has a transfer function:

$$H(z) = \frac{z - 1}{z^2 - z + 0.25}$$

Calculate:

(a) The difference equation describing the system.

(b) The impulse response of the system.

(c) The system output for input $x[n] = u[n]$

Answer: (a) We will calculate the transfer function from the equation $H(z) = Y(z)/X(z)$, so we have:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z - 1}{z^2 - z + 0.25} = \frac{z^{-1} - z^{-2}}{1 - z^{-1} + 0.25z^{-2}}$$

We cross-multiply the fractions and get:

$$Y(z) - z^{-1}Y(z) + 0.25z^{-2}Y(z) = z^{-1}X(z) - z^{-2}X(z)$$

We apply inverse Z- transformation and get the LDECC:

$$y[n] - y[n-1] + 0.25y[n-2] = x[n-1] - x[n-2] \Rightarrow$$

$$y[n] = y[n-1] - 0.25y[n-2] + x[n-1] - x[n-2]$$

(b) Because in the expression of the transfer function $H(z)$ in terms z^{-n} the degree of the numerator is the same as the degree of the denominator we will develop the function in some fractions $\tilde{H}(z)$:

$$\tilde{H}(z) = \frac{H(z)}{z} = \frac{z-1}{z(z^2 - z + 0.25)} = \frac{(z-1)}{z(z-0.5)^2}$$

The development is:

$$\tilde{H}(z) = \frac{H(z)}{z} = \frac{R_1}{z} + \frac{R_2}{(z-0.5)^2} + \frac{R_3}{(z-0.5)} \qquad (1)$$

Using equation (9.38) we find the rest R_1 and R_2:

$$R_1 = \left[z\,\tilde{H}(z)\right]_{z=0} = \left[\frac{(z-1)}{(z-0.5)^2}\right]_{z=0} = -4$$

$$R_2 = \left[(z-0.5)^2\,\tilde{H}(z)\right]_{z=0.5} = \left[\frac{(z-1)}{z}\right]_{z=0.5} = -1$$

To find the remainder R_3 we substitute random values of z, which must not be poles. We put $z = 1$ in equation (1) and we have:

$$\tilde{H}(1) = \frac{H(1)}{1} = \frac{(1-1)}{1(1-0.5)^2} = 0 \Rightarrow \frac{-4}{1} + \frac{-1}{(1-0.5)^2} + \frac{R_3}{(1-0.5)} = 0 \Rightarrow$$
$$= -4 - 4 + \frac{R_3}{0.5} \Rightarrow R_3 = 4$$

Therefore:

$$H(z) = \frac{R_1\,z}{z} + \frac{R_2\,z}{(z-0.5)^2} + \frac{R_3\,z}{(z-0.5)}$$
$$= -4 - 2\frac{0.5\,z}{(z-0.5)^2} + 4\frac{z}{(z-0.5)}$$

Based on Table 9.1 the impulse response is:

$$h[n] = -4\delta[n] - 2n(0.5)^n\,u[n] + 4(0.5)^n\,u[n]$$

(c) We can compute the output by either convolution $y[n] = x[n] * h[n]$ or inverse transformation $y[n] = Z^{-1}\{X(z)H(z)\}$. We will follow the second way. The Z-transform of the input is:

$$X(z) = \frac{1}{1-z^{-1}} = \frac{1}{z-1}, |z| > 1$$

Therefore:

$$Y(z) = X(z)H(z) = \frac{1}{z-1} \cdot \frac{z-1}{z^2 - z + 0.25} = \frac{1}{z^2 - z + 0.25} = \frac{z^{-2}}{(1 - z^{-1} + 0.25z^{-2})}$$
$$= \frac{z^{-2}}{(1 - 0.5z^{-1})^2}$$

Since the expression in the denominator is quadratic we use equation (9.39) and obtain the expansion:

$$Y(z) = \frac{R_1}{1 - 0.5z^{-1}} + \frac{R_2 z^{-1}}{(1 - 0.5z^{-1})^2}, (1)$$

We calculate it R_2 from the equation (9.38):

$$R_2 = [(1 - 0.5z^{-1})^2 \, Y(z)]_{z=2} = [z^{-2}]_{z=2} = \frac{1}{4}$$

To calculate it R_1 we choose a value of z which is not a pole of the function $Y(z)$. Let's say that $z = 1$. We calculate it $Y(1)$ from the definition. Is:

$$Y(1) = \frac{1^{-2}}{(1 - 0.5 \, 1^{-1})^2} = \frac{1}{0.5^2} = 4$$

We calculate it $Y(1)$ from the expansion (equation 1). Is:

$$Y(1) = \frac{R_1}{1 - 0.5 \, 1^{-1}} + \frac{R_2 \, 1^{-1}}{(1 - 0.5 \, 1^{-1})^2} = \frac{R_1}{0.5} + \frac{0.25}{0.25} = \frac{R_1}{0.5} + 1$$

Apply:

$$\frac{R_1}{0.5} + 1 = 4 \Rightarrow R_1 = \frac{3}{2}$$

So the development is:

$$Y(z) = \left(\frac{3}{2}\right)\frac{1}{1 - 0.5z^{-1}} + \left(\frac{1}{2}\right)\frac{0.5 \, z^{-1}}{(1 - 0.5z^{-1})^2}$$

and from Table 9.1 we find:

$$y[n] = \frac{3}{2}(0.5)^n \, u[n] + \frac{1}{2}(0.5)^n \, nu[n] = \frac{1}{2}[3 + n] \, (0.5)^n \, u[n]$$

📖**Example 4**

To solve the LDECC with zero initial conditions and input $x[n] = u[n]$:

$$y[n] = 1.6 \, y[n - 1] - 0.64 \, y[n - 2] + x[n]$$

Answer: We write the LDECC, transferring the terms $y[\,]$ to the left member:

$$y[n] - 1.6 \, y[n - 1] - 0.64 \, y[n - 2] = x[n]$$

The Z-transform of the input is:

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{1}{z - 1}, |z| > 1$$

We calculate the Z transformation of both members of the LDECC. Using the time shift property of the Z-transform we get:

$$Y(z) - 1.6 \, z^{-1}Y(z) - 0.64 \, z^{-2}Y(z) = X(z) \Rightarrow Y(z)[1 - 1.6z^{-1} - 0.64z^{-2}] = \frac{1}{1 - z^{-1}} \Rightarrow$$

$$Y(z) = \frac{1}{(1 - z^{-1})(1 - 1.6z^{-1} - 0.64z^{-2})} = \frac{1}{(1 - z^{-1})(1 - 0.8z^{-1})^2}$$

We use equation (9.39) and obtain the expansion:

$$Y(z) = \frac{R_1}{1 - z^{-1}} + \frac{R_2}{1 - 0.8z^{-1}} + \frac{R_3 z^{-1}}{(1 - 0.8z^{-1})^2}, (1)$$

We calculate R_1 and R_3 from the equation (9.38):

$$R_1 = [(1 - z^{-1})^2 \, Y(z)]_{z^{-1}=1} = \left[\frac{1}{(1 - 0.8z^{-1})^2}\right]_{z^{-1}=1} = \cdots = 25$$

$$R_3 = [(1 - 0.8z^{-1})^2 \, Y(z)]_{z^{-1}=1/0.8=1.25} = \left[\frac{1}{(1 - z^{-1})}\right]_{z^{-1}=1.25} = \cdots = -4$$

To calculate it R_2 we choose a value of z which is not a pole of the function $Y(z)$. Let's say that $z = 2$. We calculate it $Y(2)$ from the definition. Is:

$$Y(2) = \frac{1}{(1 - 2^{-1})(1 - 0.8 \, 2^{-1})} = \cdots = \frac{1}{0.18}$$

We calculate it $Y(2)$ from the expansion (equation 1). Is:

$$Y(2) = \frac{25}{1 - 2^{-1}} + \frac{R_2}{(1 - 0.8 \, 2^{-1})} - \frac{4 \, 2^{-1}}{(1 - 0.8 \, 2^{-1})^2} = \frac{25}{0.5} + \frac{R_2}{0.6} - \frac{2}{0.36}$$

Doing the operations, we find:

$$R_2 = -\frac{5}{6}$$

So the development is:

$$Y(z) = 25\frac{1}{1 - z^{-1}} + \left(-\frac{5}{6}\right)\frac{1}{1 - 0.8z^{-1}} + (-5)\frac{0.8 \, z^{-1}}{(1 - 0.8z^{-1})^2}$$

and from Table 9.1 we find:

$$y[n] = 25(1)^n u[n] - \frac{5}{6}(0.8)^n u[n] - 5(0.8)^n n \, u[n] = \left[25 - 5\left[n + \frac{1}{6}\right](0.8)^n\right] u[n]$$

3. Solving Differential Equations

> **📖 Example 5**
>
> An LSI system is described by LDECC:
>
> $$y[n] = 0.2y[n - 1] + 0.8y[n - 2] + x[n]$$
>
> Find the system's response to input $x[n] = (0.5)^n \, u[n]$ and initial conditions $y[-1] = 5$ and $y[-2] = 10$.

<u>Answer:</u> We calculate the one-sided transformation Z^+ of each of the LDECC terms:

$$Y^+(z) = 0.2\left[z^{-1}Y^+(z) + y[-1]\right] + 0.8\left[z^{-2} \, Y^+(z) + z^{-1}y[-1] + y[-2]\right] + X^+(z)$$

Substituting the values of the initial conditions, we have:

$$Y^+(z) = 0.2 \, [z^{-1} \, Y^+(z) + 5] + 0.8 \, [z^{-2} \, Y^+(z) + 5z^{-1} + 10] + X^+(z) \Rightarrow$$

$$Y^+(z) = 0.2 \, z^{-1} \, Y^+(z) + 1 + 0.8 \, z^{-2} \, Y^+(z) + 4z^{-1} + 8 + X^+(z)$$

Transferring the terms containing the $Y^+(z)$ to the left member, we have:

$$Y^+(z)[1 - 0.2z^{-1} - 0.8z^{-2}] = 9 + 4z^{-1} + X^+(z)$$

The sequence $x[n] = (0.5)^n u[n]$ has a one-sided transformation $X^+(z)$ where is given by the equation:

$$X^+(z) = \frac{1}{1 - 0.5z^{-1}}$$

So it holds:

$$Y^+(z)[1 - 0.2z^{-1} - 0.8z^{-2}] = (9 + 4z^{-1}) + \frac{1}{1 - 0.5z^{-1}}$$

Solving for $Y^+(z)$ we have:

$$Y^+(z) = Y_{zi}(z) + Y_{zs}(z) = \frac{(9 + 4z^{-1})}{1 - 0.2z^{-1} - 0.8z^{-2}} + \frac{\frac{1}{1 - 0.5z^{-1}}}{1 - 0.2z^{-1} - 0.8z^{-2}} \quad (1)$$

By adding the fractions and then factoring the denominator, we have:

$$Y^+(z) = \frac{10 - 0.5z^{-1} - 2z^{-2}}{(1 - 0.2z^{-1} - 0.8z^{-2})(1 - 0.5z^{-1})} = \frac{10 - 0.5\,z^{-1} - 2z^{-2}}{(1 - z^{-1})(1 + 0.8z^{-1})(1 - 0.5z^{-1})}$$

The region of convergence is $|z| > 1$ and the poles of the system are $z_1 = 1, z_2 = -0.8, z_3 = 0.5$. Because one pole lies on the unit circle (and the rest are inside the unit circle) the system is marginally stable.

For its development $Y^+(z)$ in some fractions we will need to calculate the remainders R_1, R_2 and R_3:

$$Y^+(z) = \frac{R_1}{1 - z^{-1}} + \frac{R_2}{1 + 0.8z^{-1}} + \frac{R_3}{1 - 0.5z^{-1}}$$

Since the poles are simple and distinct ( ), we $z_1 = 1, z_2 = -0.8, z_3 = 0.5$ also R_3 calculate remainders R_1, R_2 from the equation (9.38) and we have:

$$R_1 = [Y^+(z)(1 - z^{-1})]_{z=1} = \left[\frac{10 - 0.5\,z^{-1} - 2z^{-2}}{(1 + 0.8z^{-1})(1 - 0.5z^{-1})}\right]_{z=1} = \frac{25}{3}$$

$$R_2 = [Y^+(z)(1 + 0.8z^{-1})]_{z=-0.8} = \left[\frac{10 - 0.5\,z^{-1} - 2z^{-2}}{(1 - z^{-1})(1 - 0.5z^{-1})}\right]_{z=-0.8} = \frac{80}{39}$$

$$R_3 = [Y^+(z)(1 - 0.5z^{-1})]_{z=0.5} = \left[\frac{10 - 0.5\,z^{-1} - 2z^{-2}}{(1 - z^{-1})(1 + 0.8z^{-1})}\right]_{z=0.5} = -\frac{10}{26}$$

Therefore, its development $Y^+(z)$ in some fractions it is:

$$Y^+(z) = \left(\frac{25}{3}\right)\frac{1}{1 - z^{-1}} + \left(\frac{80}{39}\right)\frac{1}{1 + 0.8z^{-1}} + \left(-\frac{10}{26}\right)\frac{1}{1 - 0.5z^{-1}}$$

Performing an inverse Z-transform yields the requested solution:

$$y[n] = \left(\frac{25}{3}\right)(1)^n u[n] + \left(\frac{80}{39}\right)(-0.8)^n u[n] + \left(-\frac{10}{26}\right)(0.5)^n u[n]$$
$$= \left[\left(\frac{25}{3}\right) + \left(\frac{80}{39}\right)(-0.8)^n + \left(-\frac{10}{26}\right)(0.5)^n\right]u[n]$$

Comments: The overall solution can be expressed in the following ways:

- As a sum of **homogeneous solution** and the **partial solution** :

$$y[n] = \left[\left(\frac{25}{3}\right) + \left(\frac{80}{39}\right)(-0.8)^n\right]u[n] + \left(-\frac{10}{26}\right)(0.5)^n u[n]$$

The homogeneous solution is due to the poles and the partial solution to the zeros of the input signal.

- As the sum of the **transient state** and the **permanent state** :

$$y[n] = \left[\left(\frac{80}{39}\right)(-0.8)^n + \left(-\frac{10}{26}\right)(0.5)^n\right]u[n] + \left(\frac{25}{3}\right)u[n]$$

The transient state is due to the poles (single or multiple) lying within the unit circle and the permanent state is due to the simple poles lying on the unit circle. If there are poles (single or multiple) outside the unit circle, then the response tends to infinity.

- As the sum of the **response zero entry** (or initial state) and **response zero initial state** . Specifically, the above equation (1) is a sum of two terms. The first term can be written as:

$$Y_{zi}(z) = H(z)\, X_{ic}(z)$$

represents the response for the given input, assuming a zero initial state and is called the **zero-state response.** The function $X_{ic}(z)$ can be thought of as an equivalent initial state input which produces the same output $Y_{zi}(z)$ that is created by the initial conditions. In our example and based on equation (1) it follows: $x_{ic}[n] = \{\hat{9}, 4\}$.

The second term can be written as:

$$Y_{zs}(z) = H(z)\, X(z)$$

represents the response for zero input, with only the initial state applied, and is called the **zero-input response**.

Calculating the inverse Z-Transform of equation (1), we have (equation 11.28):

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

where $y_{zi}[n]$ is the **zero-input response** and is given by:

$$y_{zi}[n] = \left(\frac{65}{9}\right)(1)^n u[n] + \left(\frac{116}{45}\right)(-0.8)^n u[n] = \left[\left(\frac{65}{9}\right) + \left(\frac{116}{45}\right)(-0.8)^n\right]u[n]$$

and $y_{zs}[n]$ is the **zero-state response**, which is given by:

$$y_{zs}[n] = \left(\frac{10}{9}\right)(1)^n u[n] + \left(\frac{32}{117}\right)(-0.8)^n u[n] + \left(\frac{5}{13}\right)(0.5)^n u[n]$$

$$= \left[\left(\frac{32}{117}\right)(-0.8)^n + \left(\frac{5}{13}\right)(0.5)^n + \left(\frac{10}{9}\right)\right]u[n]$$

4. Solving Differential Equations

📖 Example 6

A continuous-time GHS system that is in a state of initial rest is described by GDESS:

$$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

To find the response of the system for input $x(t) = u[t]$.

<u>Answer:</u> Applying Laplace transform to the differential equation we find:

$$s^2 Y(s) + 5sY(s) + 6Y(s) = X(s) = \frac{1}{s} \Rightarrow Y(s) = \frac{1}{s(s^2 + 5s + 6)} = \cdots$$

$$= \left(\frac{1}{6}\right)\frac{1}{s} - \left(\frac{1}{2}\right)\frac{1}{s+2} + \left(\frac{1}{3}\right)\frac{1}{s+3}$$

Therefore the response for step input is:

$$y(t) = \left[\frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}\right]u(t)$$

Applying the first and second derivative approximations given in relations (11.29) and (11.30) to the differential equation, we then set $t = nT_s$ and obtain:

$$\frac{y(t) - 2y(t - T_s) + y(t - 2T_s)}{T_s^2} + 5\frac{y(t) - y(t - T_s)}{T_s} + 6y(t) = x(t)$$

$$\Rightarrow \left(6 + \frac{5}{T_s} + \frac{1}{T_s^2}\right)y(t) - \left(-\frac{5}{T_s} - \frac{2}{T_s^2}\right)y(t - T_s) + \left(\frac{1}{T_s^2}\right)y(t - 2T_s) = x(t)$$

$$\Rightarrow \left(6 + \frac{5}{T_s} + \frac{1}{T_s^2}\right)y(nT_s) - \left(-\frac{5}{T_s} - \frac{2}{T_s^2}\right)y((n-1)T_s) + \left(\frac{1}{T_s^2}\right)y((n-2)T_s) = x(nT_s)$$

$$\Rightarrow a_1 y(nT_s) + a_2 y((n-1)T_s) + a_3 y((n-2)T_s) = b_1 x(nT_s)$$

where:

$$a_1 = \left(6 + \frac{5}{T_s} + \frac{1}{T_s^2}\right), a_2 = -\left(-\frac{5}{T_s} - \frac{2}{T_s^2}\right), a_3 = \left(\frac{1}{T_s^2}\right), b_1 = 1$$

The sampling period must have a sufficiently small value calculated according to the Nyquist criterion. For simplicity we set $T_s = 1$ and get:

$$a_1 = 12, a_2 = 7, a_3 = 1, b_1 = 1$$

so the following difference equation is obtained:

$$12y[n] + 7y[n - 1] + y[n - 2] = x[n], n > 0$$

For zero initial conditions we have $y[0] = 1/12$ and from the final value theorem we find $y[n] = 1/20$ for $n \to \infty$. The Z-transform of the output is:

$$Y(z)(12 + 7z^{-1} + z^{-2}) = \frac{1}{1 - z^{-1}} \Rightarrow Y(z) = \frac{z^3}{(z-1)(12z^2 + 7z + 1)}$$

Calculating the inverse Z-transform by the method shown in Example 9.15, we find:

$$y[n] = \left[\frac{1}{240} + \frac{1}{12}\left(\frac{1}{3}\right)^n - \frac{1}{20}\left(\frac{1}{4}\right)^n\right]u[n]$$

This solution is approximate and not exact because we chose a large value for the sampling period $T_s = 1$. Choosing a smaller value makes the solution more accurate.