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# DIGITAL SIGNAL PROCESSING

## Solved Examples

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**SET #3 – Discrete Time Systems**

- Categorization of Discrete Time Systems
- System Description with Convolutional Sum
- Study of Systems with the Method of Convolution

## 1. Categorization of Discrete Time Systems

**Example 1**

Examine whether the following systems are time shift invariant.

(a) $y[n] = \sum_{k=-\infty}^{n} x[k]$   (b) $\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$

(a) From the input–output relationship and considering that $n \to \infty$, we find the time shifted response:

$$y[n-n_0] = \sum_{k=-\infty}^{n} x[k-n_0] = \sum_{k=-\infty}^{+\infty} x[k-n_0]$$

The response of the system to the time-shifted input $x'[n] = x[n-n_0]$, is:

$$y'[n] = \sum_{k=-\infty}^{+\infty} x'[k] = \sum_{k=-\infty}^{+\infty} x[k-n_0]$$

Because $y'[n] = y[n-n_0]$ the system is time-shift invariant.

(b) The shifted response by $n_0$ is:

$$y[n-n_0] = \sum_{k=0}^{N} a_k\, y[n-n_0-k] = \sum_{m=0}^{M} b_m x[n-n_0-m]$$

The response of the system to the shifted input $x'[n] = x[n-n_0]$, is:

$$y'[n] = \sum_{k=0}^{N} a_k\, y'[n-k] = \sum_{m=0}^{M} b_m x'[n-m] = \sum_{m=0}^{M} b_m x[n-n_0-m]$$

Because $y'[n] = y[n-n_0]$ the system is time-shift invariant.

**Example 2**

Examine whether the discrete-time system is linear with an input-output relationship:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

Answer: For inputs $x_1[n]$and $x_2[n]$ the corresponding outputs $T\{x_1[n]\}$ and $T\{x_2[n]\}$are:

$$y_1[n] = T\{x_1[n]\} = \sum_{k=0}^{N} a_k\, y_1[n-k] = \sum_{m=0}^{M} b_m x_1[n-m]$$

$$y_2[n] = T\{x_2[n]\} = \sum_{k=0}^{N} a_k\, y_2[n-k] = \sum_{m=0}^{M} b_m x_2[n-m]$$

For the combined entry $x[n] = \alpha x_1[n] + \beta x_2[n]$ applies:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{m=0}^{M} b_m x[n-m] = \sum_{m=0}^{M} b_m(\alpha x_1[n-m] + \beta x_2[n-m])\ (1)$$

The combined output $y[n] = \alpha y_1[n] + \beta y_2[n]$ of the system are:

$$a\sum_{k=0}^{N} a_k\, y_1[n-k] + \beta \sum_{k=0}^{N} a_k\, y_2[n-k] = a\sum_{m=0}^{M} b_m x_1[n-m] + \beta \sum_{m=0}^{M} b_m x_2[n-m] \Rightarrow$$

$$\sum_{k=0}^{N} \alpha a_k\, y_1[n-k] + \sum_{k=0}^{N} \beta a_k\, y_2[n-k] = \sum_{m=0}^{M} \alpha b_m x_1[n-m] + \sum_{m=0}^{M} \beta b_m x_2[n-m] \Rightarrow$$

$$\sum_{k=0}^{N} a_k\, (\alpha y_1[n-k] + \beta y_2[n-k]) = \sum_{k=0}^{N} b_m\, (\alpha x_1[n-k] + \beta x_2[n-k])\ (2)$$

Since the second members of equations (1) and (2) are equal, it follows that the first members are also equal, i.e.:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{N} a_k\, (\alpha y_1[n-k] + \beta y_2[n-k])$$

Because the output for combined input equals the combined output, the system is linear.

**Example 3**

Check if the system is linear with the following input-output relationship:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, x[n+k]$$

Answer: We observe that it is $y[n]$ formed by the sum of its products $x[n]$ with shifted versions of itself. E.g.:

$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]\, x[k] = \sum_{k=-\infty}^{+\infty} x^2[k]$$

The squared term is expected to make the system non-linear. We use an example:

- If $x[n] = \delta[n]$, then $y[n] = \delta[n]$.
- If $x[n] = 2\delta[n]$, then $y[n] = 4\delta[n]$.

Therefore, the system is not homogeneous. Therefore it is not linear either, because homogeneity is a condition of linearity.

**📖 Example 4**

Examine whether the system is stable with an input-output relationship:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Answer: To judge the stability of the system we will set a blocked input and examine if the output is also blocked (BIBO stability). If the input is blocked, that is $|x[n]| \leq A < \infty$, then the measure of the output is:

$$|y[n]| = \left|\sum_{k=-\infty}^{n} x[k]\right| < \sum_{k=-\infty}^{n} |x[k]| < \sum_{k=-\infty}^{n} A$$

This sum tends to infinity for $n \to \infty$. Therefore, the outlet is not blocked so the system is not BIBO-stable.

## 2. Description of a System using the Convolutional Sum

**📖 Example 5**

Find the impulse response of an LSI and causal system when for input $x[n] = u[n]$ the system produces output $y[n] = \delta[n]$.

Answer: We write the equatio $y[n] = \sum_{k=0}^{N-1} x[k]\, h[n-k]$ as follows:

$$y[n] = x[0]h[n] + \sum_{k=1}^{N-1} x[k]\, h[n-k]$$

We solve for $h[n]$ and we find:

$$h[n] = \frac{1}{x[0]}\left[y[n] - \sum_{k=1}^{N-1} x[k]\, h[n-k]\right]$$

This process is called **deconvolution** and offers a recursive way to calculate the shock response through the following steps for various values of $n$:

$$n = 0, h[0] = \frac{1}{x[0]}[y[0]]$$

$$n = 1, h[1] = \frac{1}{x[0]}[y[1] - h[0]x[1]]$$

$$n=2, h[2]=\frac{1}{x[0]}\left[y[2]-h[0]x[2]-h[1]x[1]\right]$$

....

We apply for the given input and output functions and get:

$$n=0, h[0]=\frac{1}{u[0]}\left[\delta[0]\right]=1$$

$$n=1, h[1]=\frac{1}{u[0]}\left[\delta[1]-h[0]u[1]\right]=(0-1)=-1$$

$$n=2, h[2]=\frac{1}{u[0]}\left[\delta[2]-h[0]u[2]-h[1]u[1]\right]=(0-1+1)=0$$

$$n=3, h[3]=\frac{1}{u[0]}\left[\delta[3]-h[0]u[3]-h[1]u[2]-h[2]u[1]\right]=(0-1+1-0)=0$$

....

More generally, the solution is $h[n]=\delta[n]+\delta[n-1]$

## 3. Study of Systems using Convolution

**Example 6**

Calculate the convolution between $x[n]=(0.9)^n u[n]$ and $h[n]=n\,u[n]$.

Answer: Convolution is:

$$y[n]=\sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]=\sum_{k=-\infty}^{+\infty}\{(0.9)^k\,u[k]\}\,\{[n-k]\,u[n-k]\}$$

Since $u[k]=0$ για $k<0$ and $u[n-k]=0$ για $k>n$, we have:

$$y[n]=\sum_{k=0}^{n}[n-k](0.9)^k \;=n\sum_{k=0}^{n}(0.9)^k\;-\sum_{k=0}^{n}k(0.9)^k \text{ για } n\geq 0$$

Using the formulas:

| |
|---|
| $\sum_{n=0}^{N-1}a^n=\frac{1-a^N}{1-a}$ |
| $\sum_{n=0}^{N-1}n\,a^n=\frac{(N-1)a^{N+1}-Na^N+a}{(1-a)^2}$ |

we get:

$$y[n]=n\frac{1-(0.9)^{n+1}}{1-0.9}-\frac{n(0.9)^{n+2}-[n+1](0.9)^{n+1}+0.9}{[1-0.9]^2}$$
$$=10n\{1-(0.9)^{n+1}\}-100\{n(0.9)^{n+2}-[n+1](0.9)^{n+1}+0.9\}\; n\geq 0$$
$$=\{10n-90+90(0.9)^n\}\,u[n]$$

### Example 7

Calculate the convolution between signals $x[n] = \{\hat{1}, -2, 0, 3, -1\}$ and $h[n] = \{2, \hat{3}, 0, 1\}$ using the Toeplitz table method.

<u>Answer:</u> The signal $x[n]$ is of finite duration in the space $[0, 4]$ of length $L_x = 5$, while the signal $h[n]$ is of finite duration in the space $[-1, 2]$ of length $L_h = 4$. Therefore, the convolution is of finite length in the time interval $[0 + (-1), 4 + 2] = [-1, 6]$ and has a length equal to $L_y = L_x + L_h - 1 = 5 + 4 - 1 = 8$ samples.

The vector $\boldsymbol{x}$ has dimensions $[L_x, 1] = [5, 1]$ and are: $x = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ -1 \end{bmatrix}$

The Table $\boldsymbol{H}$ has dimensions $[L_y, L_x] = [8, 5]$ and are: $H = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 1 & 0 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

We calculate the vector $\boldsymbol{y}^T$:

$$\boldsymbol{y}^T = \boldsymbol{H}\,\boldsymbol{x}^T = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ 2 & 1 & 3 & 2 & 0 \\ 0 & 2 & 1 & 3 & 2 \\ 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ -1 \end{bmatrix} = \cdots = \begin{bmatrix} 2 \\ -1 \\ -6 \\ 7 \\ 5 \\ -3 \\ 3 \\ -1 \end{bmatrix}$$

Therefore the convolution is:

$$y[n] = \{2, -\hat{1}, -6, 7, 5, -3, 3, -1\}$$