



University of the Peloponnese

Electrical and Computer  
Engineering Department

# Digital Signal Processing

## Unit 09: Digital FIR Filters

Dr. Michael Paraskevas

Professor

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# Lecture Contents

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# Introduction to Digital Filters

- Filter gain control
- Minimum, maximum, mixed and linear phase filters
- Ideal frequency selection filters
- Specifications of real digital filters

# Introduction to Digital Filters

- If  $h[n]$  a complex exponential sequence is applied to the input of an LSI system with impulse response  $x[n] = A e^{jn\omega_0}$ ,  $-\infty < n < +\infty$ , digital frequency  $\omega_0$ , then the output is calculated from the convolution:

$$y[n] = A e^{jn\omega_0} \sum_{k=-\infty}^{+\infty} h[k] e^{-j\omega_0 k}$$

- The complex function  $H(e^{j\omega})$  is called **the frequency response** and is calculated from the DTFT of the impulse response:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} h[k] e^{-j\omega k}$$

- The output  $y[n]$  is written  $y[n] = A e^{jn\omega_0} H(e^{j\omega}) = x[n]H(e^{j\omega})$ . Therefore if the input is a complex sequence of frequency output  $\omega_0$ , then the output is also a complex sequence of the same frequency output  $\omega_0$  multiplied by the function  $H(e^{j\omega})$ .

# Introduction to Digital Filters

- The frequency response is written in polar form (magnitude, phase) as:

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\varphi_H(\omega)}$$

- Their plots (usually expressed in dB) are the **magnitude** and **phase spectra**, respectively.
- If the impulse response is a **real sequence** then the real part and the amplitude of the filter's frequency response exhibit **even symmetry**:

$$H_R(e^{j\omega}) = H_R(e^{-j\omega}) \text{ and } |H_R(e^{j\omega})| = |H_R(e^{-j\omega})|$$

- The imaginary part, phase and group delay exhibit **redundant symmetry**:

$$H_I(e^{j\omega}) = -H_I(e^{-j\omega}), \varphi_H(\omega) = -\varphi_H(-\omega) \text{ and } \tau_H(\omega) = -\tau_H(-\omega)$$

- A digital filter to be **practically implementable** must be **stable** and obviously **causal**.
- Practical digital filters are also desirable to be **linear and shift-invariant** (LSI).

# Filter Gain Control

- If  $|H(e^{j\omega_0})| > 1$  then the filter causes **amplification** of the input signal.
- If, then  $|H(e^{j\omega_0})| < 1$  **attenuation** of the input signal occurs.
- This process is called **filtering** the input signal and can lead to the production of an output with desired spectral characteristics.
- **Question:** By what digital filter design can the amplitude of the frequency response at a given frequency be appropriately determined  $\omega_0$  to result in either amplification or attenuation of the input signal at that frequency?
- The frequency response of the filter can be written:

$$H(e^{j\omega}) = \frac{\sum_{m=0}^M b[m]e^{-jm\omega}}{\sum_{k=0}^N a[k]e^{-jk\omega}} = b[0] e^{j(N-M)\omega} \frac{\prod_{m=1}^M (e^{j\omega} - z_m)}{\prod_{k=1}^N (e^{j\omega} - p_k)}$$

where  $z_m$  are the zeros **and** are  $p_k$  the poles of the frequency response.

# Filter Gain Control

- The magnitude response can be expressed as the quotient of the sum of the distances of the zeros from the unit circle to the sum of the distances of the poles from the unit circle, for each value of the digital frequency.
- Considering that a causal LSI system is stable if and only if all its poles lie inside the unit circle, it follows that for a certain frequency  $\omega_0$ :
  - If it is desired to **amplify** the input signal, then we place a **pole** very close to the unit circle (inside it) at an angle equal to the frequency  $\omega_0$ .
  - If **attenuation** of the input signal is desired, then we place a **null** very close to the unit circle at an angle equal to the frequency  $\omega_0$ .
- In any case, it is imperative that **all filter poles** lie **within** the unit circle so that the filter remains stable.



# Minimum and Maximum Phase Filters

The position of the filter **nulls** affects the phase of the frequency response. Specifically, the system is:

- **Minimum phase** when all zeros are within the unit circle.
- **Maximum phase** when all zeros are outside the unit circle.
- **Mixed phase** when some zeros are inside the unit circle and the rest outside it.

E.g. the transfer function filter:

$$H(z) = \frac{(z - 0.2)(z + 0.4)}{(z + 0.5)(z - 0.7)}$$

is of minimum phase, because the positions of the zeros  $z_1 = 0.2$  and  $z_2 = -0.4$  lie inside the unit circle.

# Linear Phase Filters

- Digital filters have **linear phase**, when the frequency response can be written as:

$$H(e^{j\omega}) = A(e^{j\omega}) e^{-ja\omega}, \quad a \in R$$

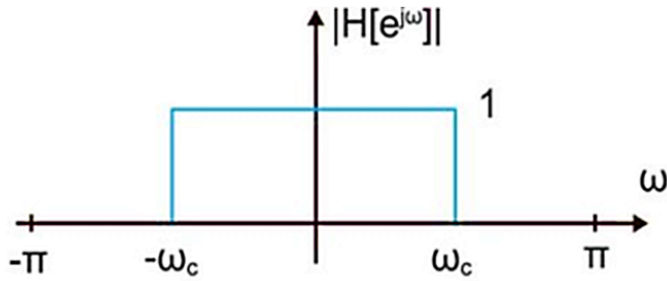
where  $A(e^{j\omega})$  is a real function of frequency  $\omega$ .

- For the phase of the frequency response  $H(e^{j\omega})$ :

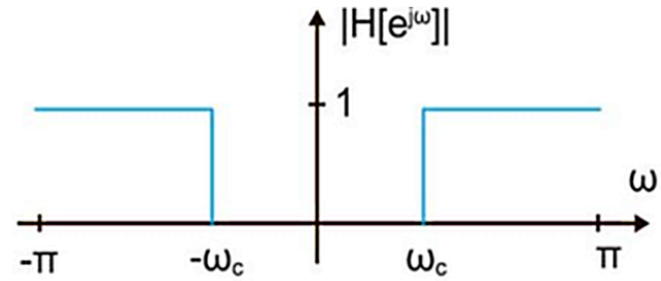
$$\varphi_H(\omega) = \begin{cases} -\alpha\omega & \text{when } A(e^{j\omega}) \geq 0 \\ -\alpha\omega + \pi & \text{when } A(e^{j\omega}) < 0 \end{cases}$$

- The sufficient and necessary condition for achieving a **linear phase** is the **symmetry** of the impulse response coefficients.
- In linear phase filters the group delay is **constant**. So all frequencies of the input signal experience the same delay when passing through the digital filter and thus the signal structure is not altered.
- Linear phase is an important **intrinsic characteristic** of FIR filters.

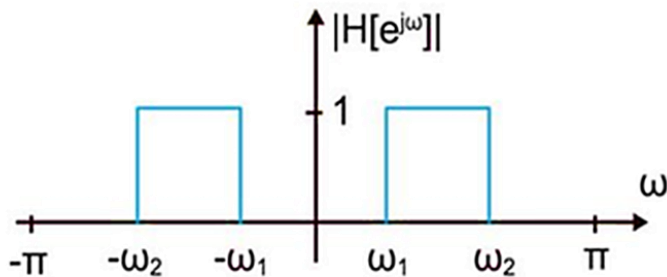
# Ideal Frequency Selection Filters



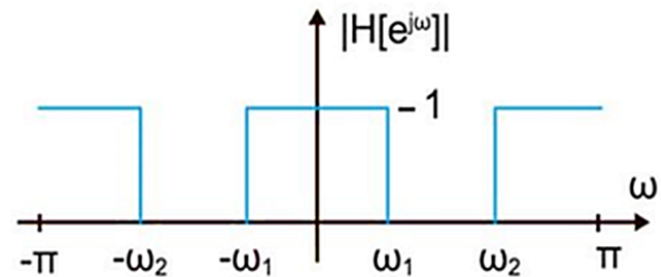
(a) Ideal Low-Pass Filter



(b) Ideal High-Pass Filter



(c) Ideal Band-Pass Filter



(d) Ideal Band-Stop Filter

- **Passband:** the frequency range in which  $|H(e^{j\omega})| = 1$ .
- **Stopband:** the frequency range in which  $|H(e^{j\omega})| = 0$ .
- **Cut-off frequencies:** the cut-off frequencies that mark the ends of the pass and stop bands.

# Ideal Frequency Selection Filters

- **Low Pass Filter (LPF):**

$$|H(e^{j\omega})| = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

- **High Pass Filter (HPF):**

$$|H(e^{j\omega})| = \begin{cases} 1, & \omega_c < |\omega| < \pi \\ 0, & 0 \leq |\omega| \leq \omega_c \end{cases}$$

- **Band Pass Filter (BPF):**

$$|H(e^{j\omega})| = \begin{cases} 1, & \omega_1 \leq |\omega| \leq \omega_2 \\ 0, & 0 < |\omega| < \omega_1 \text{ or } \omega_2 < |\omega| < \pi \end{cases}$$

- **Band Stop Filter (BSF):**

$$|H(e^{j\omega})| = \begin{cases} 1, & 0 < |\omega| < \omega_1 \text{ or } \omega_2 < |\omega| < \pi \\ 0, & \omega_1 \leq |\omega| \leq \omega_2 \end{cases}$$

- Caution! Ideal filters [ FIR, IIR ] are **non-causal**, thus **non-implementable**.

# Example 1

Let be  $h[n]$  the impulse response of an ideal low-pass filter with cutoff frequency  $\omega_c$ . What type of ideal filter has a impulse response  $g[n] = (-1)^n h[n]$  ?

Answer: The frequency response  $G(e^{j\omega})$  is:

$$\begin{aligned} G(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} g[n]e^{-jn\omega} = \sum_{n=-\infty}^{\infty} (-1)^n h[n]e^{-jn\omega} = \sum_{n=-\infty}^{\infty} h[n]e^{-jn(\omega-\pi)} \\ &= H(e^{j(\omega-\pi)}) \end{aligned}$$

Therefore, the function  $G(e^{j\omega})$  is formed by shifting it  $H(e^{j\omega})$  in frequency by  $\pi$ .

Thus, if the passband of the ideal low-pass filter is  $|\omega| \leq |\omega_c|$ , then the passband of the frequency-responsive filter  $G(e^{j\omega})$  will be  $\pi - \omega_c < |\omega| \leq \pi$ .

Therefore, the impulse response filter  $g(n)$  is an ideal high-pass filter.

# Specifications of Real Digital Filters

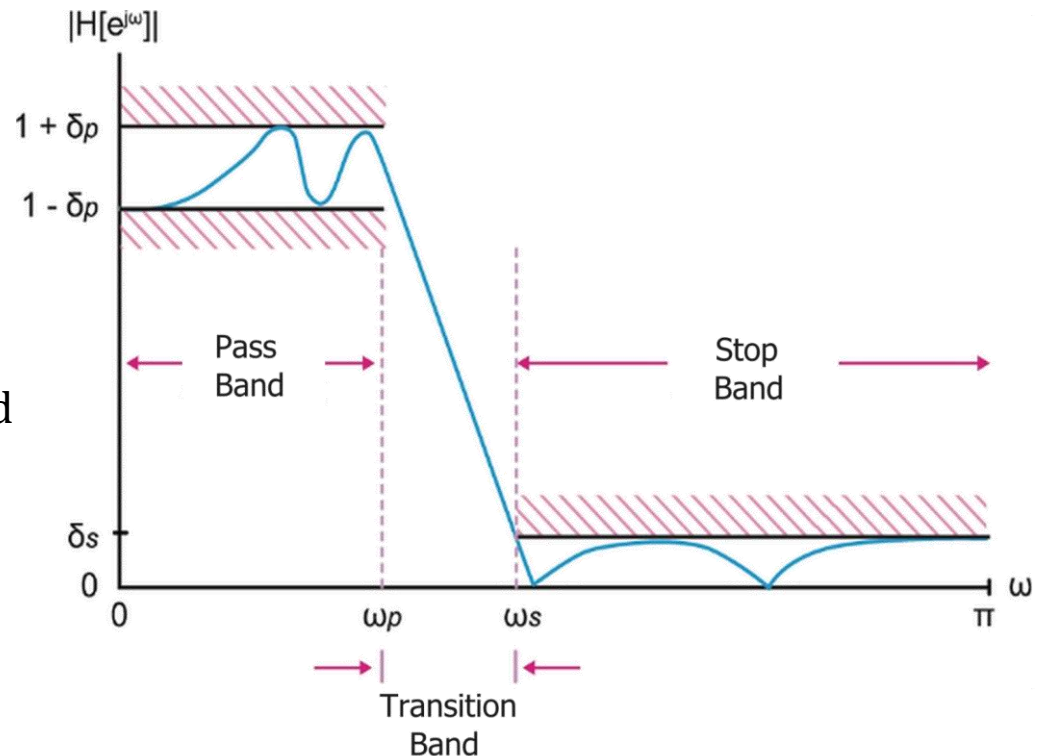
By accepting deviations from the ideal response, we make **real filters**.

For example to design a real low-pass filter we set:

- $1 - \delta_p < |H(e^{-j\omega})| \leq 1 + \delta_p$       $0 \leq |\omega| < \omega_p$  (transit zone)
- $|H(e^{-j\omega})| \leq \delta_s$       $\omega_s \leq |\omega| < \pi$  (cutoff zone)

Absolute specifications:

- $\omega_p$ : cutoff frequency in the passband
- $\omega_s$ : cutoff frequency in the stopband
- $\delta_p$ : deviation in the passband
- $\delta_s$ : deviation in stopband
- $[\omega_p, \omega_s]$ : transition band



# Specifications of Real Digital Filters

Related specifications:

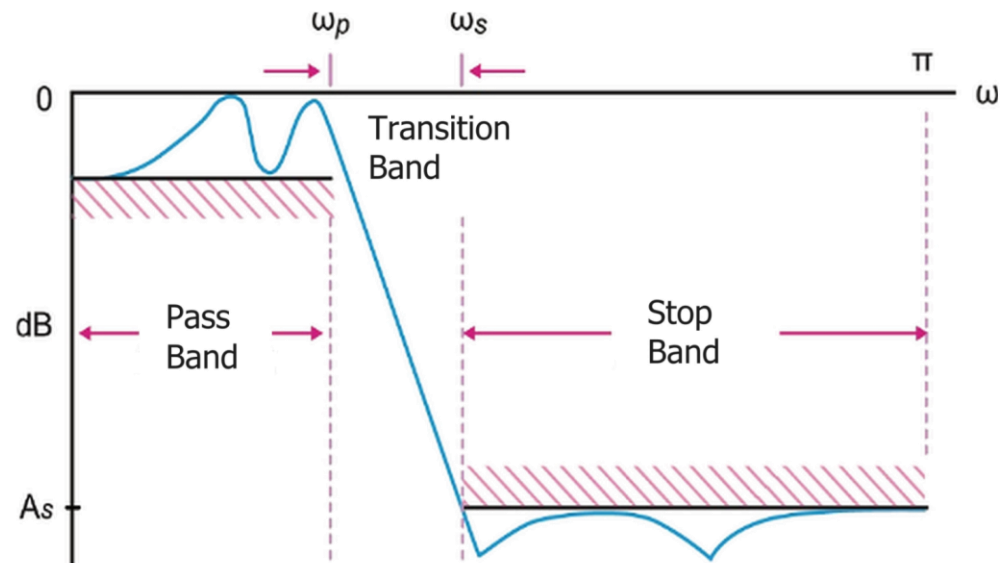
- $R_p$ : passband ripple

$$R_p = -20 \log_{10} \left( \frac{1 - \delta_p}{1 + \delta_p} \right) \text{ (dB)}$$

- $A_s$ : stopband attenuation

$$A_s = -20 \log_{10} \left( \frac{\delta_s}{1 + \delta_p} \right) \text{ (dB)}$$

- **N**: filter order, is the maximum number of coefficients of  $h[n]$



# Finite Impulse Response Filters

Finite Impulse Response  
(FIR) Filters



# Description of FIR filters in the time domain

- They are described in the time domain by the **impulse response**:

$$h[n] = \sum_{m=0}^{M-1} b[m]\delta[n - m]$$

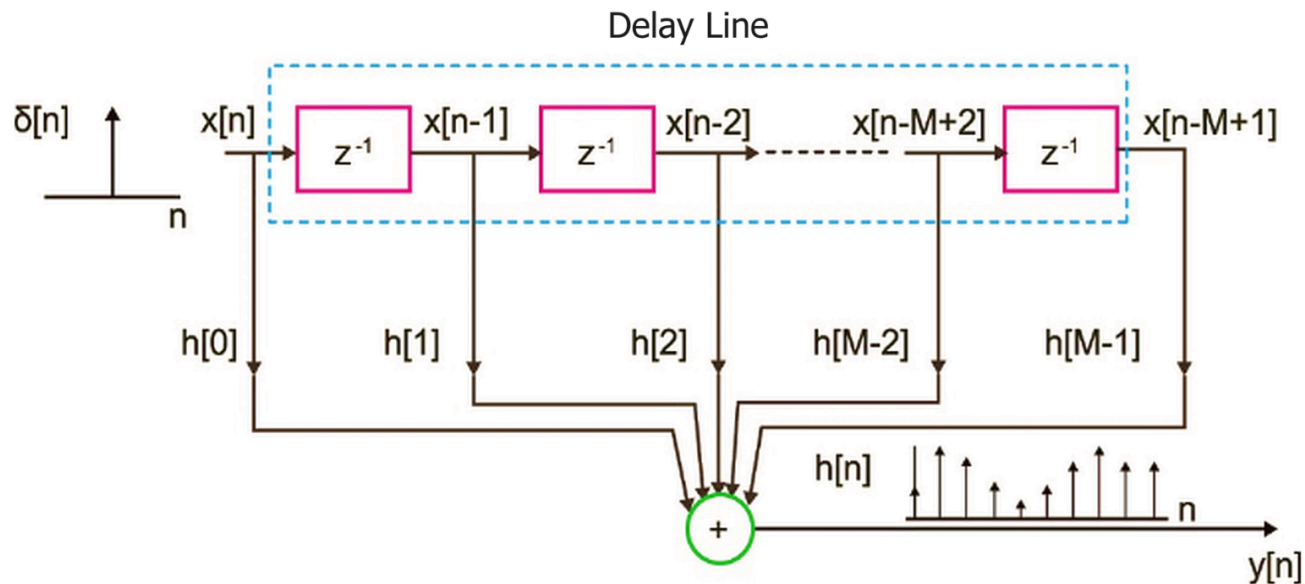
- And from LDECC:

$$y[n] = \sum_{m=0}^{M-1} b[m]x[n - m] = \sum_{m=0}^{M-1} h[m]x[n - m]$$

- The order of the filter is equal to the length  **$M$**  of the impulse response.
- The number of input samples that must be stored to compute an output sample determines the **order** of the filter.
- For each output sample are required  **$M$**  multiplications and  **$M + 1$**  additions.

# FIR filter as a "Delay Line"

- We consider an FIR filter as a **delay line**, in which the samples of the input signal are shifted  $x[n]$  and by the coefficients  $h[n]$  multiplied by delays  $x[n - m]$ .
- The results of the multiplications are added to give the final output of the filter  $y[n]$ .
- E.g 10<sup>th</sup> order filter will store ten incoming samples that precede the current sample. All eleven samples will affect the output sample of the FIR filter.



# Description of FIR filters in the Frequency Domain

- **Transfer function:**

$$H(z) = \sum_{m=0}^{M-1} b_m z^{-m}$$

- **Frequency response:**

$$H(e^{j\omega}) = \sum_{m=0}^{M-1} h[m] e^{-j\omega m}$$

- **Advantages of FIR filters:**

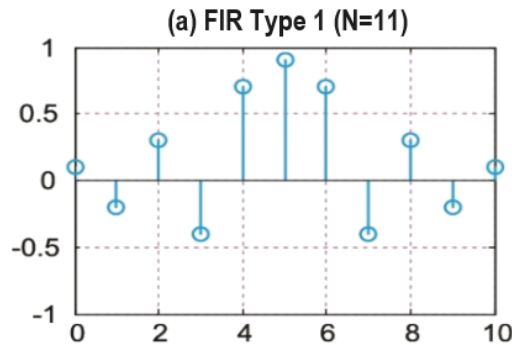
- Always stable, even after trimming decimal places of their coefficients.
- Always linear phase.
- Simple in design, they are implemented with simple calculations.
- They find many practical applications

- **Disadvantages of FIR filters:**

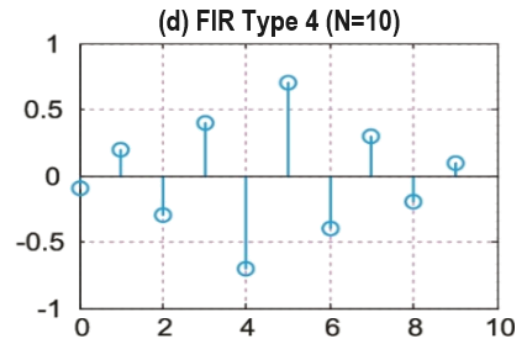
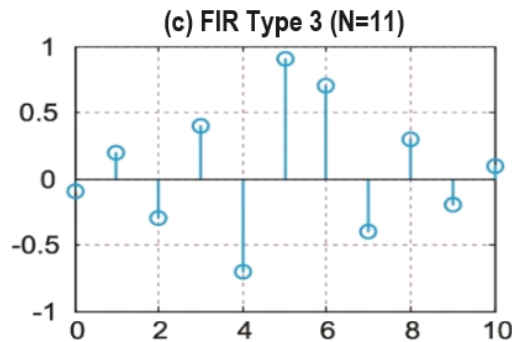
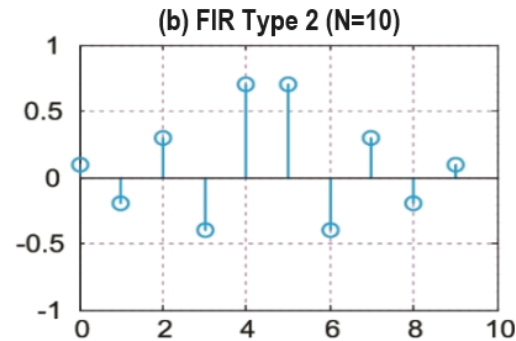
- They are implemented with more calculations, so they need more computing resources.
- Some responses cannot be implemented with FIR filters.

# Types of FIR Linear Phase Filters

**Type 1:** order  $N$  is odd and  $h[n]$  symmetric with respect to the point  $(N - 1)/2$ .



**Type 2:** order  $N$  is even and  $h[n]$  symmetric with respect to a central point that does not coincide with some value of  $h[n]$ .



**Type 3:** The order  $N$  is odd, and  $h[n]$  is antisymmetric with respect to the point  $(N-1)/2$ .

**Type 4:** order  $N$  is even and  $h[n]$  antisymmetric with respect to a central point that does not coincide with some value of  $h[n]$ .

# Frequency Responses of FIR Filters

**Type 1:** order  $N$  odd and  $h[n]$  is symmetric about the point  $(N - 1)/2$ :

$$H(e^{j\omega}) = e^{-jN\omega/2} \sum_{k=1}^{(N+1)/2} c[k] \cos(\omega(k - 1/2))$$
$$c[k] = 2h\left[\frac{N+1}{2} - k\right], \quad k = 1, 2, \dots, \frac{N+1}{2}$$

**Type 2:** order  $N$  even and  $h[n]$  is symmetric with respect to a central point that does not coincide with some value of  $h[n]$ :

$$H(e^{j\omega}) = e^{-jN\omega/2} \sum_{k=0}^{N/2} c[k] \cos(\omega k)$$
$$c[k] = 2h[N/2 - k], \quad k = 1, 2, \dots, N/2$$
$$c[0] = h[N/2]$$

# Frequency Responses of FIR Filters

**Type 3:** order  $N$  odd and  $h[n]$  is anti-symmetric with respect to the point  $(N - 1)/2$ :

$$H(e^{j\omega}) = e^{-jN\omega/2} \sum_{k=1}^{(N+1)/2} c[k] \sin(\omega(k - 1/2))$$
$$c[k] = 2h\left[\frac{N+1}{2} - k\right], \quad k = 1, 2, \dots, \frac{N+1}{2}$$

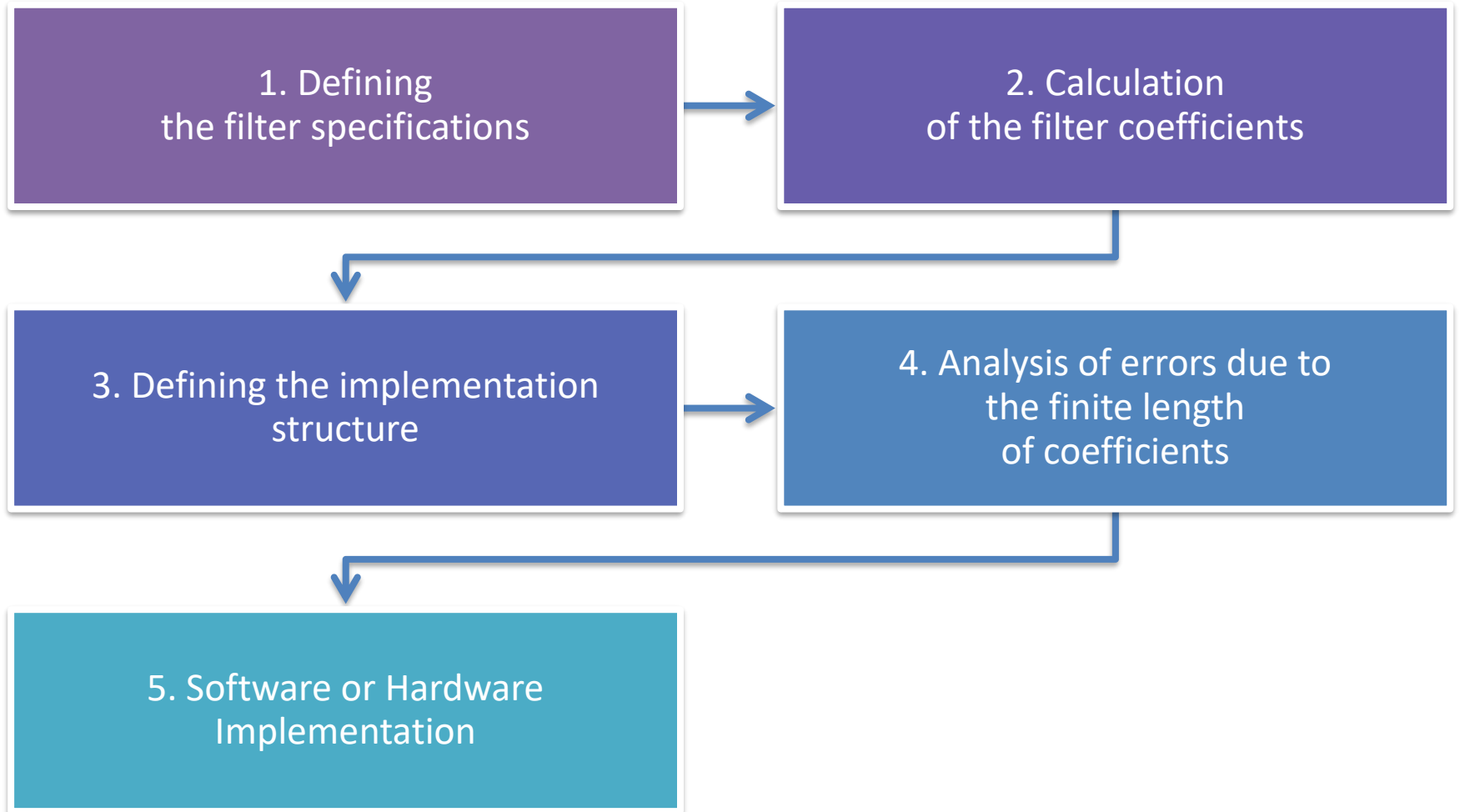
**Type 4:** order  $N$  even and  $h[n]$  is anti-symmetric with respect to a central point that does not coincide with some value of  $h[n]$ :

$$H(e^{j\omega}) = e^{-jN\omega/2} \sum_{k=0}^{N/2} c[k] \sin(\omega k)$$
$$c[k] = 2h\left[\frac{N}{2} - k\right], \quad k = 1, 2, \dots, \frac{N}{2}$$

# FIR Filter Design Methods

- Window method
- Frequency Sampling Method
- Optimal Method (Equiripple Filter)

# Stages of Implementation of Digital Filters





# FIR Filter Design Methods

- **Planning filter** = calculate the coefficients  $h[n]$  so that the filter meets the desired magnitude and phase specifications.
- FIR filter design methods:
  - **Windows method** - Due to its simplicity and efficiency, it is the most frequently used.
  - **Frequency sampling method** - Easy to understand and use, but produces filters with low attenuation in the cutoff band.
  - **Optimal design method** - Gives the best possible frequency response for a certain number of coefficients.
- All methods can produce linear phase FIR filters.
- The calculation of the **order of the filter** results from an iterative process. If the transition band of the designed filter is:
  - wider than needed, then the order of the filter **is increased**.
  - narrower than required, then the filter order **is reduced** to save hardware and/or software resources.

# Window method

# FIR Filter design using Window Method

- Let the design of an ideal **low-pass (LPF)** FIR filter with frequency response be:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

- The impulse response  $h_d[n]$  of the ideal LPF filter is:

$$h_d[n] = F^{-1}\{H_d(e^{j\omega})\} = \dots = \frac{\sin[\omega_c(n-a)]}{\pi(n-a)}$$

- We limit the generally infinite length response  $h_d[n]$  with a window  $w[n]$ :

$$h[n] = h_d[n] w[n]$$

- The window must be symmetrical (i.e.  $w[n] = w[N-n]$ ), to have linear phase.
- We shift in time the  $h[n]$  vs  $n_0 = (N-1)/2$  samples, so that the filter becomes causal.
- The frequency response of the filter is:

$$H(e^{j\omega}) = \frac{1}{2\pi} H_d(e^{j\omega}) * W(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta$$

# Commonly used Window functions

- **Rectangle:**

$$w[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{αλλού} \end{cases}$$

- **Hamming:**

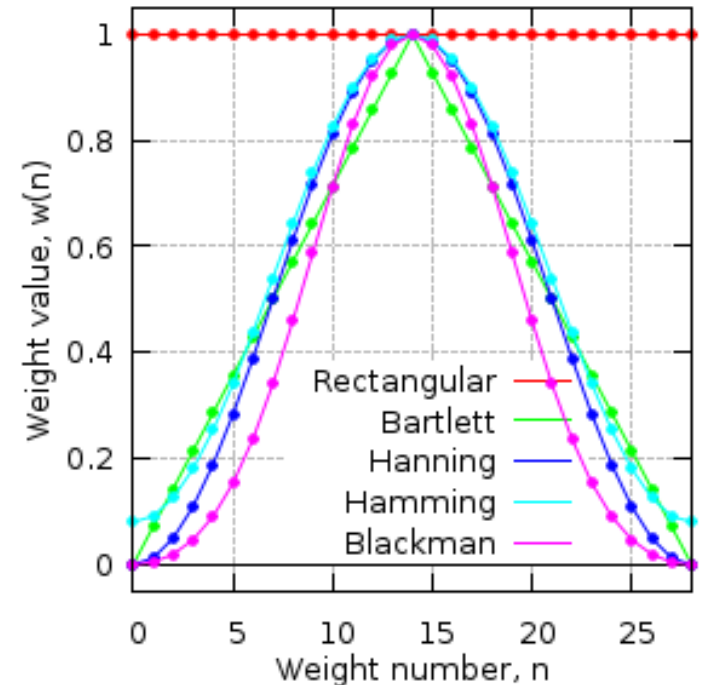
$$w[n] = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N}\right), & 0 \leq n < N \\ 0 & \text{αλλού} \end{cases}$$

- **Hannah:**

$$w[n] = \begin{cases} 0.5 - 0.5 \cos\left(\frac{2\pi n}{N}\right), & 0 \leq n < N \\ 0 & \text{αλλού} \end{cases}$$

- **Blackman:**

$$w[n] = \begin{cases} 0.42 - 0.5 \cos\left(\frac{2\pi n}{N}\right) + 0.08 \cos\left(\frac{4\pi n}{N}\right), & 0 \leq n < N \\ 0 & \text{αλλού} \end{cases}$$

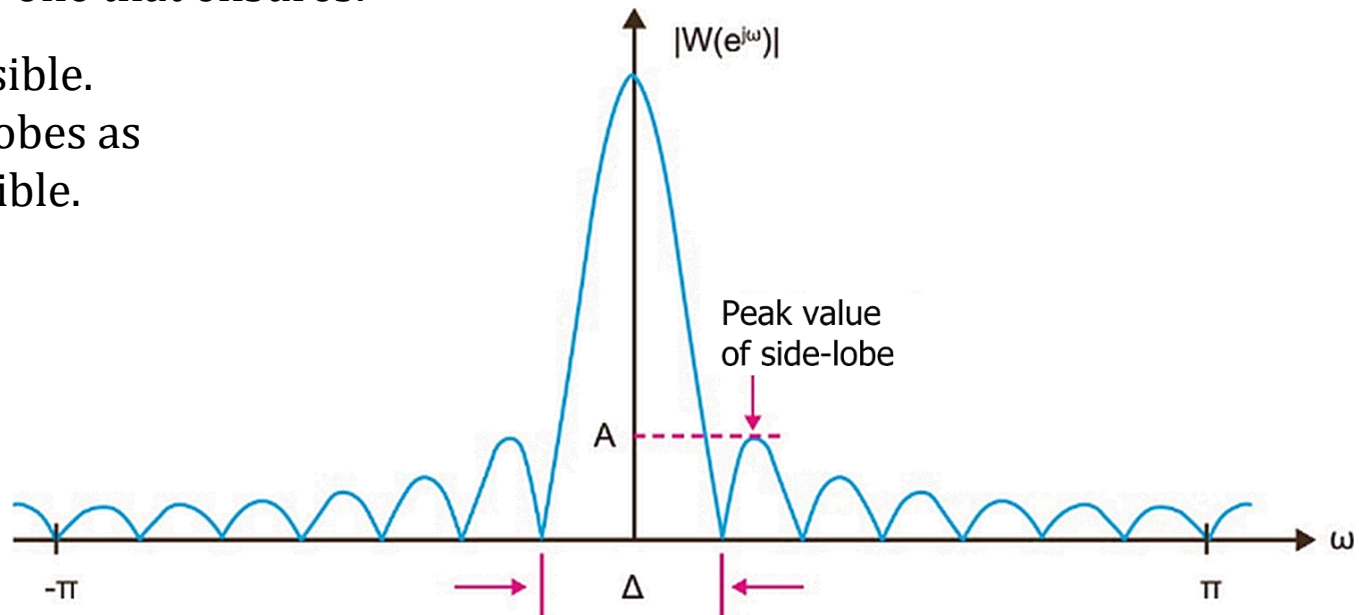


# Effects of windowing on filter characteristics

- The **width (D)** of its main lobe  $W(e^{j\omega})$  determines the width of the transition band (filter selectivity, selectivity) in the frequency response  $H_d(e^{j\omega})$ .
- The **apical width (A)** of the lateral lobe her  $W(e^{j\omega})$  determines the ripple in the frequency response  $H_d(e^{j\omega})$ .

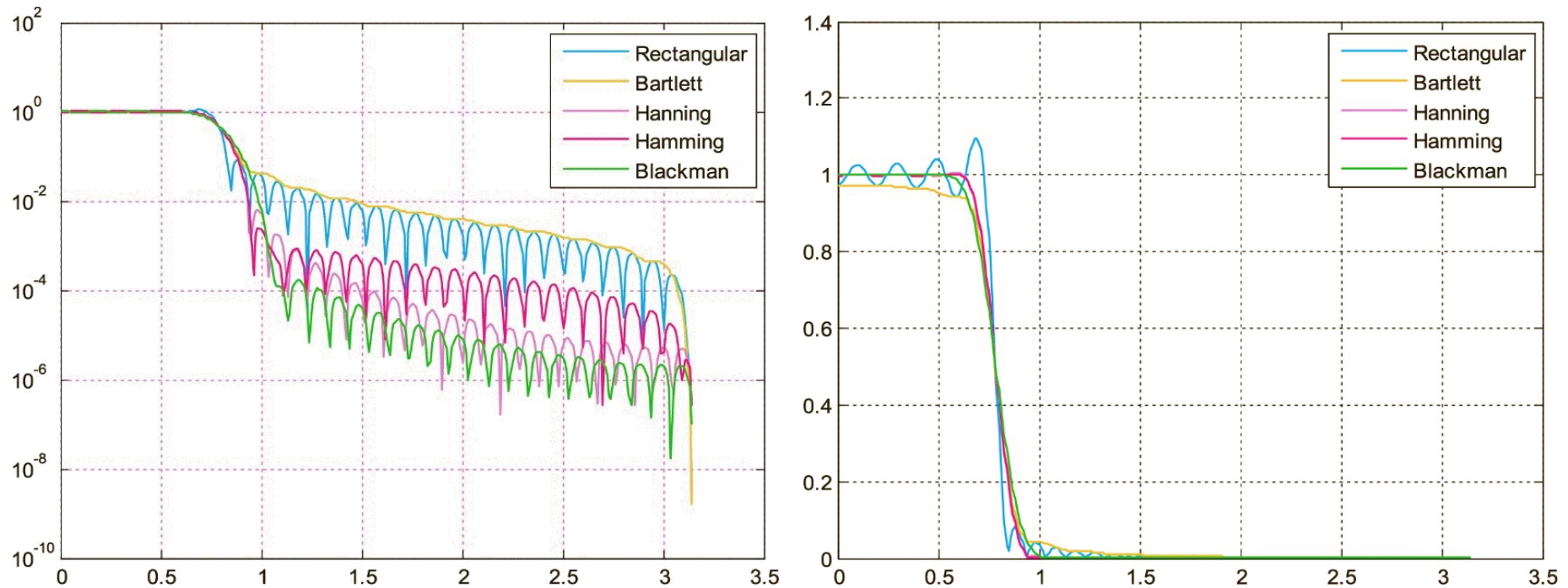
An ideal window is one that ensures:

- **Narrow** as possible.  
Width of side lobes as **limited** as possible.



DTFT of standard window function

# Frequency response of commonly used window functions



Windowed frequency response: (a) in linear, (b) in semi-logarithmic scale

# Comparison of windows' features

Window type	Normalized main lobe range $\Delta$ (N=20)	Transition Zone (N=20)	Minimal window clipping zone fading	Minimal filter cutoff band attenuation
Rectangle	0.1p	0.041p	13 dB	21dB
Bartlett	0.2p	0.11p	26dB	26dB
Hanning_	0.21p	0.12p	31dB_	44dB
Hamming	0.23p	0.14p	41dB_	53dB
Blackman	0.32p	0.2p	58dB_	75dB_

# Comparison of windows' features

- **Rectangular window:** very low attenuation in the cutoff band (13 dB). Not preferred for FIR filter design.
- Likewise the **triangular one**. However, when high attenuations are not required it can be used because it provides an easy way to calculate its coefficients.
- **Hanning:** has stronger attenuation than triangular, while for the same attenuation requirements it has a narrower transition region, which is considered its advantage.
- **Hamming:** ensures minimum attenuation in the cutoff band of 53 dB, which is sufficient for most applications. The transition zone is somewhat larger than Hanning 's. Very popular.
- **Blackman:** along with Kaiser and Hamming windows are the most popular windows. The high attenuation of the designed filter ( $\sim 75$  dB), makes this window suitable for almost all applications.
- In all cases **increasing the order of the filter reduces the width of the transition band, but does not affect the attenuation in the cutoff band.**



# Frequency Sampling Method

# Frequency Sampling Method

- The desired frequency response  $H_d(e^{j\omega})$  of the ideal filter is subject to sampling at evenly spaced frequencies in the frequency range  $[0, 2\pi]$ , that is:

$$\omega_k = \frac{2\pi}{N} k, \quad k = 0, 1, \dots, N - 1$$

and the sequence is produced  $H[k]$ :

$$H[k] = H_d(e^{j2\pi k/N}), \quad k = 0, 1, \dots, N - 1$$

- The sequence  $H[k]$  constitutes an N-point DFT transform, from which the impulse response of the approximation filter can be derived.
- The frequency response  $H(e^{j\omega})$  of the approximation filter is obtained by interpolating its samples  $H[k]$  and is:

$$H(e^{j\omega}) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{H[k]}{1 - e^{-j\omega} e^{j2\pi k/N}}$$

- The transfer function of the filter is:

$$H(z) = \sum_{n=0}^{N-1} h[n] z^{-n} = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H[k]}{1 - z^{-1} e^{j2\pi k/N}}$$

# Frequency Sampling Method

- If it is desired that the filter has a linear phase, then the sequence  $H[k]$  must have conjugate symmetry:

- N: odd:  $H[k] = H^*[N - k], k = 1, 2, \dots, \frac{N-1}{2}$

- N: even:  $H[k] = H^*[N - k], H\left[\frac{N}{2}\right] = 0, k = 1, 2, \dots, \frac{N}{2} - 1$

- The approximate impulse response  $h[n]$  of the order  $N - 1$  FIR filter is obtained by inverse Fourier transform:

$$h[n] = \frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{j2\pi nk/N}, \quad 0 \leq n < N - 1$$

- The impulse response  $h[n]$  for N even, (for  $n = 0, 1, \dots, N - 1$ ) is:

$$h[n] = \frac{H[0]}{N} + \frac{1}{N} \sum_{k=0}^{N/2-1} 2 (-1)^k H[k] \cos\left[\frac{\pi k(1 + 2n)}{N}\right]$$

- For N odd we replace the upper bound of the sum with  $(N - 1)/2$ .
- The approximate impulse response  $h[n]$  is related to the ideal impulse response  $h_d[n]$ :

$$h[n] = \sum_{k=-\infty}^{\infty} h_d[n + kN], \quad 0 \leq n < N - 1$$

# Frequency Sampling Method

- The above process has the following results:
  - At the frequencies for which the samples are taken  $H_d(e^{j\omega})$ , the difference between  $H_d(e^{j\omega})$  and  $H(e^{j\omega})$  is **zero**, so at the specific frequencies the approximation error of the method is **zero**.
  - At the remaining frequencies, the **approximation error** depends on its  $H_d(e^{j\omega})$  **shape** and becomes larger when it  $H_d(e^{j\omega})$  has strong changes.
  - The approximation error is largest at the **boundaries** between the pass and cutoff bands of the filter.
- The frequency sampling method **does not have an inherent mechanism to control the error** in the frequencies interpolated between the samples.
- Control of the error can be achieved by adding one or more samples to the transition regions and applying an iterative algorithm that maximizes attenuation in the cutoff band or minimizes ripple in the passband.

## Example 2

To design a low pass filter (LPF) with linear phase, cutoff frequency  $\omega_c = 0.4\pi$  and length  $N = 10$  coefficients.

**Answer:** We divide the frequency domain  $[0, 2\pi]$  into  $N=10$  equal-sized parts based on the relation  $\omega_k = (2\pi/10)k = 0.2\pi k$ ,  $k = 0, 1, \dots, 9$ .

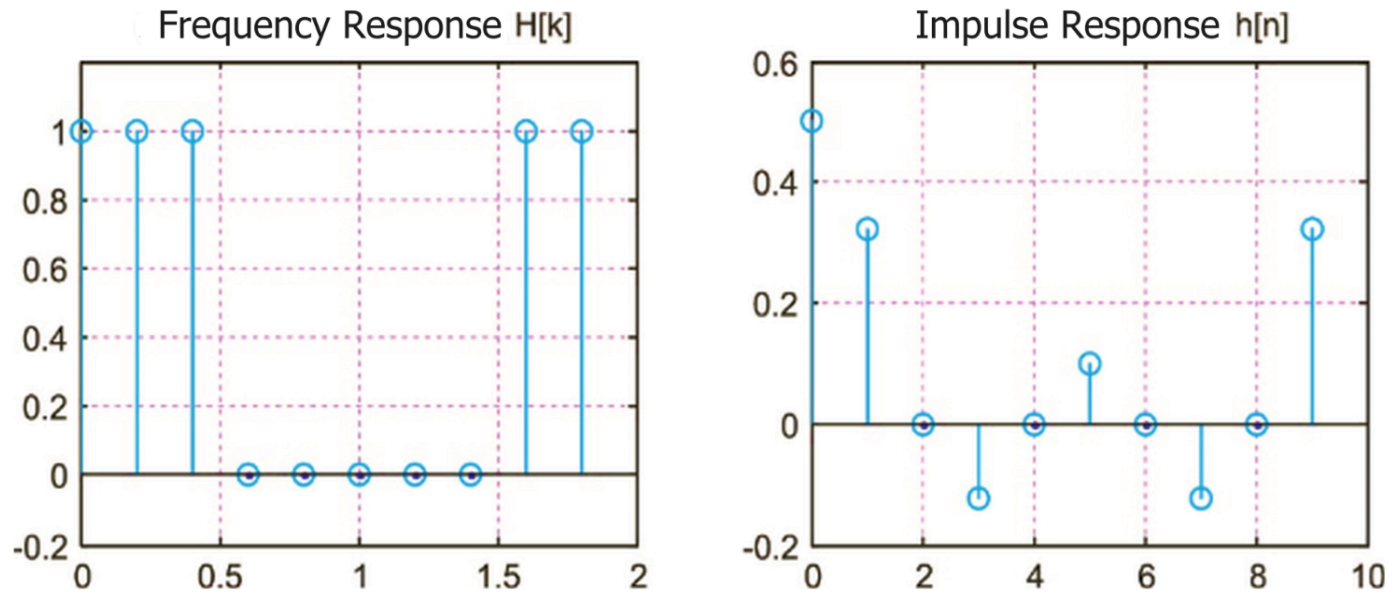
$k = 0,$	$\omega_0 = 0,$	$H[0] = 1$	$k = 5,$	$\omega_5 = \pi,$	$H[5] = 0$
$k = 1,$	$\omega_1 = 0.2\pi,$	$H[1] = 1$	$k = 6,$	$\omega_6 = 1.2\pi,$	$H[6] = 0$
$k = 2,$	$\omega_2 = 0.4\pi,$	$H[2] = 1$	$k = 7,$	$\omega_7 = 1.4\pi,$	$H[7] = 0$
$k = 3,$	$\omega_3 = 0.6\pi,$	$H[3] = 0$	$k = 8,$	$\omega_8 = 1.6\pi,$	$H[8] = 1$
$k = 4,$	$\omega_4 = 0.8\pi,$	$H[4] = 0$	$k = 9,$	$\omega_9 = 1.8\pi,$	$H[9] = 1$

The impulse response  $h[n]$  of the approximation filter for  $0 \leq n < 9$  is:

$$\begin{aligned} h[n] &= \frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{j2\pi nk/N} = \frac{1}{10} \sum_{k=0}^9 H[k] e^{j2\pi nk/10} \\ &= \frac{1}{10} (1e^0 + 1e^{j2\pi n/10} + 1e^{j4\pi n/10} + 1e^{j16\pi n/10} + 1e^{j18\pi n/10}) \end{aligned}$$

## Example 2 (continued)

- Matlab code we obtain the sampled frequency response and the impulse response of the approximation filter:

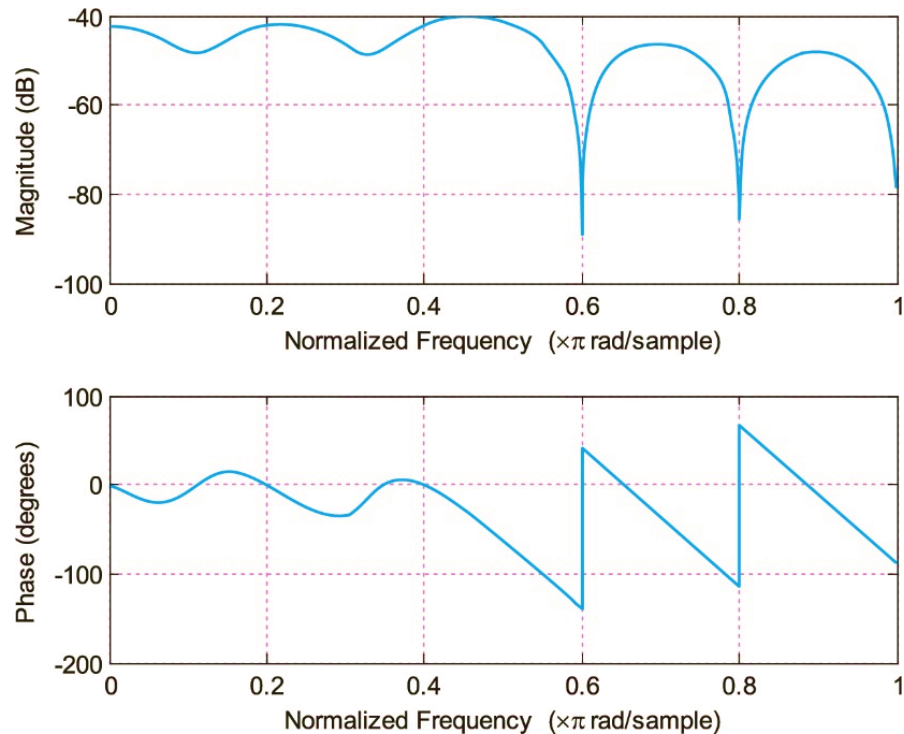


(a) Sampled frequency response (interval  $[0.2\pi]$ )

(b) Impulse response of approximating filter

## Example 2 (continued)

Matlab code we get  
the frequency response of the  
FIR approximation filter



- We find the design performs poorly, as significant ripple occurs in the passband while the attenuation in the cutoff is only  $\sim 10$ dB.
- We improve the performance of the method by increasing the order of the filter. However, even for large values of the order there still remains a ripple in the passband and a small attenuation in the cutoff band.
- An improvement of the method is based on the addition of samples in the transition zone.

# **Optimal Design Method (Equiripple Method)**



# Optimal Design Method (Equiripple)

- Windowing and frequency sampling methods **do not produce optimal filters** because:
  - They do not offer passband and cutoff ripple control.
  - They introduce ambiguity in determining the cutoff frequencies of the pass and cutoff bands.
  - They exhibit an inability to uniformly distribute the error between the ideal and the approximate frequency response in all frequency bands.
- To control the ripple, the filter specification is overestimated, however the ripple is still **uneven** in the pass and cutoff bands.
- The requirement for **stable and controlled ripple** in the pass and cutoff bands leads to the definition of the **equiripple filter**.
- The Equiripple filter is considered the optimal FIR filter because it achieves the best possible frequency response specification with the smallest number of coefficients.

# Optimal Design Method (Equiripple)

- The frequency response of a linear phase FIR filter is written:

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega}$$

- If the filter is type 1, the amplitude response is written as a Chebyshev polynomial of order  $L$ :

$$A(e^{j\omega}) = \sum_{k=0}^L \alpha(k) \cos(k\omega) = \sum_{k=0}^L \alpha(k) (\cos\omega)^k$$

- Although  $A_d(e^{j\omega})$  are  $A(e^{j\omega})$  the amplitude responses of the ideal and approximation filters, we define the error between them:

$$E(e^{j\omega}) = W(e^{j\omega})[A_d(e^{j\omega}) - A(e^{j\omega})]$$

- $W(e^{j\omega})$  is a positive weighting function, which we can define appropriately to control the ripple in either the passband or the cutoff band.
- Parks–McClellan algorithm:** selection of appropriate  $a(k)$  filter coefficients to minimize the maximum absolute value of the error  $E(e^{j\omega})$  along a set of frequencies (peaks) in the passband and cutoff band, but not in the transition band (indifference region).
- Mathematical formulation:

$$\min_{\alpha[k]} \left| \max_{\omega \in S} |E(e^{j\omega})| \right| \text{ όπου } S \triangleq [0, \omega_p] \cup [\omega_s, \pi]$$

# Parks–McClellan Algorithm

1. We choose an initial set of  $L + 2$  extreme frequencies (extrema).
2. We calculate the maximum weighted error ( $\varepsilon$ ) by solving the system:

$$\begin{bmatrix} 1 & \cos(\omega_0) & \dots & \cos(L\omega_0) & 1/W(e^{j\omega_0}) \\ 1 & \cos(\omega_1) & \dots & \cos(L\omega_1) & -1/W(e^{j\omega_1}) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \cos(\omega_L) & \dots & \cos(L\omega_L) & (-1)^L/W(e^{j\omega_L}) \\ 1 & \cos(\omega_{L+1}) & \dots & \cos(L\omega_{L+1}) & (-1)^{L+1}/W(e^{j\omega_{L+1}}) \end{bmatrix} \begin{bmatrix} \alpha[0] \\ \alpha[1] \\ \vdots \\ \alpha[L] \\ \varepsilon \end{bmatrix} = \begin{bmatrix} A_d(e^{j\omega_0}) \\ A_d(e^{j\omega_1}) \\ \vdots \\ A_d(e^{j\omega_L}) \\ A_d(e^{j\omega_{L+1}}) \end{bmatrix}$$

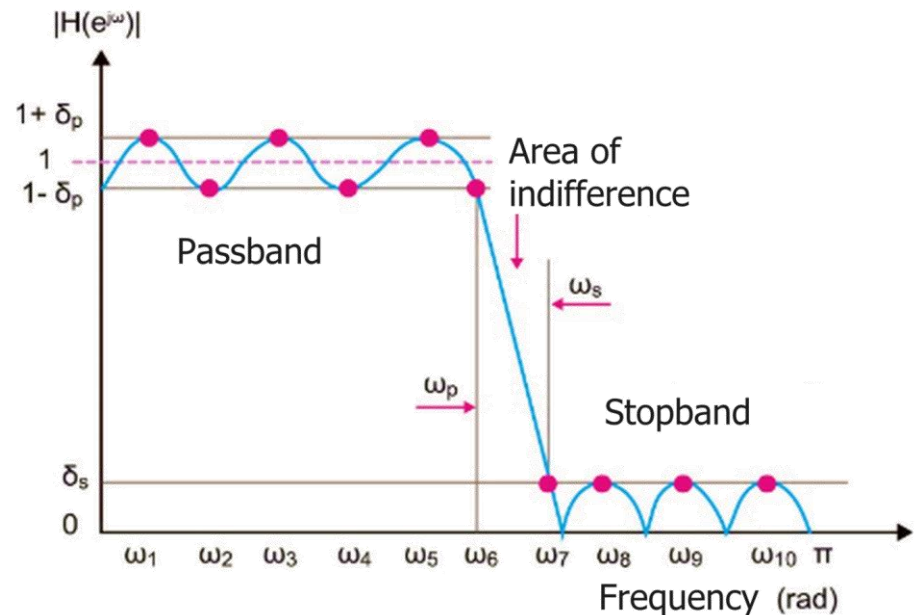
3. We estimate the error function  $E(e^{j\omega})$  over the entire set of frequencies  $S$ , by Lagrange interpolation at the extreme frequencies.
4. We find the  $L + 2$  local maxima of the function  $E(e^{j\omega})$  for the whole set  $S$  and then calculate the maximum weighted error  $\max_{\omega \in S} |E(e^{j\omega})|$ .
5. If the commutation theorem is satisfied, that is,  $\max_{\omega \in S} |E(e^{j\omega})| < \varepsilon$ , then the solution is completed and from the current set of infinite frequencies we find the coefficients  $h[n]$  of the inverse filter DTFT in the function  $H(e^{j\omega})$ .
6. If the commutation theorem is not satisfied, that is  $\max_{\omega \in S} |E(e^{j\omega})| > \varepsilon$ , then we add new frequencies to the points where the function  $E(e^{j\omega})$  has local maxima and repeat the process from step 2.

# Perfect Design Method (Equiripple)

- The algorithm keeps constant the quantities  $N$ ,  $\omega_p$  and  $\omega_s$  in order to control the limits of the zones and allows the change of  $\delta_p$  and  $\delta_s$ .
- The designer can know in advance the number of factors required.

## Park - McClellan optimization algorithm:

- We assume that the frequency extremes (extrema) are evenly spaced in the pass and cutoff bands.
- We perform polynomial interpolation and redefine the positions of the local extremities.
- We move the extremities to new positions and repeat until the extremities stop moving.



# Specify Filter Order

- Determining the appropriate order  $N$  of the FIR filter is the initial step in designing the filter.
- Various empirical formulas have been proposed for linear phase FIR filters. The simplest is the Kaiser formula:

$$N = \frac{-20 \log_{10}(\sqrt{\delta_p \delta_s}) - 13}{2.285 (\omega_s - \omega_p)} - 1$$

- Other approximate types of order calculation are reported in the literature.
- At the start of the Parks - McClellan algorithm, an initial value is given to the filter order, which may then change as the order is inversely connected according to the maximum weighted error.

# Study of Window Sequences

- Rectangular window
- Triangular window (Bartlett)
- Hanning window
- Hamming window
- Blackman window
- Kaiser window

# “Windowing” process

- In this section it will be investigated whether the use of a window to slice the signal into individual segments affects the spectral content of the signal.
- **Windowing:**  $x_w[n] = x[n] w[n]$ , where  $x[n]$  discrete-time signal,  $w[n]$  window and  $x_w[n]$  the "windowed" part of the signal.
- The DTFT of the signal  $x_w[n]$  is:  $X_w(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta$ , where  $X_w(e^{j\theta})$  and  $W(e^{j\theta})$  are the DTFTs of the signals  $x_w[n]$  and  $w[n]$ , respectively.
- **Spread Spectral:** "Windowing" causes the spectrum of the original signal to be **broadened** by the effective spectral bandwidth of the window sequence.
- The longer the length of the window sequence, the smaller its effective bandwidth and therefore the smaller the spectral spread.
- So to **reduce the spectral dispersion** it is necessary to **increase the length of the window sequence**.

# Rectangular Window

- Rectangular window (rectangular, boxcar) of  $N$  point length

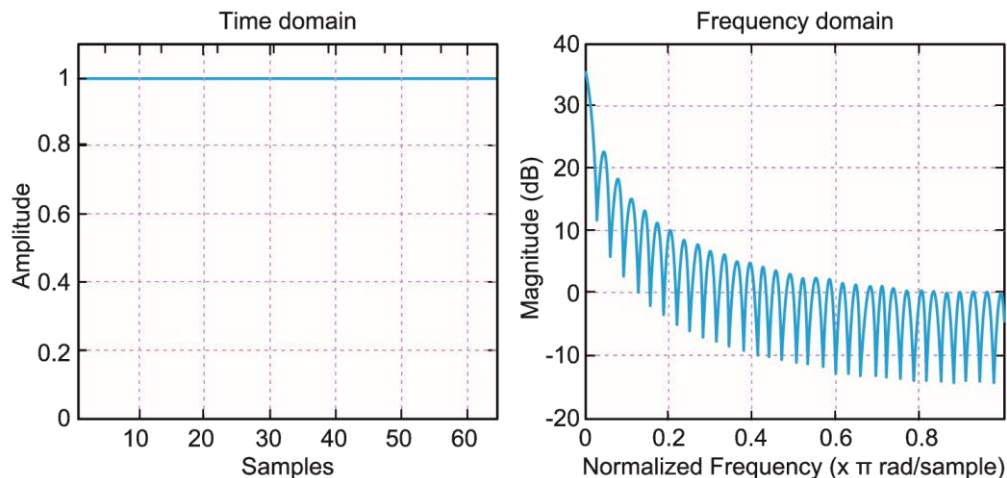
$$w[n] = \begin{cases} 1, & n = 0, 1, 2, \dots, N - 1 \\ 0, & \text{αλλού} \end{cases}$$

- DTFT:

$$W(e^{j\omega}) = e^{-j\omega(N-1)/2} \left[ \frac{\sin(\omega N/2)}{\sin(\omega/2)} \right]$$

- Amplitude Response:

$$W_r(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$



Rectangular window and width spectrum (gain, dB) ( $N = 64$ )



# Rectangular Window

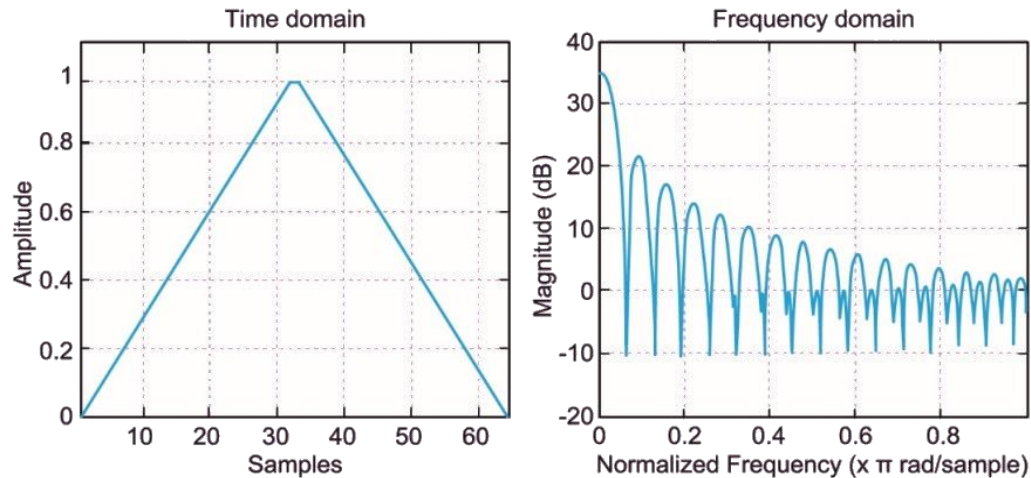
- The magnitude response  $W_r(\omega)$  has a central lobe and many lateral ones, due to the term  $\sin(\omega N/2) / \sin(\omega/2)$ .
- Center Lobe Frequency Bandwidth =  $4\pi/N$ - Width =  $N$ .
- The width of the first lateral lobe is  $\sim 2N/3\pi$  (less 13,3 dB than the width of the main lobe). The remaining sidelobes decay at a rate proportional to  $1/\omega$  (6 dB /octave).
- The side lobes cause ripples in the spectrum of the window, i.e. energy **leakage from the** central to the side lobes. This causes frequencies that were not present in the original signal to appear (distortion).
- Leakage depends on the width ratio between the center and side lobes. The higher this ratio, the lower the energy leakage.
- In order to reduce the energy leakage it is necessary to increase the degree of attenuation of the width of the sidelobes. This requirement, however, is in conflict with the requirement to reduce spectral dispersion.
- As the window duration tends to infinity, the width of the sidelobes tends to zero and the window spectrum tends to the shock function  $\delta(\omega)$ . Therefore, there is no spectral effect on the signal (ideal window). But this window is of no practical use, as it does not slice the signal into segments.

# Triangular Window

- Triangular window:

$$w[n] = 1 - \frac{2 \left| n - \frac{N-1}{2} \right|}{N-1}, \quad 0 \leq n \leq N-1$$

- The difference between the width of the center and the first sidelobe is -26.5 dB, the bandwidth of the center lobe is -26.5 dB, the bandwidth of the center lobe is  $8\pi/N$  and the attenuation rate of the sidelobes is 12 dB.



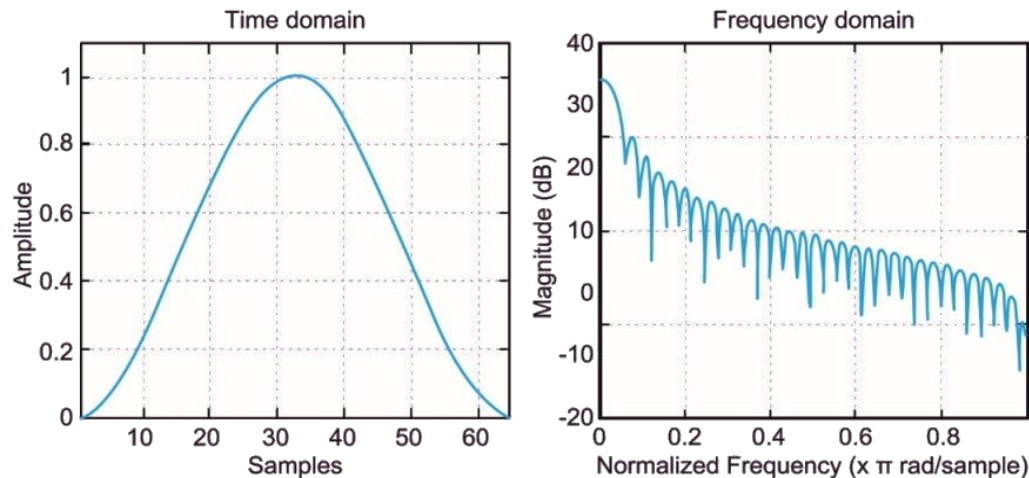
Triangular window and amplitude spectrum ( $N = 64$ )

# Hanning Window

- Hanning window:

$$w[n] = \frac{1}{2} \left[ 1 - \cos \frac{2\pi n}{N-1} \right], \quad 0 \leq n \leq N-1$$

- The difference between the width of the center and the first sidelobe is -31.5 dB, the bandwidth of the center lobe is  $8\pi/N$  and the attenuation rate of the sidelobes is 18 dB /octave.



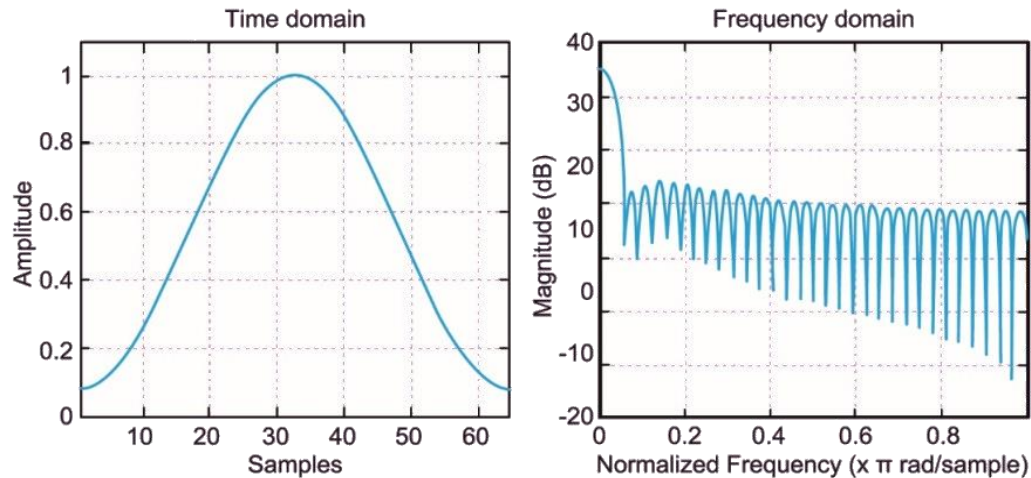
Hanning window and amplitude spectrum ( $N = 64$ )

# Hamming Window

- Hamming window:

$$w[n] = 0.54 - 0.46 \cos \frac{2\pi n}{N-1}, \quad 0 \leq n \leq N-1$$

- The difference between the width of the center and the first sidelobe is -42.7 dB, the bandwidth of the center lobe is  $8\pi/N$  and the attenuation rate of the sidelobes is 6 dB/octave.



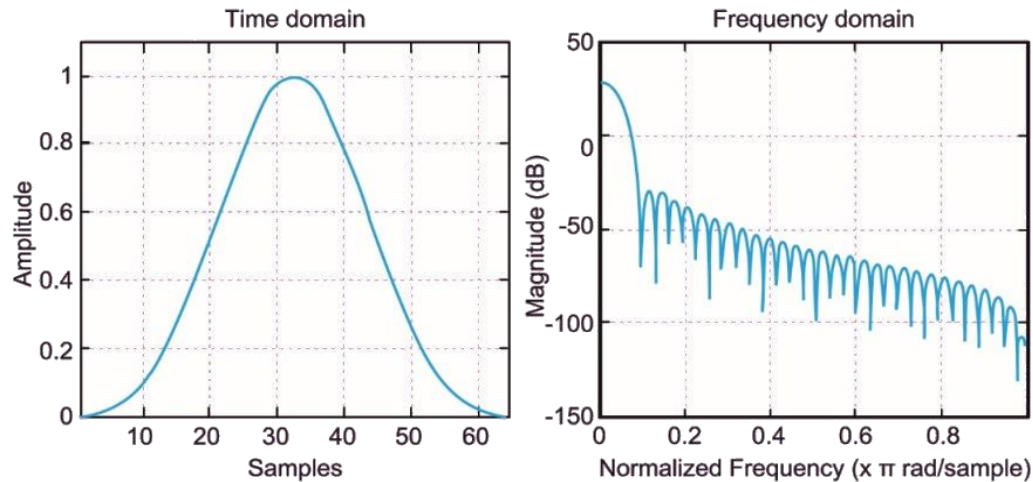
Hamming window and amplitude spectrum ( $N = 64$ )

# Blackman Window

- Blackman window:

$$w[n] = 0.42 - 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1}, \quad 0 \leq n \leq N-1$$

- The difference between the width of the central and the first lateral lobe is -58.1 dB, the bandwidth of the center lobe is  $12\pi/N$  and the attenuation rate of the side lobes is 18 dB/octave.



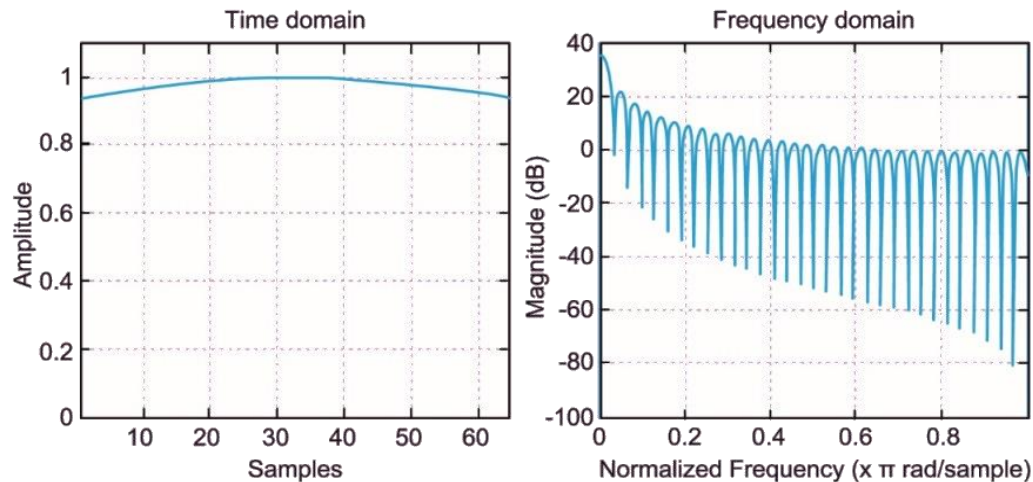
Blackman window and amplitude spectrum ( $N = 64$ )

# Kaiser Window

- Kaiser window:

$$w[n] = \frac{I_0 \left[ \beta \sqrt{1 - \left(1 - \frac{2N}{N-1}\right)^2} \right]}{I_0(\beta)}, \quad 0 \leq n \leq N - 1$$

- $I_0(\beta)$  modified Bessel function of zero order
- $\beta$  parameter that depends on  $N$  and is chosen to achieve various transition band widths and near-optimal attenuation in the cutoff band.
- For the  $\beta = 0$  Kaiser window it turns into a rectangular window.
- The difference between the width of the central and the first side lobe: 25 dB,



Kaiser window and amplitude spectrum ( $N = 64$ )

# Summary of Windows' Features

Window sequence	Main lobe range	Sidelobe Attenuation Rate (dB)	Maximum Sidelobe Magnitude (dB)
Rectangle	$4\pi/N$	-6	-13.3
Bartlett	$8\pi/N$	-12	-26.5
Hanning	$8\pi/N$	-18	-31.5
Hamming	$8\pi/N$	-6	-42.7
Blackman	$12\pi/N$	-18	-58.1