University of the Peloponnese
Electrical and Computer Engineering Department

Digital Signal Processing

Unit 08: Discrete Fourier Transform (DFT)

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Lecture Content

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  - DFT of Aperiodic Signals
  - Phase/Twiddle Factors
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Discrete Fourier Series (DFS)

Discrete Fourier Series (DFS)

A periodic sequence $x[n]$ with period N and fundamental cyclic frequency $\omega_0 = 2\pi/N$, resolves into a sum of N harmonically correlated complex exponential terms:

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{\,j2\pi nk/N}, \qquad 0 \le n \le N-1$$

where the coefficients of the discrete Fourier series $X[k]$are calculated by:

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi nk/N}, \qquad 0 \le k \le N-1$$

The coefficients $X[k]$ correspond to harmonic frequencies $k\omega_0 n, 0 \le k \le N-1$. In the intermediate frequency values the signal $\tilde{x}[n]$ has no spectral content.

The exponential Fourier series in its discrete form is **inherently periodic** with period N, because of the term$e^{-j2\pi/N}$.

Discrete Fourier Transform (DFT)

- DFT of Periodic Signals
- DFT of Aperiodic Signals
- Phase / Twiddle Factors
- Magnitude and Phase Spectra

DFT of Periodic Signals

Direct DFT (analysis equation):

$$X(k) = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi nk/N} \,, \qquad 0 \leq k \leq N-1$$

The coefficients $X[k]$ correspond to the harmonic frequencies $2\pi nk/N$, όπου $0 \leq k \leq N-1$. In the intermediate frequency values the spectral content of the signal $x[n]$ is zero.

Inverse DFT (composition equation):

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{-j2\pi nk/N} \,, \qquad 0 \leq n \leq N-1$$

If we calculate its value $x[n]$ for $n \geq N$ then we will get a periodic expansion $\tilde{x}[n]$ of it $x[n]$ and not zero values, as one might expect.

DFT of Aperiodic Signals

If $x[n]$ it is an aperiodic signal, we can calculate its DFT by sampling L points in the DTFT $X(e^{j\omega})$, i.e. at frequencies $\omega_k = 2\pi k/L,\ \ 0 \leq k \leq L-1$.

But sampling in the time domain creates periodicity in the frequency domain and vice versa. Therefore, pointwise L sampling of the DTFT $X(e^{j\omega})$ will produce **periodicity** in time, i.e. when calculating the inverse DFT the periodic signal will be produced:

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n+kL]$$

If the aperiodic signal $x[n]$ is of finite length $N$and zeros outside the interval $[0, N-1]$, then the periodic extension $\tilde{x}[n]$ of the signal $x[n]$ is:

$$\tilde{x}[n] = x[n \bmod N] = x[[n]]_N$$

The operation $x[n \bmod N]$ shifts the values of the sequence in the interval from 0 to $N-1$.

DFT of Aperiodic Signals

We distinguish the following cases depending on the values of the parameters L and N:

- If $L = N$ then the first period of the periodic expansion $\tilde{x}[n]$ exactly coincides with the signal $x[n]$.
- If $L \geq N$ then the first period of the periodic expansion $\tilde{x}[n]$ is identical to the signal $x[n]$ by adding $L - N$ zeros to its end.
- If $L < N$ then the first period of the periodic expansion $\tilde{x}[n]$ is not identical to the signal $x[n]$, as time-shifted versions of the signal are added and create the effect of folding in time (time - aliasing), as shown in the figure on the next slide.

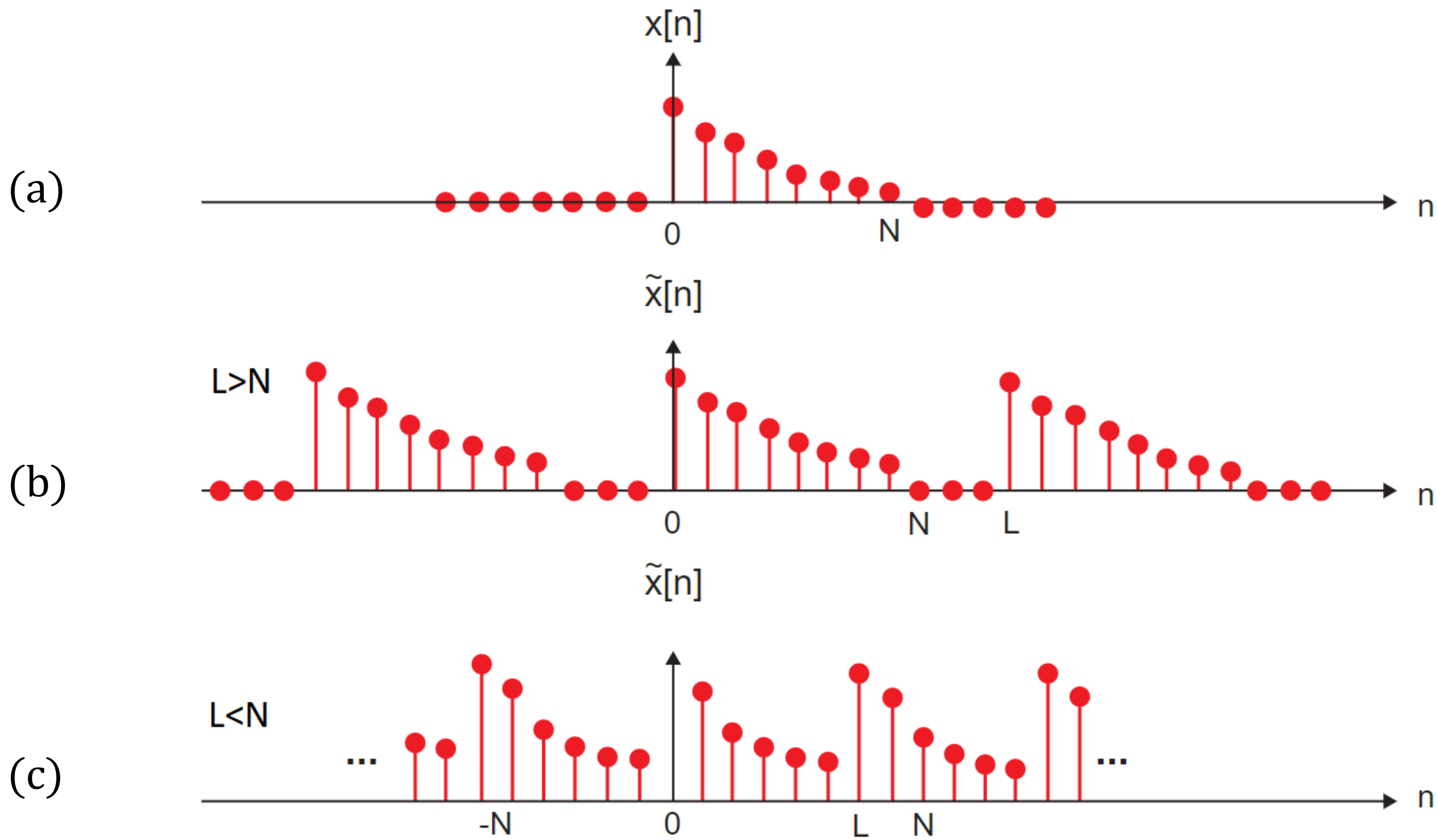
The first two cases are acceptable, while the third is not.

To avoid time folding, the length L of the DFT must be greater than or equal to the duration $N$of the aperiodic signal ($L \geq N$).

In this case the DFT of the periodic expansion $\tilde{x}[n]$ is:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j2\pi nk/L}, \qquad 0 \leq k \leq L-1$$

DFT of Aperiodic Signals



(a) Aperiodic signal $x[n]$, (b) Its periodic expansion $\tilde{x}[n]$for $L \geq N$, (c) For periodic expansion for $L < N$ appears convolution in the time domain.

DFT of Aperiodic Signals

The DFT of the aperiodic signal $x[n]$ is:

$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/L} = \sum_{n=0}^{L-1} x[n]\, e^{-\,j2\pi nk/L}, \qquad 0 \le k \le L-1$$

The inverse DFT is the exponential Fourier series of $\tilde{x}[n]$, that is:

$$x[n] = \frac{1}{L}\sum_{k=0}^{L-1} X[k]\, e^{\,j2\pi nk/L}, \qquad 0 \le n \le L-1$$

where $X[k] = X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/L}$, that is, the L samples of DTFT.

In practice we do not need to create a periodic extension of the aperiodic signal, but it is enough to add a number **of zeros** (zero padding) at the end of it to satisfy the condition $L \ge N$.

Definition of DFT

In both cases of signals (periodic, aperiodic) the term $e^{-j2\pi/N}$ makes the sequence periodic $X[k]$ of the DFT coefficients, with a period equal to the number N of samples of the sequence $x[n]$.

Its samples $X[k]$ start from $k = 0$, which corresponds to the frequency $\omega = 0$, and reach up to $k = N - 1$. They do not include $k = N$, which corresponds to frequency $\omega = 2\pi$ and is included in the next period.

Putting the term $W_N = e^{-\frac{j2\pi}{N}}$, we get the definitions:

- **Direct DFT**

$$X(k) = \sum_{n=0}^{N-1} x[n]\, W_N{}^{nk}, \qquad 0 \leq k \leq N - 1$$

- **Inverse DFT**

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N{}^{-nk}, \qquad 0 \leq n \leq N - 1$$

The complex terms $W_N = e^{-\frac{j2\pi}{N}}$ are called **phase factors** or **twiddle factors**.

Phase / Twiddle Factors

- The phase/twiddle factors W_N^k are given by the equation:

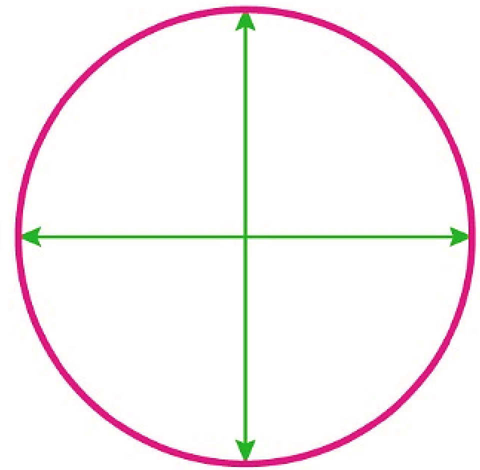
$$W_N = e^{-\frac{j2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\,\sin\left(\frac{2\pi}{N}\right)$$

- They are the N-th complex root of unity, as they have unit Magnitude and simply different phase.
- They are rendered as vectors on the unit circle in the complex plane.
- Phase / Twiddle factors for N=4:

$W_4^0 = 1$	$W_4^4 = W_4^0 = 1$	$W_4^8 = W_4^0 = 1$
$W_4^1 = -j$	$W_4^5 = W_4^1 = -j$	$W_4^9 = W_4^1 = -j$
$W_4^2 = -1$	$W_4^6 = W_4^2 = -1$	$W_4^{10} = W_4^2 = -1$
$W_4^3 = j$	$W_4^7 = W_4^3 = j$	$W_4^{11} = W_4^3 = j$

Properties of Phase / Twiddle Factors

4-point DFT
phase/twiddle factors



Properties of Phase / Twiddle Factors

- $W_N^{k+N} = W_N^k$ (periodicity)
- $W_N^{k+\frac{N}{2}} = -W_N^k$ (symmetry)
- $W_N^2 = W_{N/2}$
- If N is a power of 2, the vectors appear in pairs of conjugate complex numbers.

The implementation of the properties in the DFT calculation leads to a significant reduction in the number of operations to calculate the phase factors and thus dramatically reduces the cost of the DFT calculation.

Magnitude, Phase and Power Spectra

- **Magnitude Spectrum:**

$$|X[k]| = \sqrt{X_R^2[k] + X_I^2[k]}, \qquad 0 \le k \le N-1$$

Even symmetry because: $|X[N-k]| = |X[k]|$

- **Phase Spectrum:**

$$\varphi_X[k] = tan^{-1}\left[\frac{X_I[k]}{X_R[k]}\right], \qquad 0 \le k \le N-1$$

Odd symmetry because: $\measuredangle X[N-k] = -\measuredangle X[k]$

- **Power Spectrum:**

$$P[k] = \frac{1}{N^2}|X[k]|^2 = \frac{1}{N^2}\{X_R^2[k] + X_I^2[k]\}, \qquad 0 \le k \le N-1$$

Example 1

Compute the N-point DFT of the following sequences:

(a) $x[n] = \delta[n]$ (b) $x[n] = \delta[n - n_0]$

Answer: (a) $\delta[n]$ DFT will be calculated from the definition:

$$X[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{nk} = \delta[0] W_N^0 = 1, \qquad k = 0,1,\dots,N-1$$

That is, it is $X[k] = [1,1,1,\dots 1]$(set N).

(b) Likewise:

$$X[k] = \sum_{n=0}^{N-1} \delta[n - n_0] W_N^{nk} = \delta[0] W_N^{n_0} = W_N^{n_0}, \qquad k = 0,1,\dots,N-1$$

That is, it is $X[k] = \left[1, W_N^{n_0}, W_N^{2n_0}, \dots, W_N^{(N-1)n_0}\right]$

Comparing the results, we notice that the time shift of $\delta[n]$ by n_0 produces a DFT of the same magnitude but with a phase shift.

Example 2

Compute the N-point DFT of $x[n] = u[n] - u[n - n_0$ of, $(0 \leq n_0 \leq N - 1)$.

Answer: The DFT of the given pulse is calculated from the definition, as follows:

$$X[k] = \sum_{n=0}^{n_0-1} W_N^{nk} = \frac{1 - W_N^{kn_0}}{1 - W_N^{k}}, \qquad k = 0,1,\dots,N-1$$

Factoring out the term $W_N^{kn_0/2}$ in the numerator and the term $W_N^{k/2}$ in the denominator, the DFT is written:

$$X[k] = \frac{W_N^{kn_0/2}\left(W_N^{-kn_0/2} - W_N^{kn_0/2}\right)}{W_N^{k/2}\left(W_N^{-k/2} - W_N^{k/2}\right)} = W_N^{k(n_0-1)/2}\frac{\left(W_N^{-kn_0/2} - W_N^{kn_0/2}\right)}{\left(W_N^{-k/2} - W_N^{k/2}\right)}$$

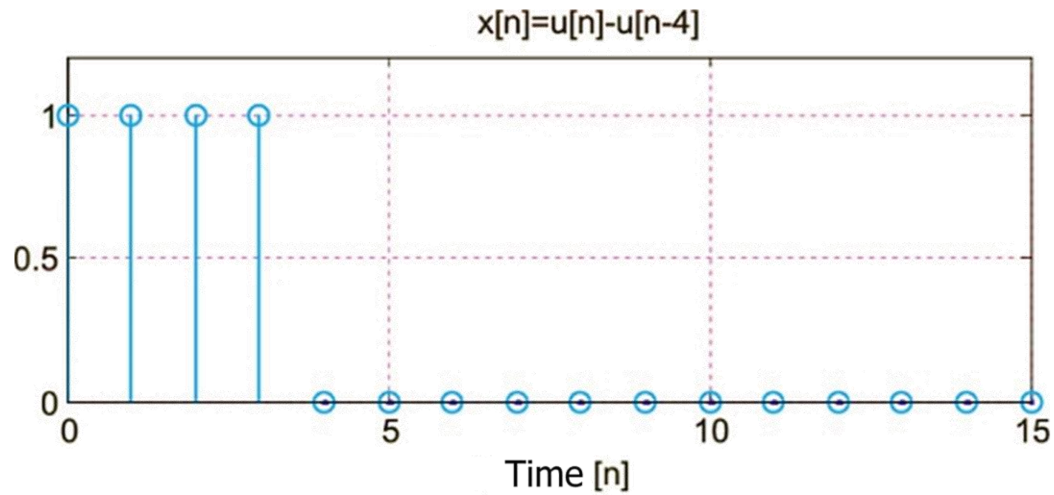
Using the Euler equation, we have:

$$X[k] = e^{-j\frac{2\pi k}{N}\left(\frac{n_0-1}{2}\right)}\frac{sin(n_0\pi k/N)}{sin(\pi k/N)}, \qquad k = 0,1,\dots,N-1$$

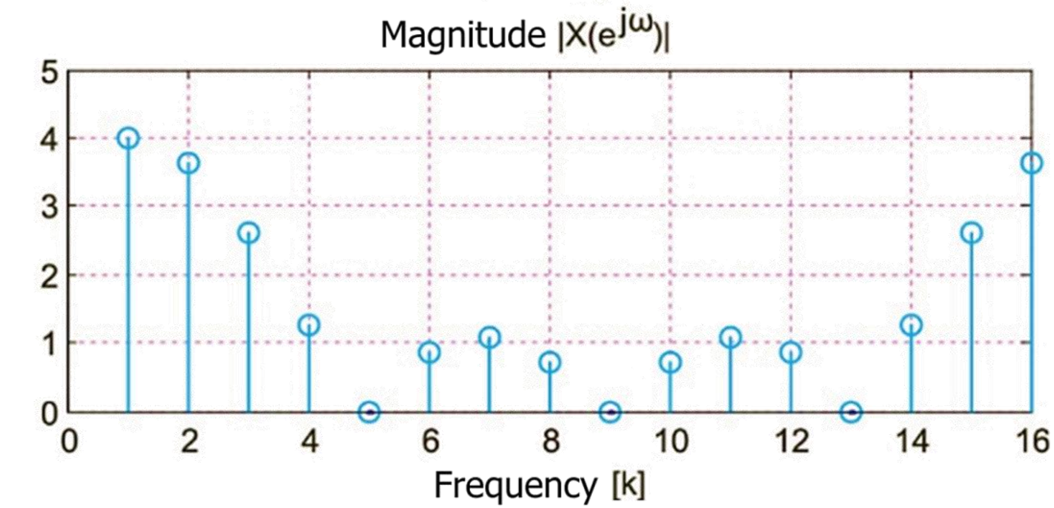
Example 2 (continued)

(a) Rectangular pulse

$x[n] = u[n] - u[n-4]$



(b) Spectrum of a 16-point DFT



Relationship of DFT with other Transforms

- With the Discrete Time Fourier Transform (DTFT)
- With the Z-transform

Relationship between DFT and DTFT

- DTFT of aperiodic sequence $x[n]$ of finite length N points:

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{N-1} x[n]\, e^{-jn\omega}, \qquad 0 \le \omega \le 2\pi$$

- We convert the continuous frequency ω to a discrete frequency ω_k:

$$\omega_k = \frac{2\pi}{N} k, \qquad k = 0,1,\dots,N-1$$

- We sample the continuous function $X(e^{j\omega})$ and obtain:

$$X[k] \equiv X\left(\frac{2\pi}{N} k\right) = X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x[n]\, e^{-\frac{j2\pi}{N}nk}, \qquad 0 \le k \le N-1$$

- Therefore, the DFT $X[k]$results from **sampling the DTFT $\boldsymbol{X\left(e^{j\omega}\right)}$**.

- Indicators: n– time, k – frequency

Relationship between DFT and DTFT

- The quantity $\Delta\omega = 2\pi/N$ is the distance between successive samples of the DTFT and is called **the density** of the DFT in the frequency domain.
- The density improves as the number of N DFT coefficients increases.
- Attention: We must not confuse the density of the spectrum with the **sharpness** of the spectrum (see next Example).

Important notes about DTFT

- If $x[n]$ it is periodic with a period then the N, DFT values are the coefficients of the exponential Fourier series, which exist only at the harmonic frequencies $2k\pi/N$.
- If $x[n]$ is aperiodic, then the number of possible frequencies depends on the length Lchosen to calculate the DFT. If $L \geq N$ the frequencies we calculate the DFT can be considered as frequencies on the unit circle.
- In both signal cases (periodic, aperiodic) it is desirable to have a significant number of points on the unit circle in order to adequately describe the frequency content of the signal.
- The number of frequencies is related to the frequency resolution of the DFT.

Important notes about DTFT

- If the signal is aperiodic, we can increase the frequency resolution of the DFT by increasing the number of samples of the signal by adding zeros to the end of the signal. Adding zeros does not affect the frequency content of the signal, but increases the number of spectral coefficients produced by the DFT.
- If the signal is periodic with period N, then the harmonic frequencies are placed in the positions $2k\pi/N$. In this case we cannot add zero points because they are not part of the periodic signal, but we can get more periods of the signal in the DFT calculation.
- The DFT values correspond to the harmonic frequencies regardless of the number of periods we will use. The more periods we include, the higher the frequency definition will be.

Example 3

(a) Compute the 4-point DFT of the pulse $x[n] = u[n] - u[n-4]$

Answer: Valid: $x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$. The 4-point DFT is:

$$X[k] = \sum_{n=0}^{3} x[n]\, W_4^{nk} = W_4^0 + W_4^{1k} + W_4^{2k} + W_4^{3k}, \qquad 0 \le k \le 3$$

We calculate the points $X[k]$for $0 \le k \le 3$. Is:

$$X[0] = W_4^0 + W_4^0 + W_4^0 + W_4^0 = 1 + 1 + 1 + 1 = 4$$
$$X[1] = W_4^0 + W_4^1 + W_4^2 + W_4^3 = 1 - j - 1 + j = 0$$
$$X[2] = W_4^0 + W_4^2 + W_4^4 + W_4^6 = 1 - 1 + 1 - 1 = 0$$
$$X[3] = W_4^0 + W_4^3 + W_4^6 + W_4^9 = 1 + j - 1 - j = 0$$

Therefore the 4-point DFT is:

$$X[k] = \begin{cases} 4, & n = 0 \\ 0, & n = 1,2,\ldots \end{cases}$$

DTFT is:

$$X(e^{j\omega}) = e^{-\frac{j3\omega}{2}} \frac{\sin(2\omega)}{\sin(\omega/2)}$$

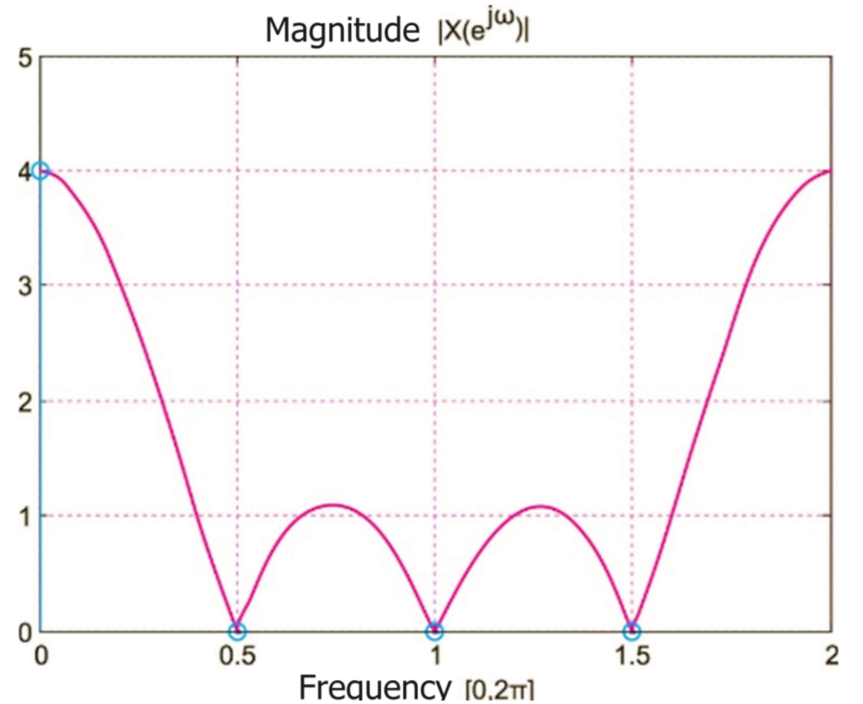
Example 3 (continued)

DTFT width spectrum (red color) and

4-point DFT (blue color) of the pulse

$x[n] = u[n] - u[n-4]$ in the

frequency range $[0, 2\pi)$. The phase is zero.



The points of the DFT are obtained by sampling the DTFT, according to the equation:

$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega=\pi k/2}, \qquad k = 0,1,2,3$$

Example 3 (continued)

(b) To repeat the solution for 8-point ($N = 8$)DFT, after applying the procedure of adding zeros (zero padding) to the sequence $x[n]$.

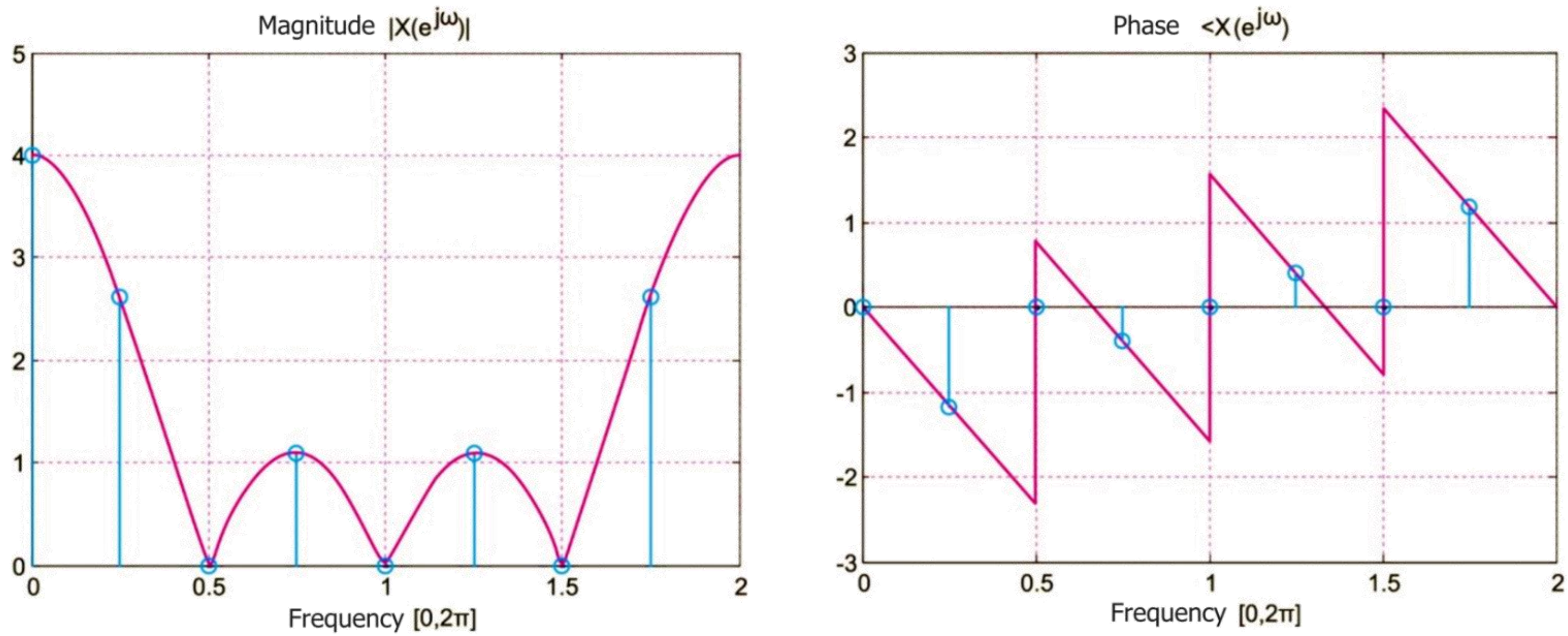
Answer: The 8-point $x[n] = u[n] - u[n-4]$DFT of the pulse is:

$$X[k] = \sum_{n=0}^{7} x[n]\, W_8^{nk}, \qquad 0 \leq k \leq 7$$

We calculate the points $X[k]$for $0 \leq k \leq 7$. Is:

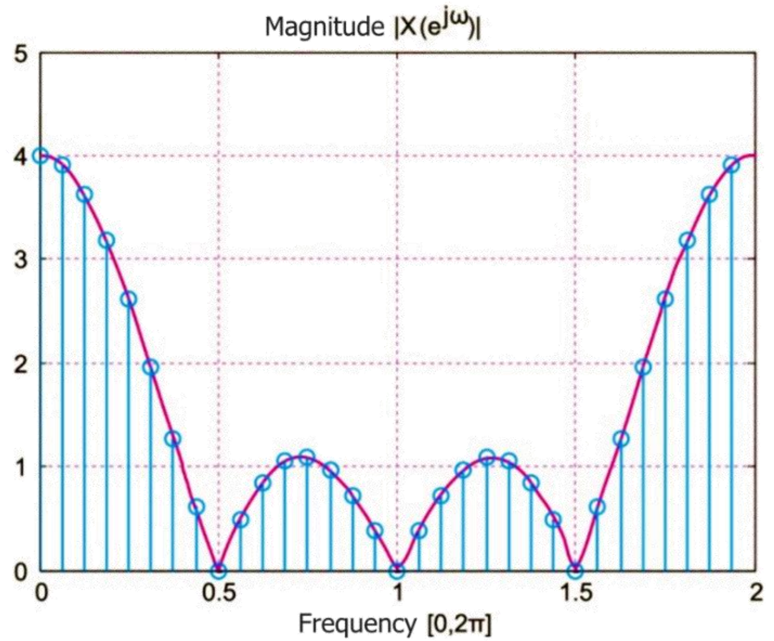
- $X[0] = 1W_8^0 + 1W_8^0 + 1W_8^0 + 1W_8^0 + 0W_8^0 + 0W_8^0 + 0W_8^0 + 0W_8^0 = 4$
- $X[1] = 1W_8^0 + 1W_8^1 + 1W_8^2 + 1W_8^3 + 0W_8^4 + 0W_8^5 + 0W_8^6 + 0W_8^7 = 1 - 2.41j$
- $X[2] = 1W_8^0 + 1W_8^2 + 1W_8^4 + 1W_8^6 + 0W_8^8 + 0W_8^{10} + 0W_8^{12} + 0W_8^{14} = 0$
- $X[3] = 1W_8^0 + 1W_8^3 + 1W_8^6 + 1W_8^9 + 0W_8^{12} + 0W_8^{15} + 0W_8^{18} + 0W_8^{21} = 1 - 0.41j$
- $X[4] = 1W_8^0 + 1W_8^4 + 1W_8^8 + 1W_8^{12} + 0W_8^{16} + 0W_8^{20} + 0W_8^{24} + 0W_8^{28} = 0$
- $X[5] = 1W_8^0 + 1W_8^5 + 1W_8^{10} + 1W_8^{15} + 0W_8^{20} + 0W_8^{25} + 0W_8^{30} + 0W_8^{35} = 1 + 0.41j$
- $X[6] = 1W_8^0 + 1W_8^6 + 1W_8^{12} + 1W_8^{18} + 0W_8^{24} + 0W_8^{30} + 0W_8^{36} + 0W_8^{42} = 0$
- $X[7] = 1W_8^0 + 1W_8^7 + 1W_8^{14} + 1W_8^{21} + 0W_8^{28} + 0W_8^{35} + 0W_8^{42} + 0W_8^{49} = 1 + 2.41j$

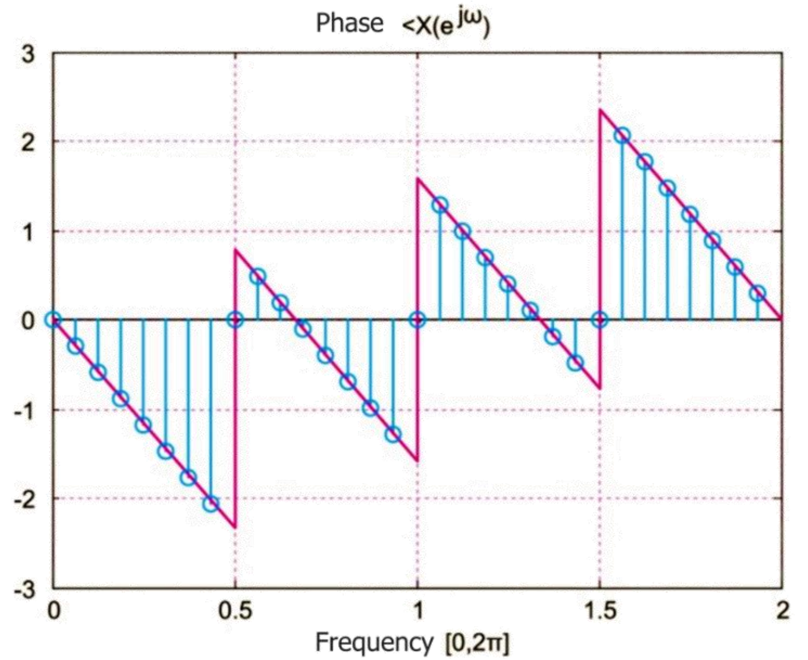
Example 3 (continued)



Spectra of magnitude and phase for DTFT (red color) and 8-point DFT (blue color) of the pulse $x[n]=u[n]-u[n-4]$ in the frequency range $[0,2\pi)$.

Example 3 (continued)





Spectra of magnitude and phase for DTFT (red color) and 32-point DFT (blue color) of the pulse $x[n]=u[n]-u[n-4]$ in the frequency range $[0,2\pi)$.

Example 3 (conclusions)

- The **magnitude** and **phase** of the DFT have **even** and **odd symmetry,** respectively. Therefore, we keep only the part $[0, \pi)$, i.e. $X[k],\ k = 0,1, \dots, (N/2) - 1$.
- For a given sequence, the **density** of DFT samples increases proportionally to **the number** of samples of $x[n]$. This is done by the process **of adding zeros** (zero-padding):
  - We create a periodic extension $\tilde{x}[n]$ of $x[n]$ length $L \geq N$, adding $L - N$ zeros to its end.
  - We compute the L-point DFT $\tilde{X}[k]$ of the sequence $\tilde{x}[n]$.
  - DFT $X[k]$ of sequence $x[n]$ is: $X[k] = \tilde{X}[k],\ \ \gamma\iota\alpha\ 0 \leq k \leq L - 1$.
- Adding zeros does not improve the sharpness of the DFT, it simply reduces the distance between its successive samples $X[k]$.
- The **sharpness** of the DTF increases according to the number of samples of the signal.
- If the sequence is **periodic,** then the sharpness of the DFT increases if more than one period of the sequence is included in the DFT calculation.

Example 4

Calculate and design in Matlab the DFT of the signal:

$$x[n] = cos\left(\frac{4\pi n}{9}\right) + \ cos\left(\frac{5\pi n}{9}\right)$$

for the following cases:

α) DFT with 10 points for $0 \le n \le 9$.

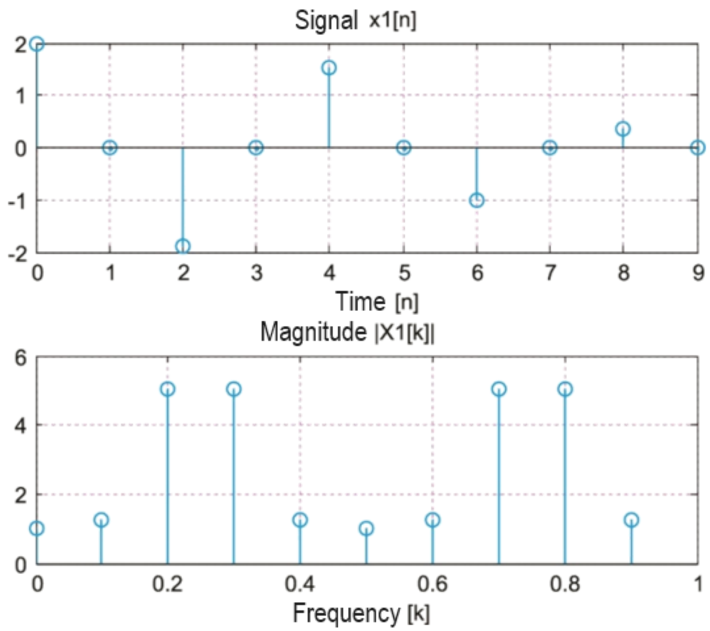
β) DFT with 100 points for $0 \le n \le 9$ and the remaining 90 points being zero.

γ) DFT with 100 points for $0 \le n \le 99$.

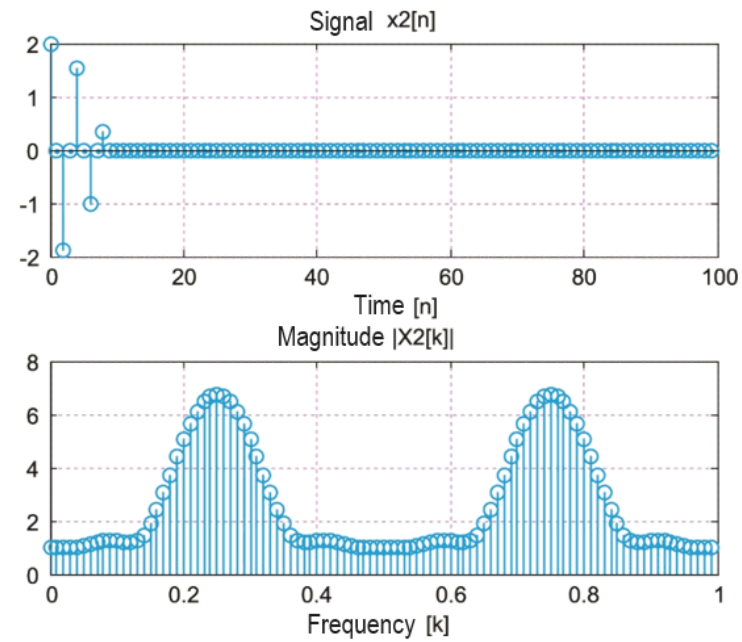
Answer: Given the Matlab code of example 10.5 of the book, the following diagrams result.

Example 4 (continued)

(a)



(b)



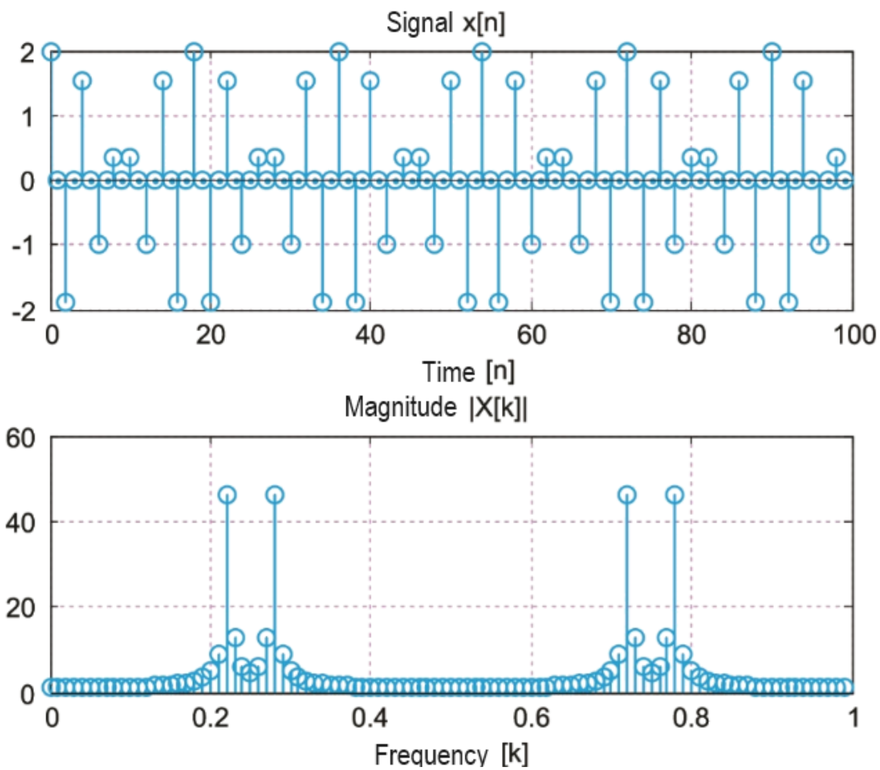
(a) Signal of length 10 points and magnitude spectrum of a 10-point DFT.

(b) Signal of length 10 points with 90 zeros and magnitude spectrum of a 100-point DFT.

Adding zeros and calculating a longer DFT **increased the density** of the spectrum, but failed to identify the two different frequencies contained in the signal $x[n]$, as both are rendered as one peak.

Example 4 (continued)

(c) 100-point length signal and 100-point DFT width spectrum



From diagrams (b) and (c), it can be seen that adding more periods of the signal (instead of zeros) and calculating a longer DFT **increased the sharpness** of the spectrum and accurately located the two different frequencies contained in the signal $x[n]$.

Relationship between DFT and Z-transform

- DFT coefficients correspond to N samples of $X(z)$, which have been taken at$N$ equidistant points on the unit circle:

$$X[k] = X(z)\Big|_{z=e^{j2\pi k/N}}$$

- The above is valid under the condition that the unit circle is contained in the region of convergence of the Z-transform.

DFT Calculation using Linear Algebra

DFT Calculation using Linear Algebra

The **straight DFT** is calculated from:

$$\boldsymbol{X}^T = \boldsymbol{W_N}\, \boldsymbol{x}^T$$

- $\boldsymbol{x}^T = \{ x[0], x[1], \dots, x[N-1]\}$ transpose input sequence table.
- $\boldsymbol{X}^T = \{ X[0], X[1], \dots, X[N-1]\}$ inverse matrix of the sequence of coefficients of the DFT.
- $\boldsymbol{W_N}$ symmetric matrix of dimensions NxNgenerated by the phase factors:

$$\boldsymbol{W_N} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^1 & W_N^2 & \dots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

DFT Calculation using Linear Algebra

Each column $\boldsymbol{W_i}$ of the matrix $\boldsymbol{W_N}$ is called **the basis vector** of the DFT:

$$\boldsymbol{W_i} = \begin{bmatrix} 1 \\ W_N^i \\ W_N^{2i} \\ \vdots \\ W_N^{(N-1)i} \end{bmatrix}$$

If the inverse matrix exists $\boldsymbol{W_N^{-1}}$, then the <u>**inverse DFT**</u> given by:

$$\boldsymbol{x}^T = \boldsymbol{W_N^{-1}}\, \boldsymbol{X}^T$$

Since holds $\boldsymbol{W_N^{-1}} = (1/N)\boldsymbol{W_N^*}$, where $\boldsymbol{W_N^*}$ is its conjugate complex matrix $\boldsymbol{W_N}$**,** it follows that the inverse DFT is:

$$\boldsymbol{x}^T = \frac{1}{N}\boldsymbol{W_N^*}\, \boldsymbol{X}^T$$

In the next section we will study a more efficient way of computing the DFT, which exploits the symmetry properties of the DFT and the phase factor and drastically reduces the number of required operations.

Example 5

Calculate the 4-point DFT of sequence $x[n] = [1, 3, 5, 7]$

Answer: The table $\boldsymbol{W_4}$ is:

$$\boldsymbol{W_4} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

The DFT values are calculated from:

$$\boldsymbol{X}^T = \boldsymbol{W_N}\, \boldsymbol{x}^T \Rightarrow \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Example 5 (continued)

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 16 \\ -4+4j \\ -4 \\ -4-4j \end{bmatrix}$$

Therefore the DFT is: $X[k] = \{16,\ -4+4j,\ -4,\ -4-4j\}$

Circular Convolution

- Periodic sequence extension
- Periodic convolution
- Circular shift sequence
- Calculation of circular convolution

The Concept of Circular Convolution

When the sequences are periodic, the procedure for calculating the convolution is different.

To explain it we will need to define:

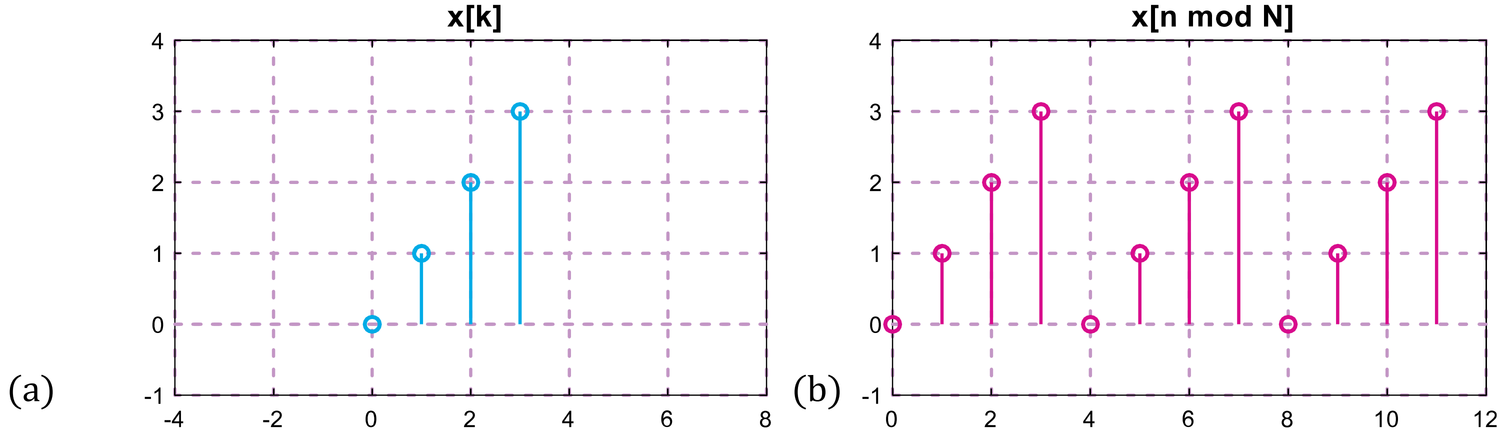
- the **periodic extension** of a sequence
- the **circular shift** of the periodic expansion

Also, we will study the terms of **periodic convolution** and **circular convolution.**

Periodic Sequence Extension

For any sequence $x[n]$of finite length, a periodic sequence $0 \leq n \leq N-1$with fundamental period N, called **the periodic extension** of the sequence by samples, N can be defined $\tilde{x}[n]$ according to the equation:

$$\tilde{x}[n] = x[n \bmod N] = x\big[[n]\big]_N$$



(a) Sequence of finite length $x[n]$
(b) Its periodic extension $\tilde{x}[n] = x[n\ mod\ 4] = x\big[[n]\big]_4$

Periodic Convolution

We define **periodic convolution** $\tilde{y}[n]$,of two discrete-time periodic signals $x_1[n]$ and $x_2[n]$ having the same fundamental period N, from the equation:

$$\tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{x}_1[k]\, \tilde{x}_2[n-k] = \sum_{k=0}^{N-1} \tilde{x}_2[n-k]\, \tilde{x}_1[k]$$

where $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$are the periodic extensions of the sequences $x_1[n]$ and $x_2[n]$, respectively. Periodic convolution is denoted as follows:

$$\tilde{y}[n] = \tilde{x}_1[n] \circledast \tilde{x}_2[n]$$

- We notice that the only difference between the two types of convolution is that in periodic convolution the sum is calculated over a **single period,** while in linear convolution it is calculated over **all values of *k***.

- Can be used to calculate the periodic convolution that were presented in the previous sections to calculate the linear convolution.

Example 6

Calculate the periodic convolution between the sequences $x[n] = \{\hat{0}, 1, 2, 3\}$ and $h[n] = \{1, \hat{2}, 0, -1\}$.

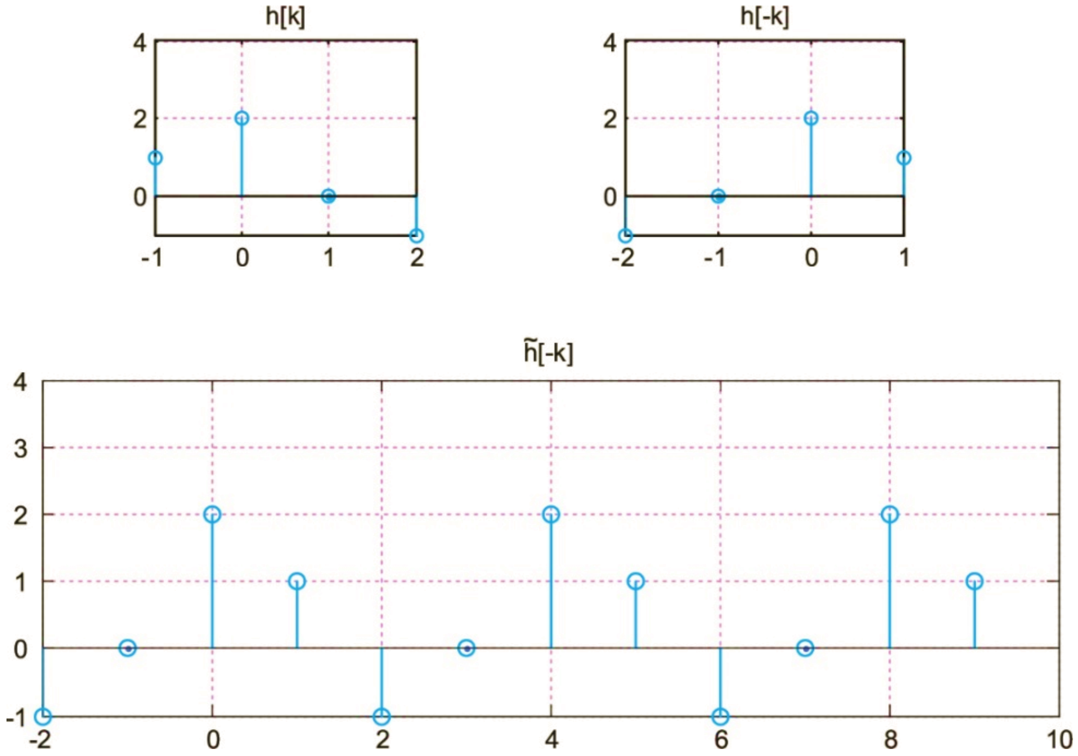
Answer: We use the graphical way of calculating the sum:

$$\tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\, \tilde{h}[n-k]$$

The graph of $x[n]$ and its periodic expansion $\tilde{x}[n]$ for $N = 4$ is:

Example 6 (continued)

The graph of $h[n]$, its reflection $h[-k]$ and its periodic expansion $\tilde{h}[-k]$ for $N = 4$, is shown in the figure:

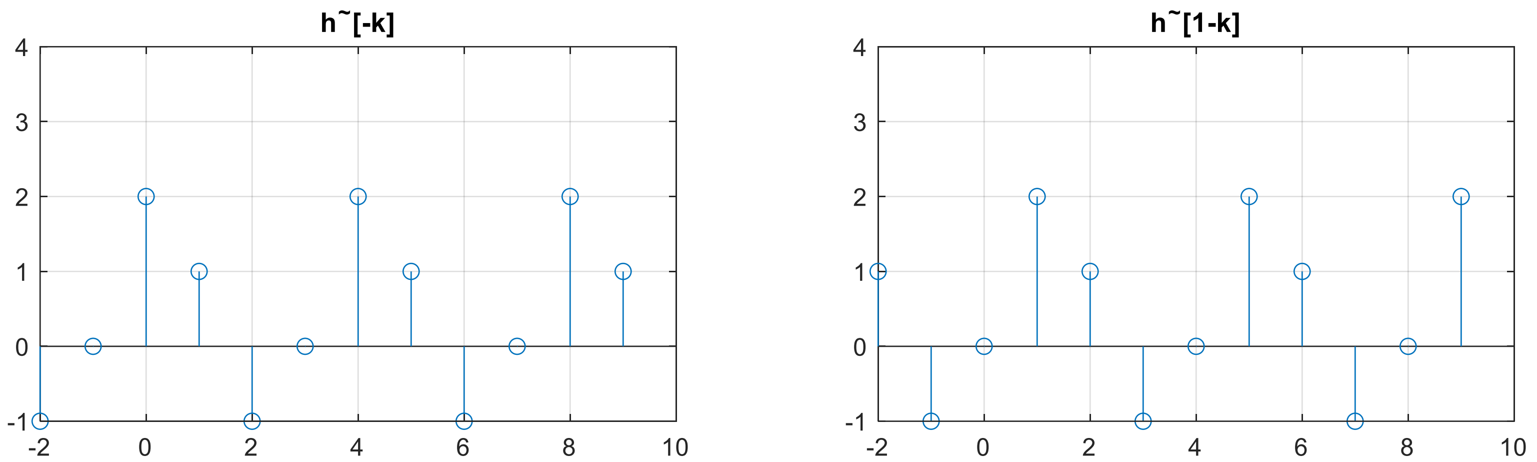


$\tilde{y}[0]$ is founded by summing the products $\tilde{x}[k]\ \tilde{h}[-k]$ from $k = 0$ to 3. They are:

$$\tilde{y}[0] = 0x2 + 1x1 + 2x(-1) + 3x0 = -1$$

Example 6 (continued)

Then, is $\tilde{h}[-k]$ shifted to the right by 1 sample, resulting in $\tilde{h}[1-k]$, as shown in the figure:



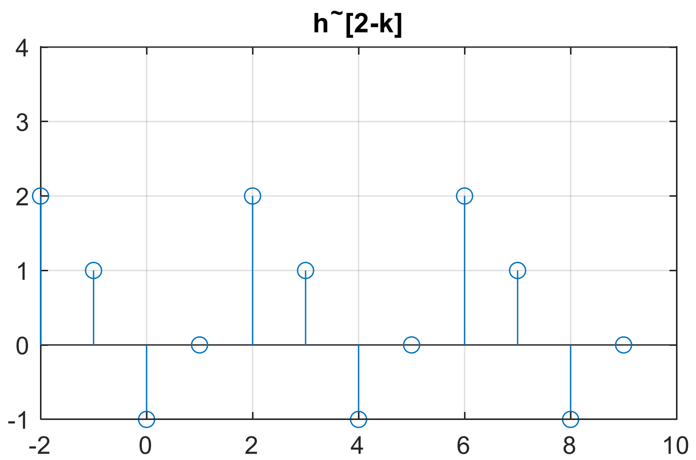
Periodic expansion $h[[-k]]_4$ and periodic expansion slip $\tilde{h}[1-k] = h[[1-k]]_4$

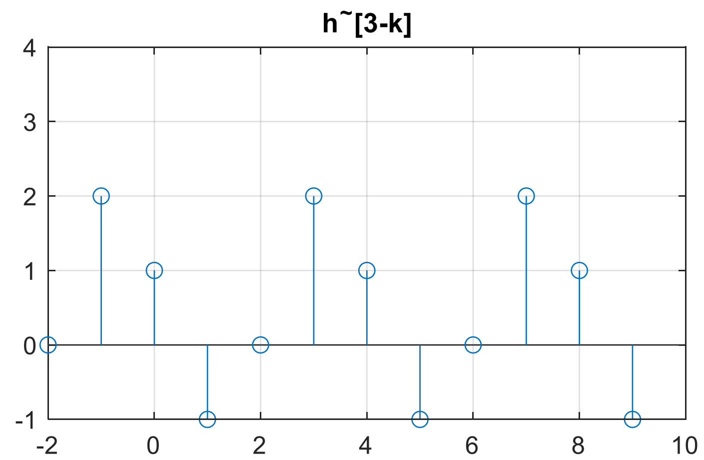
$\tilde{y}[1]$ is founded by summing the products $\tilde{x}[k]\ \tilde{h}[1-k]$ from $k = 0$ to 3. They are:

$$\tilde{y}[1] = 0x0 + 1x2 + 2x1 + 3x(-1) = 1$$

Example 6 (continued)

Shifting it $\tilde{h}[-k]$ to the right by 2 and 3 samples yields $\tilde{h}[2-k]$ and $\tilde{h}[3-k]$, as shown in the figure:





Periodic extension slip $\tilde{h}\big[[2-k]\big]_4 = h\big[[2-k]\big]_4$
and periodic extension slip $\tilde{h}[3-k] = h\big[[3-k]\big]_4$

$\tilde{y}[2]$ is founded by summing the products $\tilde{x}[k]\,\tilde{h}[2-k]$ from $k=0$ to 3. It is:

$$\tilde{y}[2] = 0x(-1) + 1x0 + 2x2 + 3x1 = 7$$

$\tilde{y}[3]$ is founded by summing the products $\tilde{x}[k]\,\tilde{h}[3-k]$ from $k=0$ to 3. It is:

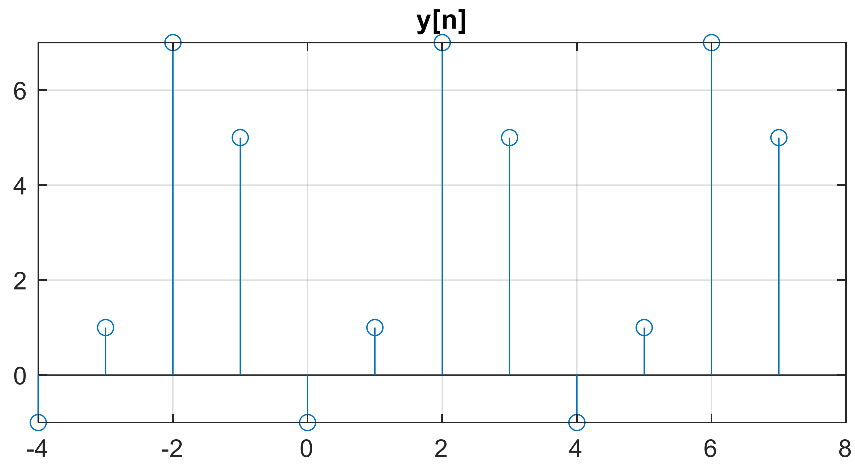
$$\tilde{y}[3] = 0x1 + 1x(-1) + 2x0 + 3x2 = 5$$

Example 6 (continued)

Therefore, the periodic convolution is:

$$\tilde{y}[n] = \{-1, 1, 7, 5, -\hat{1}, 1, 7, 5, -1, 1, 7, 5\}$$

The result is shown in the following figure:



Periodic convolution $\tilde{y}[n]$

Circular Sequence Shift

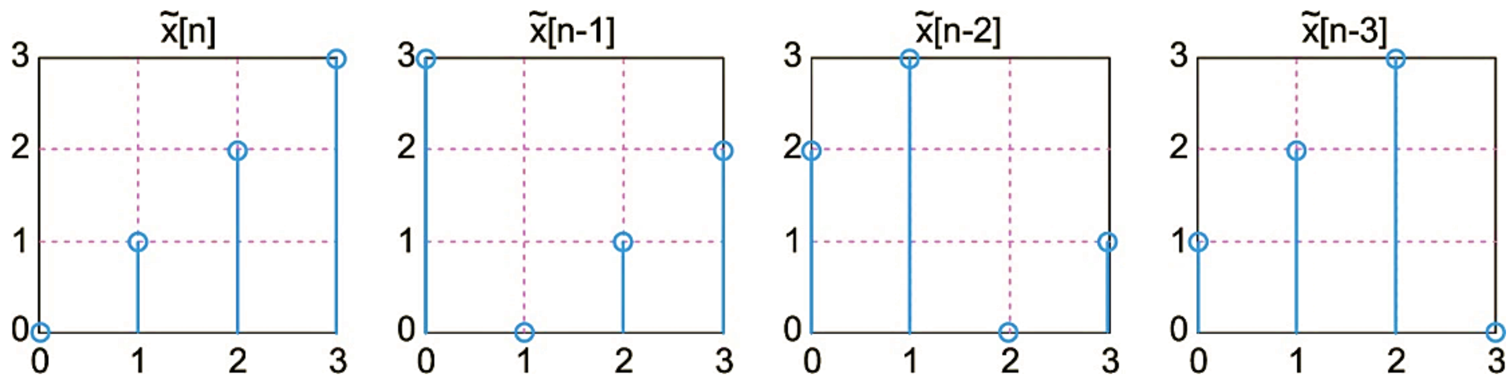
The **circular shift** of the periodic extension $\tilde{x}[n]$ of a sequence $x[n]$, by an amount of time n_0, is defined by the equation:

$$\tilde{x}[n-n_0] = x\big[[n-n_0]\big]_N R_N[n]$$

where the rectangular window $R_N[n]$ is defined by the following equation and multiplied by the signal outputs a period of the signal:

$$R_N[n] = \begin{cases} 1, & 0 \le n < N \\ 0, & \text{elsewhere} \end{cases}$$

Cyclic shifting is performed by shifting the sequence $\tilde{x}[n]$ pointwise n_0 (to the left if $n_0 < 0$ or to the right if $n_0 > 0$) and keeping only the part that lies within the fundamental period N. The process is shown in the figure:



Cyclic shift of the periodic expansion $\tilde{x}[n]$, in one period.

Circular Sequence Shift

By applying the equation:

$$\tilde{x}[n - n_0] = x\big[[n - n_0]\big]_N R_N[n]$$

it follows that the sequences depicted in the previous figure are described by the following relations:

- $\tilde{x}[n] = x\big[[n]\big]_4 R_4[n]$
- $\tilde{x}[n-1] = x\big[[n-1]\big]_4 R_4[n]$
- $\tilde{x}[n-2] = x\big[[n-2]\big]_4 R_4[n]$
- $\tilde{x}[n-3] = x\big[[n-3]\big]_4 R_4[n]$

We notice that the circular shift creates a different sequence than the simple time shift we studied in lecture 2.

Because of this difference arises the different result between linear convolution and circular convolution, which we will see next.

Circular Convolution

The point **circular convolution** of two sequences $x_1[n]$and $x_2[n]$, N each of point length, is defined as:

$$y[n] = \left[\sum_{k=0}^{N-1} \tilde{x}_1[k]\, \tilde{x}_2[n-k]\right] R_N[n] = \left[\sum_{k=0}^{N-1} \tilde{x}_2[n-k]\, \tilde{x}_1[k]\right] R_N[n]$$

where $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ are the periodic extensions of the sequences $x_1[n]$ and $x_2[n]$, respectively.

Since $\tilde{x}_1[n] = x_1[n]$for $0 \leq n \leq N-1$, the above equation is written:

$$y[n] = \left[\sum_{k=0}^{N-1} x_1[k]\, \tilde{x}_2[n-k]\right] R_N[n]$$

The sequence $y[n]$ is called **circular convolution** and is denoted as follows:

$$\tilde{y}[n] = \tilde{x}_1[n] Ⓝ\, \tilde{x}_2[n] = \tilde{x}_2[n] Ⓝ\, \tilde{x}_1[n]$$

Circular Convolution

Remarks:

- The cyclic convolution of two sequences $x_1[n]$ and $x_2[n]$ is equivalent to one period of the periodic convolution of the periodic expansions $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, that is, it holds:

$$y[n] = x_1[n] \circledN\, x_2[n] = \left[\tilde{x}_2[n] \circledast \tilde{x}_1[n]\right] R_N[n]$$

- If the signal $x_1[n]$ is of finite length N_1 and the signal $x_2[n]$ is of finite length N_2, where $N_1 \neq N_2$, then the circular convolution N points are of finite length $N \geq \max\,(N_1, N_2)$and are calculated by padding the signals at the end with zeros (zero - padding), so that they have the same length N.

- Point circular convolution $N$and point circular convolution M, where $N \neq M$, are not generally equal to each other.

- Circular convolution is not the same as linear convolution. Their difference lies in the limits of the sum and the displacement $N$of points.

Example 7

Calculate the circular convolution of 4-points between the discrete time signals $x[n] = \{\hat{0}, 1, 2, 3\}$ and $h[n] = \{1, \hat{2}, 0, -1\}$.

Answer: We calculate the circular convolution of 4-points from the equation:

$$y[n] = \left[\sum_{k=0}^{3} x[k]\,\tilde{h}[n-k]\right] R_4[n]$$

For $n = 0$:

$$y[0] = \left[\sum_{k=0}^{3} x[k]\,\tilde{h}[-k]\right] R_4[n] = \sum_{k=0}^{3} \left\{\hat{0}, 1, 2, 3\right\} \left\{\hat{2}, 1, -1, 0\right\}$$

$$= \sum_{k=0}^{3} \left\{\hat{0}, 1, -2, 0\right\} \Rightarrow y[0] = -1$$

For $n = 1$:

$$y[1] = \left[\sum_{k=0}^{3} x[k]\,\tilde{h}[1-k]\right] R_4[n] = \sum_{k=0}^{3} \left\{\hat{0}, 1, 2, 3\right\} \left\{\hat{0}, 2, 1, -1\right\}$$

$$= \sum_{k=0}^{3} \left\{\hat{0}, 2, 2, -3\right\} \Rightarrow y[1] = 1$$

Example 7 (continued)

For $n = 2$:

$$y[2] = \left[\sum_{k=0}^{3} x[k]\,\tilde{h}[2-k]\right] R_4[n] = \sum_{k=0}^{3} \{\hat{0},1,2,3\}\,\{-\hat{1},0,2,1\}$$

$$= \sum_{k=0}^{3} \{\hat{0},0,4,3\} \Rightarrow y[2] = 7$$

For $n = 3$:

$$y[3] = \left[\sum_{k=0}^{3} x[k]\,\tilde{h}[3-k]\right] R_4[n] = \sum_{k=0}^{3} \{\hat{0},1,2,3\}\,\{\hat{1},-1,0,2\}$$

$$= \sum_{k=0}^{3} \{\hat{0},-1,0,6\} \Rightarrow y[3] = 5$$

Therefore it is: $y[n] = h[n] \text{ ④ } x[n] = \{-\hat{1},1,7,5\}$

- Comparing the result with the result of Example 15, we notice that it **verifies** the relationship $y[n] = x_1[n] \text{ Ⓝ } x_2[n] = [\tilde{x}_2[n] \circledast \tilde{x}_1[n]] R_N[n]$
- The linear convolution between $h[n]$ and $x[n]$, is the sequence of six points:
$$h[n] * x[n] = \{1,\hat{4},7,5,-2,-3\}$$
- We notice that linear convolution and circular convolution of the same sequences give **different** results.

DFT properties

- Linearity
- Circular Folding in Time
- Cyclic Shift in Time
- Conjugation
- DFT Symmetry for Real Sequences
- Symmetry of DFT for Complex Sequences
- Cyclic Shift in Frequency
- Cyclic Shift in Time
- Circular Convolution
- Sequence Multiplication
- Parseval's theorem

Linearity

DFT transformations of the sequences $x_1[n]$ and $x_2[n]$ are:

$$x_1[n] \xleftrightarrow{DFT} X_1[k] \text{and} x_2[n] \xleftrightarrow{DFT} X_2[k]$$

then the DFT of the linear combination $a_1 x_1[n] + \alpha_2 x_2[n]$ is:

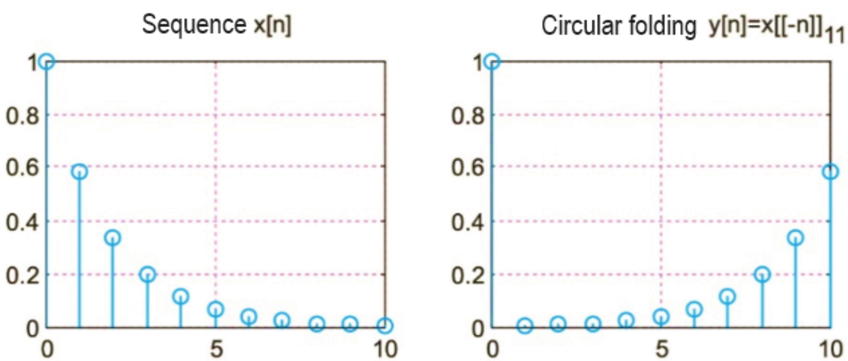
$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \xleftrightarrow{\text{DFT}} a_1 X_1[k] + \alpha_2 X_2[k]$$

- The equation holds for sequences of equal length.
- If the lengths of the sequences are different, then we pad the shorter sequence with zeroes so that it becomes the same length as the longer one.

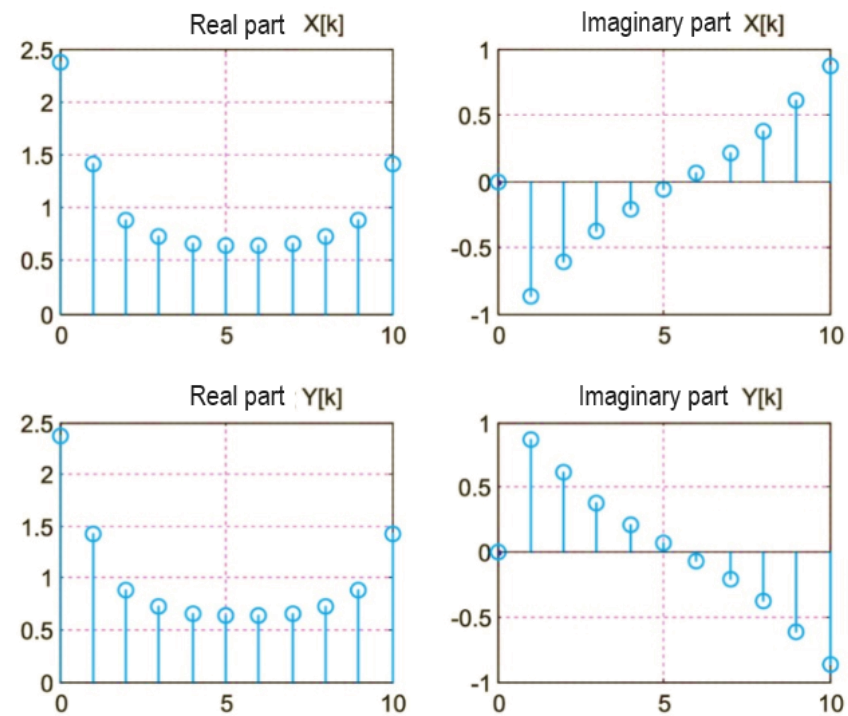
Circular Folding in Time

- If $x[n]$ is a sequence N of points and $X[k]$ is the N-point DFT of it, then for the modulo N inverted sequence $y[n]$:

$$y[n] = x[[-n]]_N \xleftrightarrow{DFT} X[[-k]]_N$$



Sequence $x[n]$ and circular folding $y[n] = x[[-n]]_{11}$



Real and imaginary parts of DFT of $X[k]$and $Y[k]$

Cyclic Shift in Time

If $x[n]$ is a sequence N of points and $X[k]$ is the N-point DFT of it, then for the circularly shifted by $n_0$sequence, defined as:

$$\tilde{x}[n-n_0] = x\big[[n-n_0]\big]_N R_N[n]$$

where the rectangular window $R_N[n]$ extracts one period of the signal and is defined by the equation:

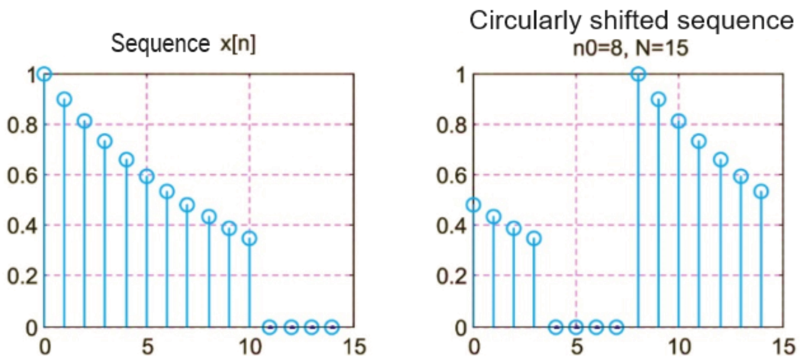
$$R_N[n] = \begin{cases} 1, & 0 \le n < N \\ 0, & \text{elsewhere} \end{cases}$$

then the DFT of the circularly shifted sequence $\tilde{x}[n-n_0]$ is given by:

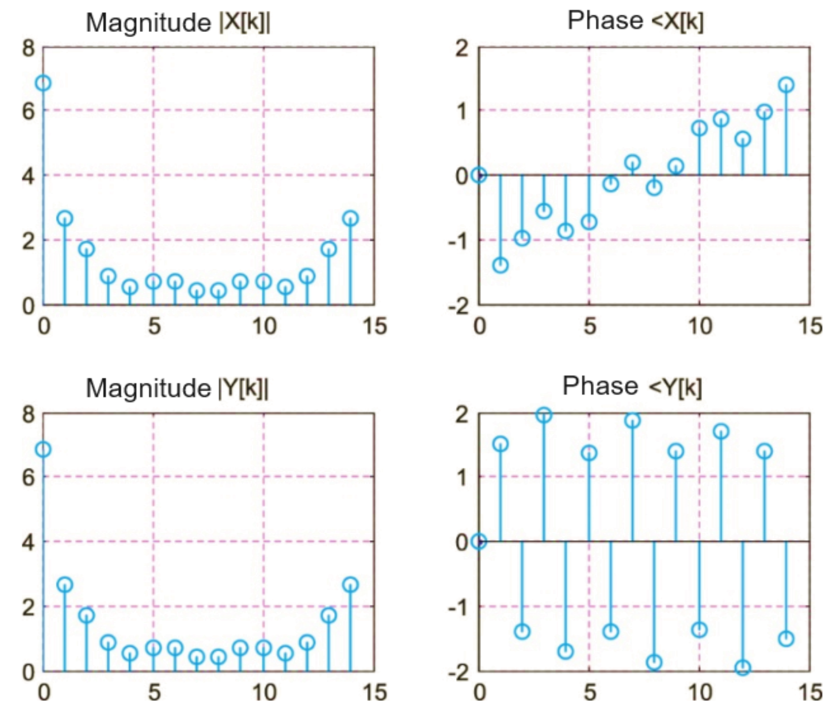
$$x\big[[n-n_0]\big]_N R_N[n] \xleftrightarrow{DFT} W_N^{kn_0}\, X[k]$$

- Since $\left|W_N^{kn_0}\right| = 1$, the Magnitude of the DFT of the circularly shifted sequence is the same as the Magnitude of the DFT of the original sequence.
- The phase of the DFT is shifted by the phase of the term $W_N^{kn_0}$.

Cyclic Shift in Time



Sequence $x[n]$ and circularly shifted sequence $y[n] = x[[n-8]]_{15}$



Magnitude and phase of DFT $X[k]$ and $Y[k]$

Cyclic Shift in Frequency

- This property is dual to the property of circular displacement in time and is described by the equation:

$$W_N^{-k_0 n} x[n] \xleftrightarrow{DFT} X\big[[k - k_0]\big]_N R_N(k)$$

- Therefore, if $x[n]$ is a sequence N-points and $X[k]$ is the N-DFT of its points, then multiplying the sequence by the term $W_N^{-k_0 n}$ has DFT the DFT of the sequence circularly shifted at frequency k_0.

Conjugation

The symmetry property is combined with the **conjugacy property,** which states that if $x[n]$is a sequence N-points and $X[k]$ is the N-point DFT of it, then for the conjugate sequence $x^*[n]$ also N-points, holds:

$$x^*[n] \xleftrightarrow{DFT} X^*\big[[-k]\big]_N = -X^*\,\big[[N-k]\big]_N$$

This property introduces the concept of **circular folding in frequency.**

Symmetry Types of Real Sequences

A real sequence $x[n]$ of point length N with $0 \leq n \leq N-1$, is called:

- **Circularly even** if:

$$x[n] = x\big[[-n]\big]_N = x\big[[N-n]\big]_N$$

- **Cyclic odd** if:

$$x[n] = -x\big[[-n]\big]_N = -x\big[[N-n]\big]_N$$

The real sequence $x[n]$ can be decomposed into a **cyclic even** $x_{ce}[n]$ and one **cyclically odd** $x_{co}[n]$ component, namely:

$$x[n] = x_{ce}[n] + x_{co}[n], \qquad 0 \leq n \leq N-1$$

where:

$$x_{ce}[n] = \frac{1}{2}\Big[x[n] + x\big[[-n]\big]_N\Big], \qquad 0 \leq n \leq N-1$$

$$x_{co}[n] = \frac{1}{2}\Big[x[n] - x\big[[-n]\big]_N\Big], \qquad 0 \leq n \leq N-1$$

DFT Symmetry for Real Sequences

- The components $x_{ce}[n]$ and $x_{co}[n]$ are sequences of length N points and their N-points DFT are:

$$X_{ec}[k] = Re\{X[k]\} = Re\left\{X\big[[-k]\big]_N\right\}$$

$$X_{oc}[k] = Im\{X[k]\} = Im\left\{X\big[[-k]\big]_N\right\}$$

- The DFT of a real sequence is **circularly symmetric,** i.e.:

$$X[k] = X^*\big[[-k]\big]_N = X^*\big[[N-k]\big]_N$$

- This equation is analyzed:

$$Re\{X[k]\} = Re\left\{X\big[[-k]\big]_N\right\}$$

$$Im\{X[k]\} = -Im\left\{X\big[[N-k]\big]_N\right\}$$

$$|X[k]| = \left|X\big[[-k]\big]_N\right|$$

$$\measuredangle X[k] = -\measuredangle X\big[[-k]\big]_N$$

DFT Symmetry for Real Sequences

- For a real sequence $x[n]$, the DFT coefficients $X[0]$ and $X[N/2]$ are real numbers since:

$$X[0] = X^*\big[[-0]\big]_N = X^*[0]$$

$$X\left[\frac{N}{2}\right] = X^*\left[\left[-\frac{N}{2}\right]\right]_N = X^*\left[\frac{N}{2}\right]$$

- The $X[N/2]$ is called the **Nyquist coefficient,** as for $k = N/2$ the frequency is $\omega_{N/2} = (N/2)(2\pi/N) = \pi$, which is the digital **Nyquist frequency.**

- The symmetry property reduces the operations for the DFT calculation by **50%.** Specifically, we calculate the values$X[k]$ only for:

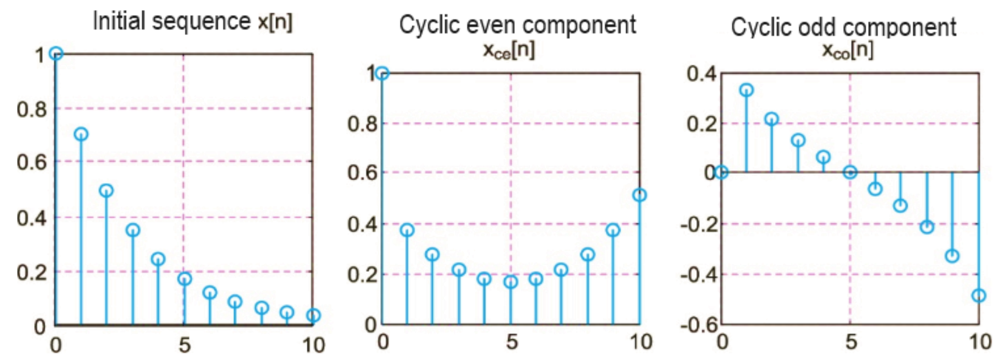
$$k = 0,1, \dots, \frac{N}{2}, \qquad \text{αν } N \text{ άρτιο}$$

$$k = 0,1, \dots, \frac{N-1}{2}, \qquad \text{αν } N \text{ περιττό}$$

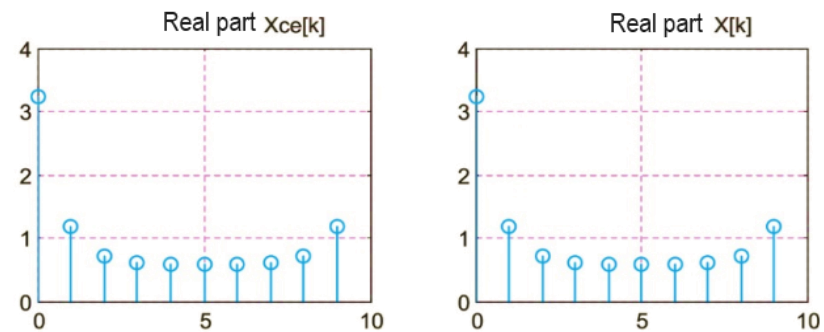
- This property is exploited by the Fast Fourier Transform (FFT).

DFT Symmetry for Real Sequences

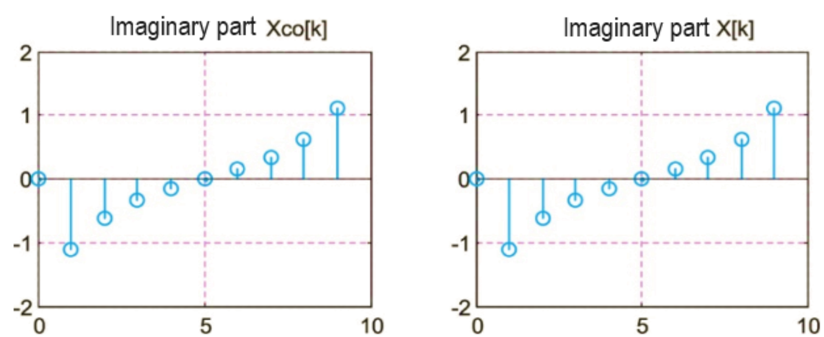
Initial sequence, cyclic even and cyclic odd components



Real parts $X[k]$ and $X_{ce}[k]$



Imaginary parts $X[k]\ and\ X_{co}[k]$



Symmetry Types of Complex Sequences

A complex sequence of $x[n]$point $0 \leq n \leq N-1$ length with N, is called:

- **Conjugate and circular symmetrical** if:

$$x[n] = x^*\big[[-n]\big]_N = x^*\big[[N-n]\big]_N$$

- **Conjugate and circular anti-symmetric** if:

$$x[n] = -x^*\big[[-n]\big]_N = -x^*\big[[N-n]\big]_N$$

The complex sequence $x[n]$ can be decomposed into a **conjugate circular symmetrical** $x_{cs}[n]$ and one **conjugate and circular anti-symmetric** $x_{ca}[n]$ sequence, namely:

$$x[n] = x_{cs}[n] + x_{ca}[n], \qquad 0 \leq n \leq N-1$$

where:

$$x_{cs}[n] = \frac{1}{2}\Big[x[n] + x^*\big[[-n]\big]_N\Big], \qquad 0 \leq n \leq N-1$$

$$x_{ca}[n] = \frac{1}{2}\Big[x[n] - x^*\big[[-n]\big]_N\Big], \qquad 0 \leq n \leq N-1$$

Symmetry of DFT for Complex Sequences

- The DFT of a real sequence is:
  - **Conjugate circularly symmetric,** if applicable:

$$X[k] = X^*\big[[-k]\big]_N = X^*\big[[N-k]\big]_N$$

  - **Conjugate cyclically anti-symmetric,** if true:

$$X[k] = -X^*\big[[-k]\big]_N = -X^*\big[[N-k]\big]_N$$

- The (complex) $X[k]$ DFT sequence can be written as:

$$X[k] = X_{cs}[k] + X_{ca}[k], \qquad 0 \le k \le N-1$$

where:

$$X_{cs}[k] = \frac{1}{2}\Big[X[k] + X^*\big[[-k]\big]_N\Big], \qquad 0 \le k \le N-1$$

$$X_{ca}[k] = \frac{1}{2}\Big[X[k] - X^*\big[[-k]\big]_N\Big], \qquad 0 \le k \le N-1$$

Circular Convolution

- When two sequences $x_1[n]$ και $x_2[n]$ N-points length each, are convoluted, then the resulting sequence has <u>a longer length.</u>

- If we want the result of the convolution to be strictly bounded in the space $0 \leq n \leq N-1$, then we use **circular convolution,** which is defined by:

$$x_1[n]\ (\mathrm{N})\ x_2[n] = \left[\sum_{k=0}^{N-1} x_1[k]\, x_2[[n-k]]_N\right] R_N[n], \qquad 0 \leq n \leq N-1$$

- Circular convolution produces sequence of length N points. It has the same structure as linear convolution, but differs in the limit of summation and the use of circular displacement.

- The DFT property for circular convolution is:

$$x_1[n]\ (N)\ x_2[n] \xleftrightarrow{DFT} X_1[k]\, X_2[k]$$

- Therefore, if we multiply two DFTs N-point each, in the frequency domain, the result is **circular convolution** (rather than linear convolution) in the time domain.

Sequence Multiplication

- This property is a binary of the previous property and states that the DTFT of the product of two sequences $x[n]$ and $y[n]$ is the periodic convolution of the individual DTFTs $X(e^{j\omega})$ and $Y(e^{j\omega})$ the signals.

$$x[n]y[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\, Y(e^{j(\omega-\theta)})d\theta = X(e^{j\omega}) \circledast Y(e^{j\omega})$$

- Because of the periodicity of the DTFT there is no perfect duality between time-domain convolution and frequency-domain multiplication. In particular, the multiplication of two aperiodic sequences is equivalent to the periodic (rather than linear) convolution of DTFTs.

Parseval's Theorem

- The well-known Parseval equation that applies to the Fourier transform, the Z-transform and the DTFT, and which calculates the energy of the signal in the time and frequency domains, also applies to the DFT:

$$E_x = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

- $|X[k]|^2/N$: **spectral energy density,** expresses the amount of energy per spectral coefficient.

- If the sequence is periodic, then the quantity $\left|\tilde{X}[k]/N\right|^2$ is called **the power spectral density.**

Relationship between
Circular and Linear Convolution

Calculation of linear convolution using DFT

Relationship between Circular and Linear Convolution

- Linear **convolution** offers the most important possibility of calculating the output of a LSI system when the input $x[n]$ and the impulse response $h[n]$ of the system are known, but its calculation requires a high computational cost.

- DFT offers **efficient tools** for analyzing signals and systems in the frequency domain through fast computational implementations such as the FFT algorithm.

- Since circular convolution can be easily calculated by DFT, the question arises, how can **DFT be used to calculate linear convolution?**

Relationship between Circular and Linear Convolution

If $x[n]$και $h[n]$ sequences of duration N_x and N_h-points respectively, then:

- The **linear convolution** between $x[n]$ and $h[n]$, is:

$$y_L[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=0}^{N_x+N_h} x[k]\, h[n-k]$$

- The **circular convolution** between the $x[n]$και $h[n]$, duration $N = N_x + N_h - 1$ points each, is:

$$y_C[n] = x[n] \qquad h[n] = \left[\sum_{k=0}^{N_x-1} x[k]\, h\big[[n-k]\big]_N\right] R_N[n]$$

It turns out that:

$$y_C[n] = \left[\sum_{r=-\infty}^{\infty} y_L[n+rN]\right] R_N[n]$$

- So circular convolution $y_c[n]$ is an altered form of linear $y_L[n]$.
- If the sequence $y_L[n]$has **duration $N \geq N_x + N_h - 1$**, then the linear convolution **is identical** to the circular one.

Calculation of Linear Convolution using DFT

- The linear convolution of two sequences $x[n]$ and $h[n]$ with duration N_x and N_h samples respectively, is calculated by DFT with the following steps:
  - The sequences $x[n]$ and $h[n]$ are extended with an appropriate number of zeros so that each one has a length of $N \geq N_x + N_h - 1$ samples.
  - The point $x[n]$ DFTs of the sequences $N$and are calculated $h[n]$ and the sequences $X[k]$ and are generated $H[k]$.
  - The product is calculated $Y[k] = X[k]\ H[k]$.
  - The inverse DFT N-points of is calculated $Y[k]$, so the circular convolution $y_C[n] = x[n]$ (N) is found $h[n]$.
  - Since $N \geq N_x + N_h - 1$ linear convolution is equal to circular convolution.
- If the length of the circular convolution is set $N = \max(N_x, N_h)$, then the first $(M - 1)$samples of the circular convolution are <u>different</u> from the corresponding samples of the linear convolution, where $M = \min(N_x, N_h)$. All samples coincide.

Convolution Calculation by Blocks

- Overlap – Save Method
- Overlap – Add Method

Convolution Calculation by Blocks

- If the sequence $x[n]$ is long, then the DFT for a large value of N does not provide significant information about the spectrum, because its calculation results as an average of a long calculation and does not clearly render the spectrum of the transition regions of the signal.
- In this case we prefer to **slice** the signal into individual **blocks** of finite duration and calculate the DFT of each block.
- Similarly we do to calculate the output of a system for input a signal of long length.
- Slicing the signal into individual blocks $x_r[n]$ of finite duration is done by multiplying it by a window $w_N[n]$ of length N:

$$x_r[n] = x[n]\, w_N[n - rN]$$

- This process is called block **convolution** and is implemented with the techniques:
  - **overlap - save**
  - **overlap - add**

Overlap – Save Method

The method is described by the following algorithm:

1. We generate the first point $x_1[n]$ N-length block from the total signal $x_1[n]$, through the equation:

$$x_1[n] = \begin{cases} 0, & 0 \le n < M-1 \\ x[n-M+1], & M-1 \le n \le N-1 \end{cases}$$

2. We calculate the N-point DFTs $X_1[k]$ of the sequence $x_1[n]$ and $H[k]$ of impulse response $h[n]$ of the system.

3. We calculate the product $Y_1[k] = X_1[k]\, H[k]$.

4. By inverse N-point DFT $Y_1[k]$ we obtain the $y_1[n]$, equivalent of circular convolution $x_1[n] \circledast h[n]$. The first $(M-1)$ values of the sequence $y_1[n]$ are false and the remaining $(N-M+1)$ values correspond to linear convolution $x_1[n] * h[n]$. The last $(N-M+1)$ values of $y_1[n]$ are the first $(N-M+1)$ values of the output sequence $y[n]$, that is:

$$y[n] = y_1[n+M-1], \qquad 0 \le n < M-1$$

Overlap – Save Method

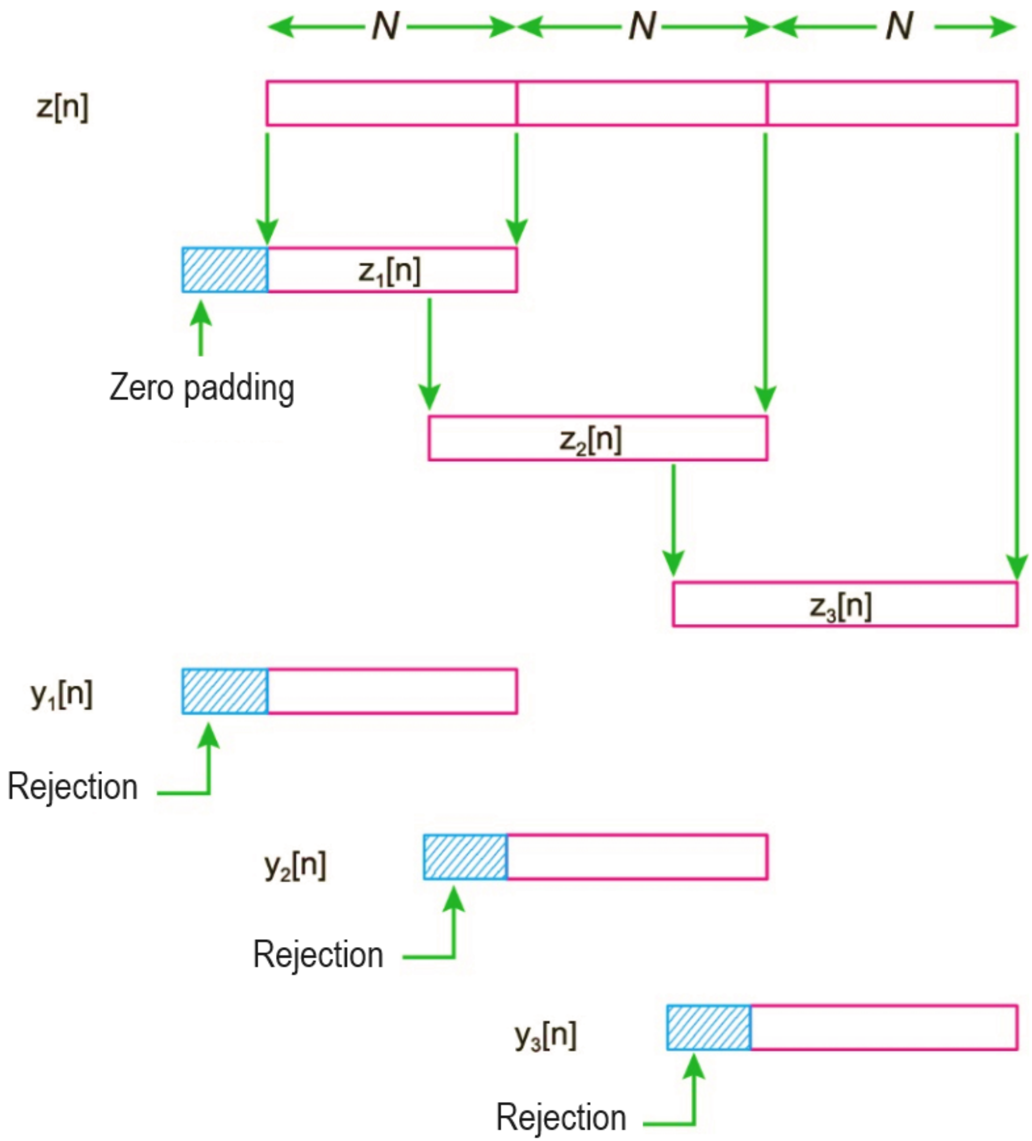
5. Let be $x_2[n]$ a sequence $N$of -points extracted from $x[n]$ with $(M-1)$ its first values overlapping with those of $x_1[n]$.

6. We perform steps 3 and 4 and obtain $y_2[n]$. Its first $(M-1)$ values $y_2[n]$ are discarded and its last values are kept and combined with its $(N-M+1)$ reserved:

$$y[n+N-M+1] = y_2[n+M-1], \qquad 0 \leq n < N-M$$

7. We repeat steps 5 and 6 until all values of the linear convolution are calculated.

Overlap – Save Method



Overlap – Add Method

The method is described by the following algorithm:

- We slice the sequence $x[n]$ into time-shifted blocks **N-point** length:

$$x[n] = \sum_{i=0}^{\infty} x_i[n - Ni]$$

where $x_i[n] = \begin{cases} x[n + Ni], & n = 0,1, \dots , N - 1 \\ 0 & elsewhere \end{cases}$

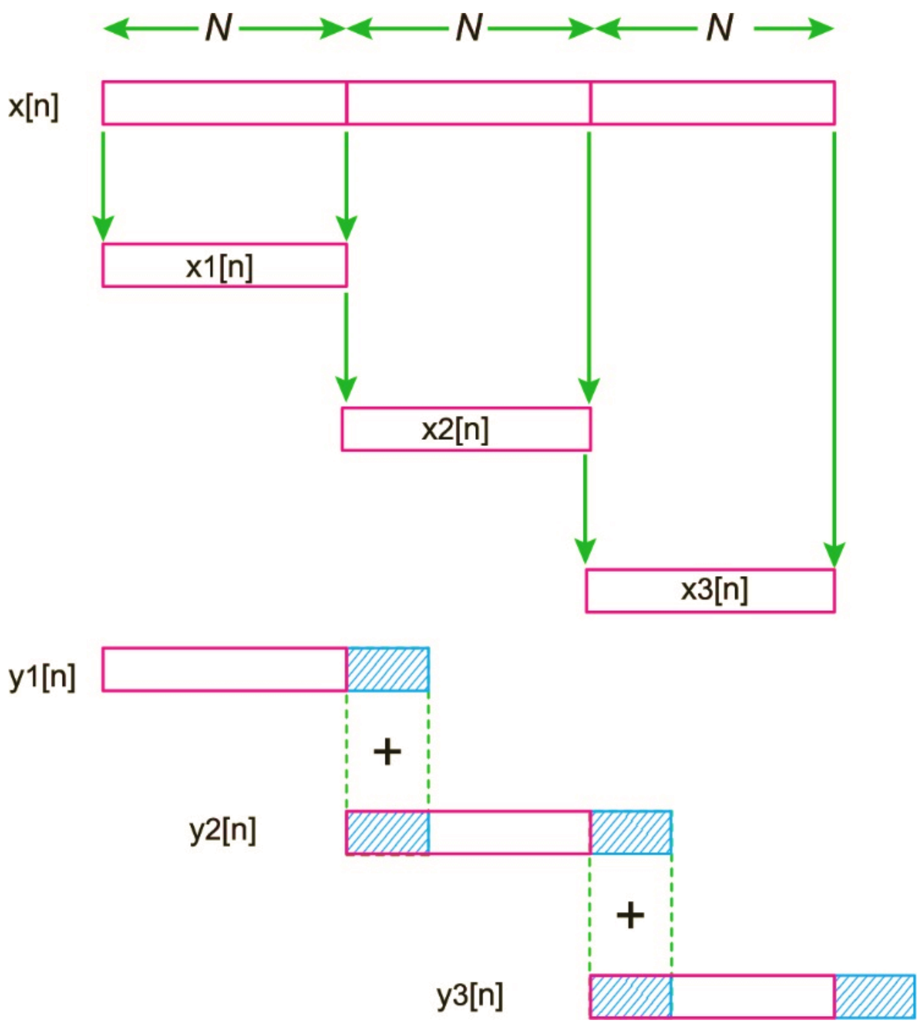
- If $h[n]$ is the impulse response length M of a LSI system, then its output to input signal $x[n]$ will be:

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k] = \sum_{i=0}^{\infty} x_i[n - Ni] * h[n] = \sum_{i=0}^{\infty} y_i[n - Ni]$$

where $y_i[n] = x_i[n] * h[n]$

- Each subsequence $y_i[n]$ can be easily calculated by N-points DFT of $x_i[n]$ and $h[n]$ and will have length $(N + M - 1)$ points.
- The consecutive sequences $y_i[n]$ and $y_{i+1}[n]$ overlap at $(N - M)$ points and the overlapping points are added.

Overlap – Add Method



Fast Fourier Transform

- Decimation in Time FFT Algorithm (Radix-2)
- Decimation in Frequency FFT Algorithm (Radix-2)

Computational Cost DFT

The N-point DFT of a N-points sequence $x[n]$ is:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk} = x[0]W_N^{0k} + x[1]W_N^{1k} + \cdots + x[N-1]W_N^{(N-1)k}, \qquad k = 0,1, \ldots N-1$$

- To calculate each point $X[k]$ you need:
  - N complex multiplications
  - $(N-1)$ complex additions
- Calculating all N DFT values requires:
  - N^2 complex multiplications and
  - $N(N-1) \cong N^2$ complex additions
- To store the phase factors W_N^{nk} you need:
  - N^2 seats
- The computational cost of DFT is $\boldsymbol{O}(\boldsymbol{N^2})$ and becomes prohibitively high for large values of N.

Strategy for constructing an efficient DFT algorithm

- The N-point DFT calculation is based on DFTs of successively shorter length, e.g. $N/2$-points. It is done by divide of the sequence $x[n]$ of length N-points (Neven) into two subsequences $x_1[n]$ and $x_2[n]$, N/2-points, each.
- DFT has a computational $O(N^2/4)$ cost $N/2$. For both sequences is 2 $O(N^2/4)$, significantly smaller than $O(N^2)$, especially for large values of N.
- When N is a power of 2, then the computational cost is just $\boldsymbol{O}(\boldsymbol{N/2\ \ logN})$.
- A way to calculate the point DFT N from the point DFT is requested $N/2$. The point DFT $N/2$can then be decomposed into point DFTs $N/4$, etc.
- The most popular techniques for implementing the FFT are:
    - **Decimation in Time**
    - **Decimation in Frequency**

Decimation in Time FFT Algorithm (Radix-2)

A sequence of $x[n]$ length $N = 2^{\nu}$ is split into two sequences, length $N/2$ each:

- $g_1[n] = x[2n],\ 0 \leq n \leq N/2 - 1$ Samples of it $x[n]$ with an even index
- $g_2[n] = x[2n+1], 0 \leq n \leq N/2 - 1$ Samples of it $x[n]$ with an unnecessary index

N-point DFT of $x[n]$ has computational cost $O(N^2)$ and $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$

Because $x[n] = g_1[n] + g_2[n]$, it is true:

$$X[k] = \sum_{m=0}^{N/2-1} g_1[m]\, W_N^{2mk} + W_N^k \sum_{m=0}^{N/2-1} g_2[m]\, W_N^{(2m+1)k}$$

Because $W_N^{2mk} = W_{N/2}^{mk}$, the above equation is written:

$$X[k] = \sum_{m=0}^{N/2-1} g_1[m]\, W_{N/2}^{mk} + W_N^k \sum_{m=0}^{N/2-1} g_2[m]\, W_{N/2}^{mk}$$

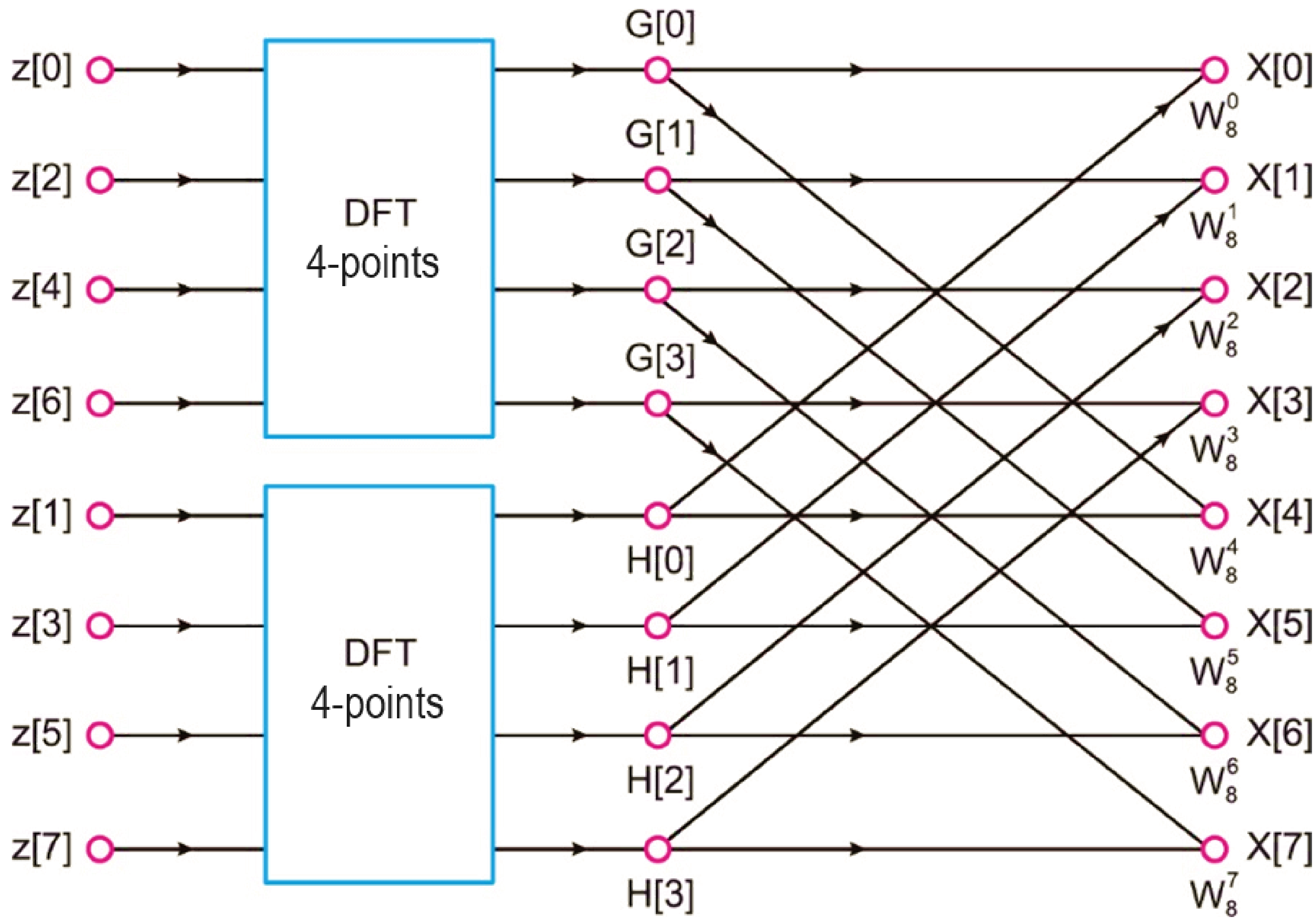
or briefly:

$$X[k] = G_1[k] + W_N^k G_2[k], \qquad 0 \leq k \leq N-1$$

Decimation in Time FFT Algorithm (Radix-2)

- Therefore, the N-point DFT of $x[n]$ is equal to the sum of the point $g_1[n]$ DFTs of $N/2$ and $g_2[n]$.
- DFT has a computational $O(N^2/4)$ cost $N/2$. So the total computational cost is 2 $O(N^2/4)$(significantly less than $O(N^2)$).
- The above process of splitting the input sequence into sequences of even and odd terms is repeated νtimes, until we arrive at a 2-point DFT.
- This process is called "**decimation in time"** (DIT - FFT) and is shown in the next figure, for an 8-point FFT.
- Total computational cost of FFT is $O(N \log_2 N)$. If N is large then it can be further reduced, to $O\left(\frac{N}{2} \log_2 N\right)$.

Decimation in Time FFT Algorithm (Radix-2)



FFT structure (DIT-FFT) for $N = 8$

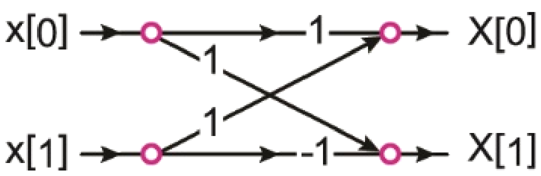
2- and 4-point DFT study

- **DFT of 2 points:**

$$X[k] = \sum_{n=0}^{1} x[n]\, W_2^{nk} = x[0]W_2^{0k} + x[1]W_2^{1k} = x[0]e^{-j0} + x[1]e^{-j\pi k}$$

Therefore $X[k] = \; x[0] + (-1)^k x[1],\; 0 \le k \le 1$, which resolves into $X[0] = x[0] + x[1]$ and $X[1] = x[0] - x[1]$

Butterfly diagram
2-point FFT



- **DFT of 4 points:**

$$X[k] = \sum_{n=0}^{3} x[n]\, W_4^{nk} = x[0]W_4^{0k} + x[1]W_4^{1k} + x[2]W_4^{2k} + x[3]W_4^{3k}, \qquad 0 \le k \le 3$$

In tabular form they are:

$$\boldsymbol{X}^T = \boldsymbol{W_4}\, \boldsymbol{x}^T \Rightarrow \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Therefore the DFT calculation requires 16 complex multiplications.

2- and 4-point DFT study

The phase factors we will need are $W_4^0, W_4^1, W_4^2, W_4^3, W_4^4, W_4^6, W_4^9$.

Due to the symmetry properties of the phase factor we find:

$$W_4^0 = W_4^4 = 1, W_4^1 = W_4^9 = -j, W_4^2 = W_4^6 = -1, \quad W_4^3 = j$$

Therefore matrix multiplication is written:

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

We break down the calculation for each coefficient $X[k]$:

- $X[0] = x[0] + x[1] + x[2] + x[3] = \big[x[0] + x[2]\big] + \big[x[1] + x[3]\big] = g_1 + g_2$
- $X[1] = x[0] - j\,x[1] - x[2] + j\,x[3] = \big[x[0] - x[2]\big] - j\big[x[1] - x[3]\big] = h_1 - jh_2$
- $X[2] = x[0] - x[1] + x[2] - x[3] = \big[x[0] + x[2]\big] - \big[x[1] + x[3]\big] = g_1 - g_2$
- $X[3] = x[0] + j\,x[1] - x[2] - j\,x[3] = \big[x[0] - x[2]\big] + j\big[x[1] - x[3]\big] = h_1 + jh_2$

2- and 4-point DFT study

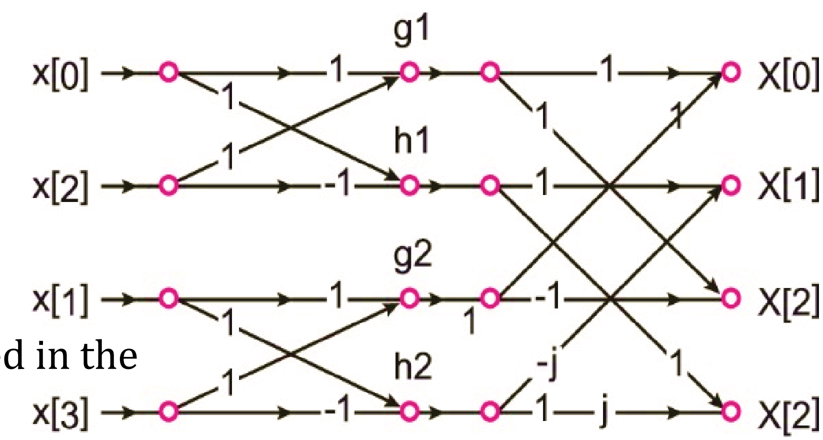
- The coefficients $X[0]$and $X[2]$can be calculated by adding and subtracting $g_1$and g_2, respectively. Depending on $X[1]$and $X[3]$.

- Therefore, we can perform the calculation of $X[k]$from the table:

Step 1	Step 2
$g_1 = x[0] + x[2]$ $g_2 = x[1] + x[3]$ $h_1 = x[0] - x[2]$ $h_2 = x[1] - x[3]$	$X[0] = g_1 + g_2$ $X[1] = h_1 - jh_2$ $X[2] = g_1 - g_2$ $X[3] = h_1 + jh_2$

- This way of calculating the 4-point DFT requires only **2 complex multiplications,** compared to **16 from the definition.**

The calculation process is graphically rendered in the 4-point FFT flowchart



Example 8

Compute the 4-point DFT $x[n] = \{1, 3, 5, 7\}$ of the sequence with the DIT - FFT algorithm.

Answer: Based on the previous DIT - FFT 4 point flowchart we find:

Step 1	Step 2
$g_1 = x[0] + x[2] = 1 + 5 = 6$	$X[0] = g_1 + g_2 = 6 + 10 = 16$
$g_2 = x[1] + x[3] = 3 + 7 = 10$	$X[1] = h_1 - jh_2 = -4 + 4j$
$h_1 = x[0] - x[2] = 1 - 5 = -4$	$X[2] = g_1 - g_2 = 6 - 10 = -4$
$h_2 = x[1] - x[3] = 3 - 7 = -4$	$X[3] = h_1 + jh_2 = -4 - 4j$

Therefore the DFT is$X[k] = \{16, -4 + 4j, -4, -4 - 4j\}$

Decimation in Frequency FFT Algorithm (Radix-2)

- Another approach: Separate calculation of even and odd samples of the DFT.
- For sequence $x[n]$ length $N = 2^{\nu}$ and N-points DFT , the even DFT samples are:

$$X[2k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk} = \sum_{n=0}^{N/2-1} x[n]\, W_{N/2}^{nk} + \sum_{n=N/2}^{N-1} x[n]\, W_{N/2}^{nk}$$

$$= \sum_{n=0}^{N/2-1} x[n]\, W_{N/2}^{nk} + \sum_{n=0}^{N/2-1} x\left[n + \frac{N}{2}\right] W_{N/2}^{(n+N/2)k}$$

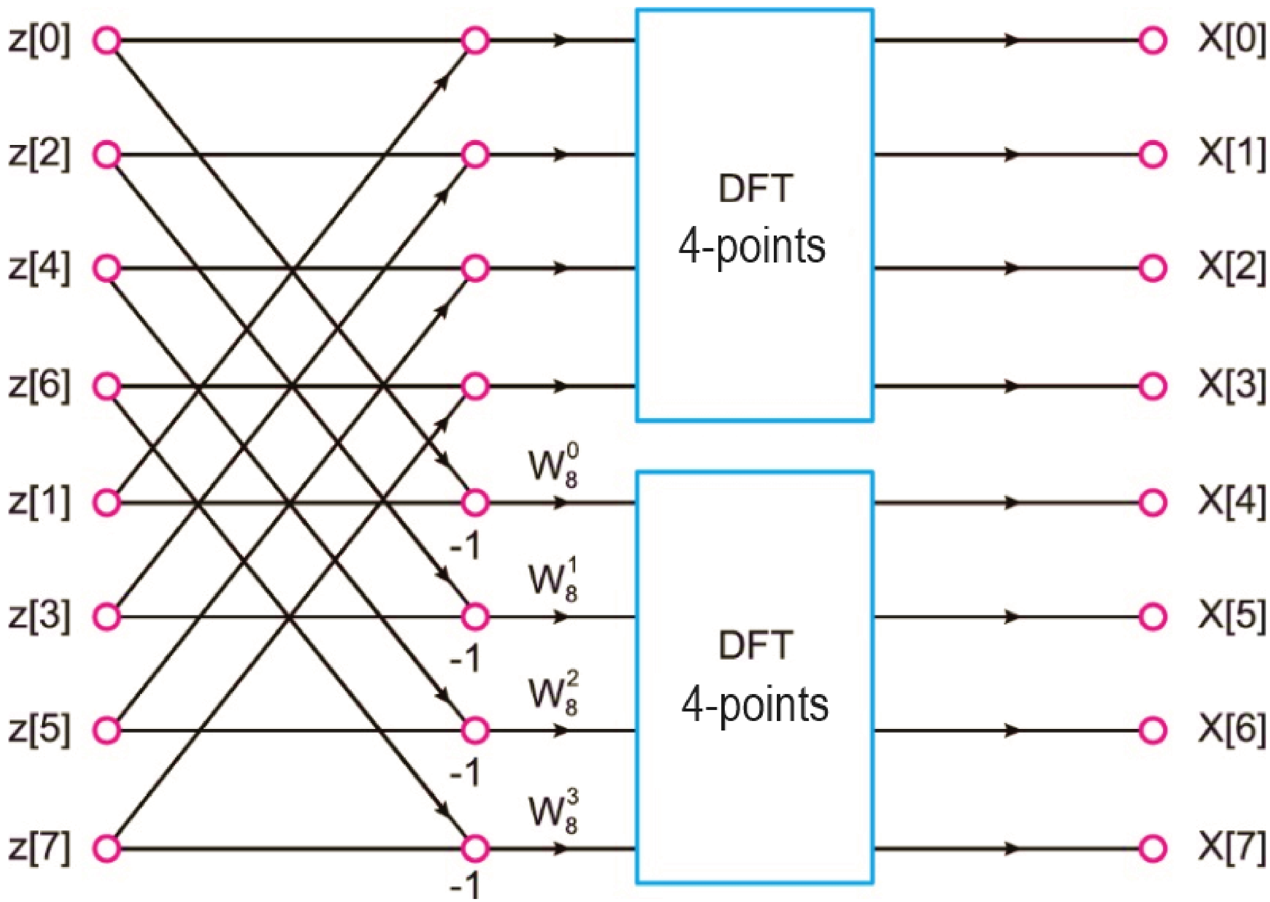
- Since $W_{N/2}^{(n+N/2)k} = W_{N/2}^{nk}$, the above equation is written:

$$X[2k] = \sum_{n=0}^{N/2-1} \left[x[n] + x\left[n + \frac{N}{2}\right]\right] W_{N/2}^{nk}$$

- So the even samples of the N-point DFT are calculated from the point DFT $N/2$ in a sequence formed by its first $N/2$ and last $N/2$ points $x[n]$.
- Accordingly, the redundant samples of the N-point DFT are given by:

$$X[2k+1] = \sum_{n=0}^{N/2-1} W_N^{n} \left[x[n] - x\left[n + \frac{N}{2}\right]\right] W_{N/2}^{nk}$$

Decimation in Frequency FFT Algorithm (Radix-2)



Decimation-in-frequency FFT (DIF-FFT) structure for $N = 8$

Example 9

Suppose that a complex pulse takes 1 μs and that the total running time of the DFT is determined by the computation time of the multiplications alone.

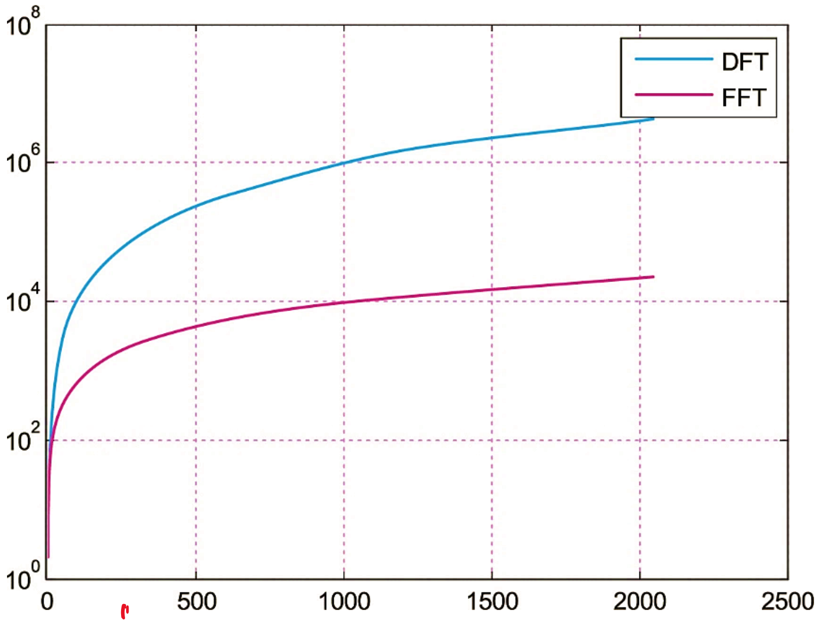
(a) How long would it take to compute a 1024-point DFT directly?

(b) How long will it take if we use FFT algorithm ?

(c) Plot in Matlab the number of complex multiplications for DFT and FFT for values of N from 1 to 2048.

Answer: (a) Number of complex DFT multiplications: N^2

(c)



DFT time -1024 points:
$t_{DFT} = 1024^2\ 10^{-6}\ \text{sec} \approx 1{,}05\ sec$

(b) Number of complex multiplications for radix -2 FFT: $(N/2)\ log\ N$

Time FFT -1024 points:
$t_{FFT} = 5120.10^{-6}\ \text{sec} = 5{,}12\ msec$