University of the Peloponnese
Electrical and Computer
Engineering Department

Digital Signal Processing

Unit 06: Discrete-Time Fourier Transform (DTFT)

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Fourier Series
of Discrete Time Signals

Fourier Series of Discrete Time Signals

A periodic discrete-time signal $x[n]$ with period N and fundamental frequency $\omega_0 = 2\pi/N$, is resolved into **an exponential Fourier series** with the equation:

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}, \qquad 0 \leq n \leq N-1$$

where the coefficients $X[k]$ of the exponential Fourier series are calculated from the equation:

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad 0 \leq n \leq N-1$$

Fourier Series of Discrete Time Signals

If the input of a LSI system is a complex exponential signal $x[n] = Ae^{jn\omega_0}, \quad -\infty < n < +\infty$ of frequency ω_0, then the output of the system is given by the convolution:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k] \; Ae^{j\omega_0(n-k)} = Ae^{jn\omega_0} \sum_{k=-\infty}^{+\infty} h[k]e^{-j\omega_0 k} = H\left(e^{j\omega_0}\right) x[n]$$

where $H\left(e^{j\omega_0}\right)$ is the value of the frequency response of the system to the frequency ω_0.

If the input of the system is a sum of complex exponential signals $x[n] = \sum_{k=-\infty}^{+\infty} A_k e^{j\omega_0 n}$ then the output of the system will be:

$$y[n] = \sum_{k=-\infty}^{+\infty} A_k e^{j\omega_0 n} H\left(e^{j\omega_0}\right)$$

Therefore, the analysis of a periodic signal in the form of an exponential Fourier series, gives us the possibility of easy calculation of the output of discrete-time Linear Shift Invariant (LSI) systems.

Remarks (1/2)

- The analysis of a real periodic discrete-time signal into Fourier series allows us to write this signal as a sum of complex discrete-time exponential sequences, or equivalently as a sum of sines in conjugate pairs.
- The Fourier series expansion of a periodic discrete-time signal **always converges,** since it consists of a finite number of terms according to its definition.
- Because discrete Fourier Series always converge, the Gibbs effect does not occur, in contrast of continuous-time signals.
- Both the signal sequence $x[n]$ and the coefficients $X[k]$ are **periodic sequences** with the same period N.
- A **non-periodic** signal (continuous or discrete time) has **a continuous spectrum.**
- A **periodic signal** (continuous or discrete time) has **a discrete spectrum.**

Remarks (2/2)

- The fact that we can describe a periodic signal (continuous or discrete time) and its spectrum in discrete form is of great practical value because it can be easily implemented in a computer.

- The basis vectors $e^{-j(2\pi/N)kn}$ and $e^{j(2\pi/N)kn}$ are periodic with period N and are **orthonormal** to a period, i.e. they satisfy the equation:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn}\left[e^{-j\frac{2\pi}{N}mn}\right]^{*} = \begin{cases} 1 & m = k \\ 0 & m \neq k \end{cases}$$

- The term orthonormal means that the basis vectors$e^{-j(2\pi/N)kn}$ and $e^{j(2\pi/N)mn}$ are **orthogonal** when $m \neq k$, i.e. their sum in one period is zero. It is also **normal,** i.e. for $m = k$ the same sum it is one.

- Likewise, the vectors $\cos(2\pi/N)n$ and $\sin(2\pi/N)n$.

Discrete-Time Fourier Transform (DTFT)

Discrete Time Fourier Transform

- The Discrete Time Fourier Transform (DTFT) is applied to **discrete signals** and produces their (usually complex) representation in **the frequency** domain.
- If $x[n]$ a discrete signal, then the **straight DTFT** is defined by the equation:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}, \qquad -\pi \leq \omega \leq \pi$$

- The digital (cyclic) frequencyω (rad) is a continuous variable resulting from the equation $\boldsymbol{\omega = \Omega T_s}$, where Ω (rad/sec) is the proportional (cyclic) frequency.
- To calculate the DTFT the sum must **converge** to an absolute value, that is:

$$\left|X\left(e^{j\omega}\right)\right| = \sum_{n=-\infty}^{\infty} \left|x[n]\, e^{-jn\omega}\right| = \sum_{n=-\infty}^{\infty} |x[n]| = S < \infty$$

- In the next section we will show how we can calculate the DTFT of signals that are not completely summable, i.e. they do not satisfy the above equation.

Discrete Time Fourier Transform

- The function $X\left(e^{j\omega}\right)$ is generally **complex.**
- **Cartesian form:**

$$X\left(e^{j\omega}\right) = X_R\left(e^{j\omega}\right) + jX_I\left(e^{j\omega}\right)$$

$X_R(e^{j\omega})$ and $X_I(e^{j\omega})$ is the real and imaginary part of DTFT.

- **Polar form:**

$$X\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right| e^{j\varphi_x(\omega)}$$

  - Magnitude:

$$\left|X\left(e^{j\omega}\right)\right| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})}$$

  - Phase:

$$\varphi_X(\omega) = tan^{-1}\left[\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right]$$

- The graphical representations of the above functions are called magnitude spectrum and phase spectrum.

Spectral Symmetries of the DTFT

If the signal $x[n]$ is real then its spectra have the same **spectral properties of symmetry** with those of the spectra of continuous-time signals, namely:

- The magnitude $|X(e^{j\omega})|$ and real part $Re\{X(e^{j\omega})\}$ are even functions of frequency ω,
- The phase $\measuredangle X(e^{j\omega})$ and imaginary part $Im\{X(e^{j\omega})\}$ are odd functions of frequency ω,

$$\left|X(e^{j\omega})\right| = |X(-e^{j\omega})|$$

$$Re\{X(e^{j\omega})\} = Re\{X(-e^{j\omega})\}$$

$$\measuredangle X(e^{j\omega}) = -\measuredangle X(-e^{j\omega})$$

$$Im\{X(e^{j\omega})\} = -Im\{X(-e^{j\omega})\}$$

- The discrete-time Fourier transform (DTFT) is **a periodic function.** The periodicity of the DTFT is due to the fact that discrete-time complex exponential signals when they differ in frequency by multiples of 2π are identical to each other.

Inverse DTFT

- The **inverse discrete-time Fourier transform** (IDTFT) produces the sequence $x[n]$when the function is known $X\left(e^{j\omega}\right)$, from the equation:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{jn\omega}\, d\omega$$

- The inverse DTFT can be thought of as the resolution of the signal $x[n]$ into a linear combination of all complex exponential terms that have frequencies in space $-\pi < \omega < +\pi$.

- **DTFT expressions**:
    - $X\left(e^{j\omega}\right) = DTFT\{[x[n]\}$
    - $x[n] = DTFT^{-1}\left\{X\left(e^{j\omega}\right)\right\}$
    - $x[n] \xleftrightarrow{\text{DTFT}} X\left(e^{j\omega}\right)$

Practical utility of DTFT

- It turns the computationally difficult operation of convolution into the computationally simple operation of multiplication.
- It is used to solve Linear Differential Equations with Constant Coefficients (LDECC).
- The DTFT of the impulse response $h[n]$ of an LSI system gives the frequency response $H(e^{j\omega})$ of the system.

Example 1

Find the DTFT of the discrete-time signal $x[n] = \{1, -\hat{1}, 0, 4, 2\}$.

Answer: From the definition of DTFT we have:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-jn\omega}$$

$$= 1e^{-j(-1)\omega} + (-1)e^{-j0\omega} + 0e^{-j1\omega} + 4e^{-j2\omega} + 2e^{-j3\omega}$$

$$= 1e^{j\omega} + (-1)1 + 0 + 4e^{-j2\omega} + 2e^{-j3\omega}$$

$$= e^{j\omega} - 1 + 4e^{-j2\omega} + 2e^{-j3\omega}$$

Example 2

To find the DTFT of the discrete-time signals:

(a) $x[n] = \delta[n]$ (b) $x[n] = \delta[n - n_0]$

Answer: (a) We calculate the DTFT from the definition of:

$$X(e^{j\omega}) = \Delta(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n]\, e^{-jn\omega} = \delta[0]e^{0} = 1$$

- The impulse $\delta[n]$ has a DTFT with unit magnitude and zero phase for all frequencies. That is, the sequence $\delta[n]$ contains equally all (infinite) frequencies in range $-\pi < \omega < \pi$.

(b) Similarly to case (a) we have:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0]\, e^{-jn\omega} = \delta[n_0]e^{-jn_0\omega} = e^{-jn_0\omega}$$

- In this case the DTFT has unit magnitude (as before), but its phase is now non-zero and proportional to frequency for all (infinite) frequencies in the range $-\pi < \omega < \pi$.

Example 3

Find the DTFT of the discrete-time signal$x[n] = \alpha^n \, u[n]$, $|\alpha| < 1$

Answer: DTFT is:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-jn\omega} = \sum_{n=-\infty}^{+\infty} \alpha^n \, u[n]\, e^{-jn\omega}$$

$$= \sum_{n=0}^{+\infty} \alpha^n e^{-jn\omega} = \sum_{n=0}^{+\infty} \left(\alpha\, e^{-j\omega}\right)^n$$

Since $|\alpha| < 1$, the sum converges, so the DTFT is:

$$X\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}} = \frac{1}{e^{-j\omega}(e^{j\omega} - a)} = \frac{e^{j\omega}}{e^{j\omega} - a} = \frac{1}{1 - a\cos\omega + ja\sin\omega}$$

The magnitude of the transform is:

$$\left|X\left(e^{j\omega}\right)\right| = \left|\frac{1}{1 - ae^{-j\omega}}\right| = \frac{1}{\sqrt{1 + \alpha^2 - 2\alpha\cos\omega}}$$

and the phase is:

$$\varphi_X(\omega) = tan^{-1}\left(\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right) = tan^{-1}\left(\frac{-a\sin\omega}{1 - \alpha\cos\omega}\right) = -tan^{-1}\left(\frac{a\sin\omega}{1 - \alpha\cos\omega}\right)$$

where $X_R(e^{j\omega})$ is $X_I(e^{j\omega})$ the real and imaginary part of DTFT.

Example 4

Find the DTFT of the discrete-time signal $x[n] = -\alpha^n\, u[-n-1], \qquad |\alpha| > 1$

Answer: DTFT is:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-jn\omega} = -\sum_{n=-\infty}^{+\infty} \alpha^n\, u[-n-1]\, e^{-jn\omega}$$

$$= -\sum_{n=-\infty}^{-1} \alpha^n e^{-jn\omega} = -\sum_{n=1}^{+\infty} \left(\alpha^{-1}\, e^{j\omega}\right)^n = 1 - \sum_{n=0}^{+\infty} \left(\alpha^{-1}\, e^{j\omega}\right)^n$$

Since $|\alpha| > 1$, the sum converges, so the DTFT is:

$$X(e^{j\omega}) = 1 - \frac{1}{1 - a^{-1}e^{-j\omega}} = \frac{\alpha^{-1}e^{j\omega}}{1 - a^{-1}e^{-j\omega}} = \frac{1}{1 - a\, e^{-j\omega}}$$

The magnitude of the transform is:

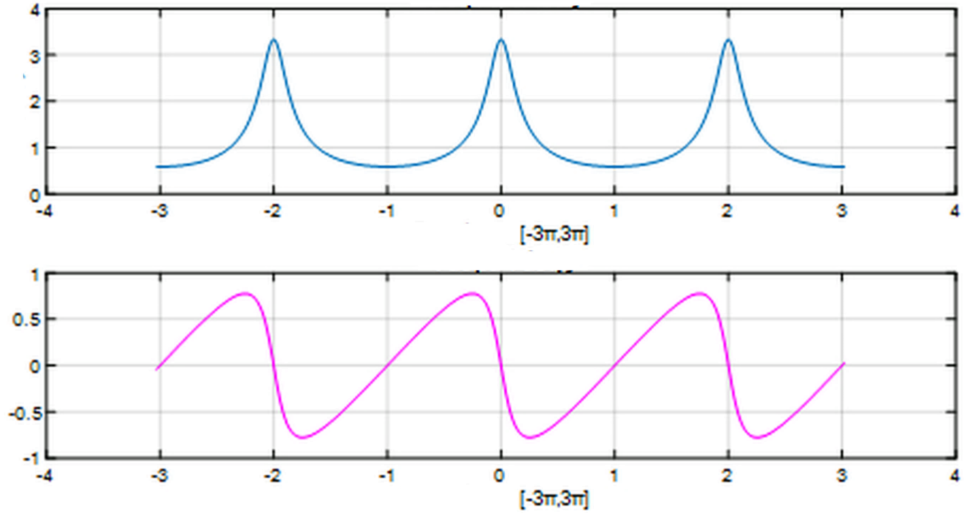
$$\left|X(e^{j\omega})\right| = \left|\frac{1}{1 - ae^{-j\omega}}\right| = \frac{1}{\sqrt{1 + \alpha^2 - 2\alpha\, \cos\omega}}$$

and the phase is:

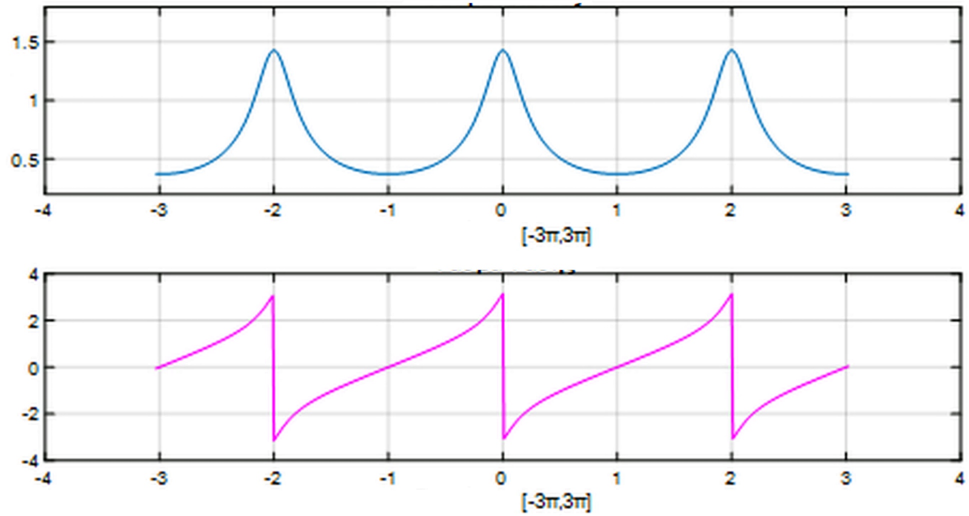
$$\varphi_X(\omega) = -tan^{-1}\left(\frac{a\, \sin\omega}{1 - \alpha\, \cos\, \omega}\right)$$

The solution is the same as Example 2, with the only difference being the value of the coefficient (a).

Example 3 and Example 4 spectra



Magnitude and phase spectra of the signal $x[n] = 0.7^n u[n]$



Magnitude and phase spectra of the signal $x[n] = 1.7^n u[n]$

Example 5

Calculate the DTFT of the sequence $x[n] = A(u[n] - u[n-N])$.

<u>Answer</u>: Using the definition of DTFT we have:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} = \sum_{n=0}^{N-1} A\, e^{-jn\omega} = A\sum_{n=0}^{N-1} e^{-jn\omega} = A\sum_{n=0}^{N-1} \left(e^{-j\omega}\right)^n = \frac{A\left(1 - e^{-j\omega N}\right)}{1 - e^{-j\omega}}$$

$$= \frac{Ae^{-j\omega N/2}\left(e^{j\omega N/2} - e^{-j\omega N/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} = \frac{Ae^{-j\omega N/2}\ 2j\ sin(\omega N/2)}{e^{-j\omega/2}\ 2j\ sin(\omega/2)} = Ae^{-j\omega(N-1)/2}\,\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

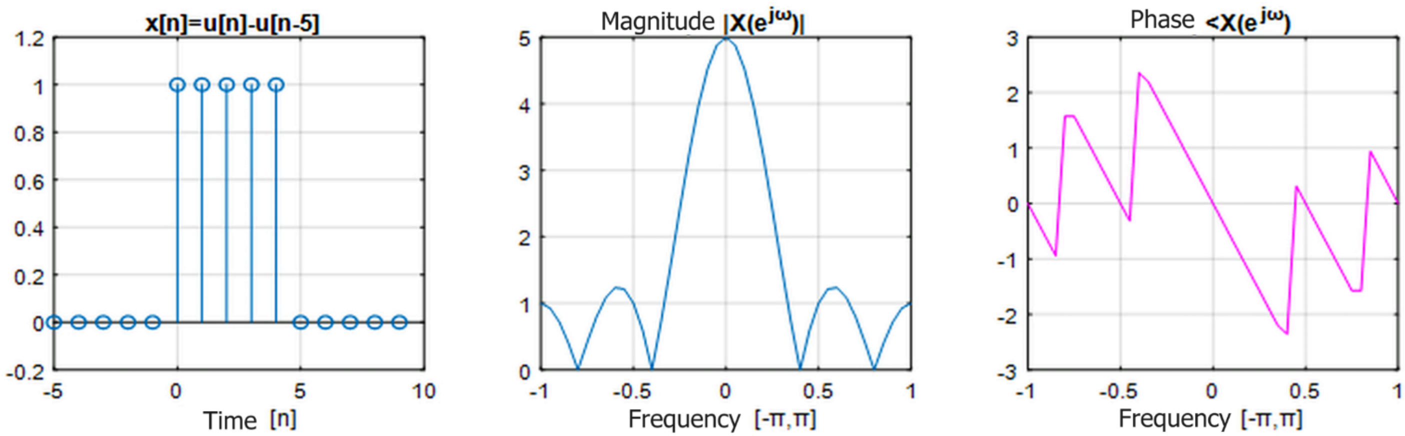
The magnitude of DTFT is:

$$\left|X\left(e^{j\omega}\right)\right| = |A|\left|\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right| \qquad (1)$$

and the phase is:

$$\varphi_X(\omega) = -\frac{\omega(N-1)}{2} \qquad (2)$$

Example 5 (continued)



(a) Signal $x[n] = u[n] - u[n-5]$, (b) Magnitude spectrum, (c) Phase spectrum, in one period

For the magnitude of the DTFT, the following observations apply:

- Since the numerator and denominator of equation (1) are odd functions, it follows that the magnitude of the DTFT is an even function, as expected.
- According de l' Hospital 's rule we find that for the frequency $\omega = 0$ the magnitude takes its maximum value, which is $\left|X\left(e^{j0}\right)\right| = A$.
- The points of zero magnitude are those that satisfy the equation $\sin(\omega N/2) = 0$, so the magnitude becomes zero at these frequencies $\omega = 2k\pi/N$.
- The magnitude of the DTFT is a function:
  - Periodic with a period of 2π, when N is odd.
  - Non-periodic when N is even.

Example 6

Find the inverse DTFT of the $X(e^{j\omega})$ with orthogonal form function, given by:

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < B \\ 0, & B < |\omega| < \pi \end{cases}, \text{ where } B = \pi/2$$

Answer: The inverse DTFT by definition is:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{jn\omega}\, d\omega = \frac{1}{2\pi}\int_{-B}^{B} 1\, e^{jn\omega}\, d\omega = \frac{1}{2\pi}\frac{1}{jn}\left[e^{jn\omega}\right]_{-B}^{B}$$

$$= \frac{1}{2\pi jn}\left(e^{jnB} - e^{-jnB}\right) = \frac{\sin(Bn)}{\pi n}, \qquad n \neq 0$$

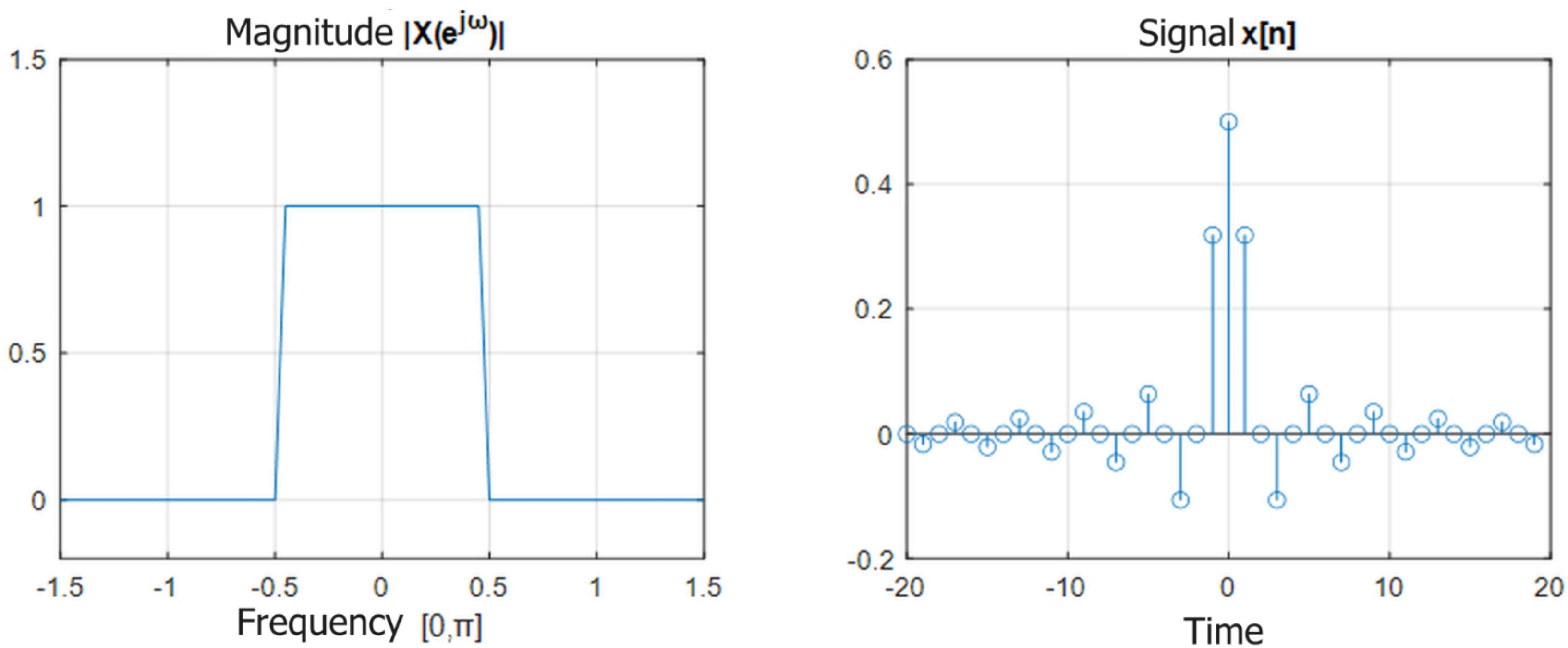
For $n = 0$ the value $x[0]$ is calculated by the rule of de L'Hospital and is:

$$x[0] = \lim_{n\to 0}\frac{B\cos(Bn)}{\pi} = \frac{B}{\pi} = 0.5$$

Therefore, the sequence $x[n]$ is:

$$x[n] = \begin{cases} 0.5, & n = 0 \\ \dfrac{\sin(\pi n/2)}{\pi n}, & n \neq 0 \end{cases}$$

Example 6 (continued)



(a) Square shape magnitude spectrum at the normalized frequency $[-\pi, \pi]$

(b) Plot of sequence $x[n] = \sin(\pi n/2)/\pi n$

Example 6 (continued)

- We observe that the inverse DTFT of an **orthogonal** spectrum produces a **non-causal** sequence. The maximum value of the sequence is $B = 0.5$ and its zero points are $n = k\pi/B = 2k$.

- If the orthogonal function $X\left(e^{j\omega}\right)$ corresponds to the spectrum of an ideal low-pass filter that we want to construct, then the time sequence corresponds to the impulse response $h[n]$ of the filter. According to the above solution the shock response is a sequence **of infinite duration and non-causal.** So the ideal depth filter **is not feasible.**

- In the next lecture we will see an approximate method of generating the impulse response of a practical filter, according to which:
  (a) we limit (arbitrarily) the infinite length of the impulse response symmetrically to zero, and
  (b) we shift the remaining part of the sequence by one amount of time shift so as to remove the non-causality of the impulse response.
  This solution leads to the creation of a <u>practical filter</u> which is close to the ideal.

DTFT Transform of Periodic Signals

DTFT of Periodic Discrete-Time Signals

The periodic discrete-time signals do not tend to zero when $n \to \infty$, so they are not absolutely summable and the DTFT cannot be calculated from its definition. We calculate the DTFT of the periodic SDX if we allow (in its calculation) the existence of shock functions with amplitudes equal to the coefficients of the exponential Fourier series.

Fourier series expansion of the train $\delta_N[n]$ is:

$$\delta_N[n] = \frac{1}{N}\sum_{k=0}^{N-1} \Delta[k]\, e^{jk\omega_0 n}$$

where $\Delta[k] = 1/\mathrm{N}$.

Since $F\left\{e^{j\omega_0 n}\right\} = 2\pi\delta(\omega - \omega_0)$, the DTFT of the periodic signal $\delta_N[n]$is:

$$\Delta\left(e^{j\omega}\right) = F\{\delta_N[n]\} = F\left\{\frac{1}{N}\sum_{k=0}^{N-1} \Delta[k]\, e^{jk\omega_0 n}\right\} = \frac{2\pi}{N}\sum_{k=0}^{N-1} \delta(\omega - k\omega_0)$$

From a periodic signal $x[n]$ we extract a period of $x[n, N]$, for which the DTFT is $X_N\left(e^{j\omega}\right)$. The periodic signal can be produced from the equation:

$$x[n] = x[n, N] * \delta_N[n]$$

DTFT of Periodic Discrete-Time Signals

- Applying DTFT to the above equation we get:

$$X\left(e^{j\omega}\right) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_N(e^{jk\omega_0})\, \delta(\omega - k\omega_0)$$

- Based on the above, the DTFT of a periodic discrete-time signal can be obtained by multiplying the DTFT of one period $x[n, N]$with the DTFT of the periodic sequence $\delta_N[n]$.
- In other words the DTFT of the periodic discrete-time signal is obtained by sampling with a sampling period ω_0 of DTFT of one period.

Example 7

Prove that the DTFT of the signal $x[n] = e^{j\omega_0 n}$, $\omega \in (-\pi, \pi]$, is given by the equation $x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 + 2\pi m)$, $m \in Z$.

__Answer:__ Since the signal is not absolutely summable the DTFT cannot be calculated from its definition. For this reason we will work in reverse, i.e. we will calculate the inverse DTFT. We notice that the function:

$$X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 + 2\pi m)$$

is an infinite sum of shock functions spaced apart $2\pi m$ on the frequency axis. In other words, its $e^{j\omega_0 n}$ DTFT contains impulse functions at frequencies $\omega_0 \pm 2\pi m$.

The **inverse DTFT** is calculated in the frequency domain $(-\pi, \pi]$ from the equation:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{ \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 + 2\pi m) \right\} e^{j\omega n}\, d\omega$$

But in the region $(-\pi, \pi]$ there is only the function $\delta(\omega - \omega_0)$, so the integral is:

$$x[n] = \int_{-\pi}^{\pi} \delta(\omega - \omega_0)\, e^{j\omega n}\, d\omega = e^{j\omega n}\Big|_{\omega=\omega_0} = e^{j\omega_0 n}$$

Useful DTFT pairs

Signal $x[n]$	DTFT transform $X(e^{j\omega})$
$\delta[n]$	$1, -\infty < \omega < \infty$
$\delta[n-n_0]$	$e^{-jn_0\omega}$
1	$2\pi\delta(\omega)$
$e^{jn_0\omega}$	$2\pi\delta(\omega-\omega_0)$
$\alpha^n u[n], \quad \lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-j\omega}}$
$-\alpha^n u[n-1], \quad \lvert a\rvert > 1$	$\dfrac{1}{1-ae^{-j\omega}}$
$[n+1]\alpha^n u[n], \quad \lvert a\rvert < 1$	$\dfrac{1}{(1-ae^{-j\omega})^2}$
$u[n]-u[n-n_0]$	$\dfrac{sin(\omega n_0/2)}{sin(\omega/2)} e^{-j(n_0-1)/2}$

Useful DTFT pairs

Signal $x[n]$	DTFT transform $X(e^{j\omega})$
$cos[n\omega_0]$	$\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi k) + \delta(\omega + \omega_0 + 2\pi k)$
$sin[n\omega_0]$	$\frac{\pi}{j} \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi k) - \delta(\omega + \omega_0 + 2\pi k)$
$a^n\, cos(n\omega_0)\, u[n]$	$\frac{1 - a\, e^{-j\omega} \cos \omega_0}{1 - 2a\, e^{-j\omega} \cos \omega_0 + a^2 e^{-2j\omega}}$
$a^n\, sin(n\omega_0)\, u[n]$	$\frac{a\, e^{-j\omega} \sin \omega_0}{1 - 2a\, e^{-j\omega} \cos \omega_0 + a^2 e^{-2j\omega}}$

DTFT properties

- Periodicity
- Symmetry and Conjugation
- Linearity
- Reversing Time
- Shift in Time
- Shift in Frequency
- Differentiation in Frequency
- Convolution Theorem
- Periodic Convolution
- Correlation
- Parseval's theorem

Periodicity

- DTFT is a periodic function with period 2π, i.e. it satisfies the equation:

$$X\left(e^{j\omega}\right) = X\left(e^{j(\omega+2k\pi)}\right)$$

- Periodicity is a result of the fact that discrete-time complex exponential signals when they differ in frequency by multiples of 2π, are identical to each other.

- This property does not apply to the Fourier transform of continuous-time signals.

- Application: Based on the property of periodicity it follows that for the analysis of the DTFT we need **only one period** of the function $X\left(e^{j\omega}\right)$, e.g. $[0,2\pi]$ or $[-\pi,\pi]$, and not the whole interval $-\infty < \omega < \infty$. This saves a lot of computation effort.

Linearity

- The DTFT is linear, that is, the DTFT of a linear combination of signals is equal to the sum of the DTFTs of the individual components of the linear combination.

- If the individual DTFT transforms are:

$$x_1[n] \xleftrightarrow{DTFT} X_1(e^{j\omega})$$

$$x_2[n] \xleftrightarrow{DTFT} X_2(e^{j\omega})$$

then the DTFT transform of the linear combination $a_1 x_1[n] + \alpha_2 x_2[n]$ will be:

$$\alpha x_1[n] + b x_2[n] \xleftrightarrow{DTFT} a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

- From this property it follows that the DTFT is a linear transform, suitable for the study of discrete-time linear systems.

Example 8

Calculate the DTFT of the signal $x[n] = \cos(\omega_0 n)$.

<u>**Answer**:</u> We know that:

$$e^{j\omega_0 n} \longleftrightarrow \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 + 2\pi m),$$

By setting $\omega = -\omega_0$ we have:

$$e^{-j\omega_0 n} \longleftrightarrow \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega + \omega_0 + 2\pi m),$$

From the Euler equation it follows:

$$x[n] = \cos(\omega_0 n) = \frac{1}{2}\left\{e^{jn\omega_0} + e^{-jn\omega_0}\right\}$$

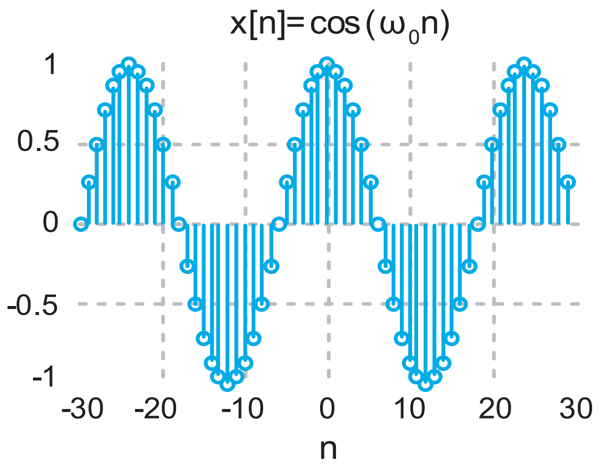
Since the DTFT is linear, we have:

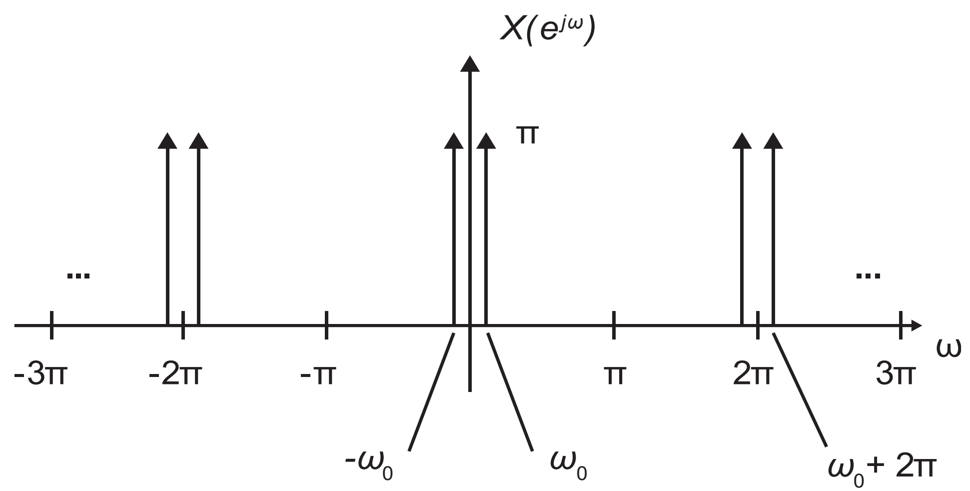
$$X\left(e^{j\omega}\right) = F\left\{\frac{1}{2}\left\{e^{jn\omega_0} + e^{-jn\omega_0}\right\}\right\} = \frac{1}{2}F\left\{e^{jn\omega_0}\right\} + \frac{1}{2}F\left\{e^{-jn\omega_0}\right\}$$

$$= \frac{1}{2}\sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 + 2\pi m) + \frac{1}{2}\sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega + \omega_0 + 2\pi m)$$

$$= \pi \sum_{m=-\infty}^{\infty} \left\{\delta(\omega - \omega_0 + 2\pi m) + \delta(\omega + \omega_0 + 2\pi m)\right\}$$

Example 8 (continued)





(a) $x[n] = \cos(\omega_0 n)$, (b) $X(e^{j\omega}) = \pi \sum_{m=-\infty}^{\infty}\{\delta(\omega - \omega_0 + 2\pi m) + \delta(\omega + \omega_0 + 2\pi m)\}$

If we limit the solution to the frequency interval [- π,π), the above equation is written:

$$X(e^{j\omega}) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

Working analogously for the sine function, we get:

$$sin(\omega_0 n) = \frac{1}{2}j\{e^{jn\omega_0} - e^{-jn\omega_0}\} \longleftrightarrow \pi j \sum_{m=-\infty}^{\infty}\{\delta(\omega - \omega_0 + 2\pi m) - \delta(\omega + \omega_0 + 2\pi m)\}$$

Symmetry and Conjugation

- For a real discrete-time signal $x[n]$, the function $X\left(e^{j\omega}\right)$ is conjugate-symmetric, that is, it satisfies the equation:

$$X\left(e^{-j\omega}\right) = X^*\left(e^{j\omega}\right)$$

- This relationship is called **Hermitian symmetry** and is equivalent to the following expressions:
  - $X_R\left(e^{-j\omega}\right) = X_R\left(e^{j\omega}\right)$ The real part has perfect symmetry
  - $X_I\left(e^{-j\omega}\right) = -X_I\left(e^{j\omega}\right)$ The imaginary part has unnecessary symmetry
  - $\left|X\left(e^{-j\omega}\right)\right| = \left|X\left(e^{j\omega}\right)\right|$ The magnitude has perfect symmetry
  - $\measuredangle X\left(e^{-j\omega}\right) = -\measuredangle X\left(e^{j\omega}\right)$ The phase has redundant symmetry
- <u>Application:</u> Based on the property of symmetry it follows that for the plotting of the function $X(e^{j\omega})$ we need **only half a period,** we usually choose $\boldsymbol{\omega} \in [\mathbf{0}, \boldsymbol{\pi}]$.

Symmetry and Conjugation Properties

Signal $x[n]$	DTFT transform $X(e^{j\omega})$
Real and even	Real and even
Real and odd	Imaginary and odd
Imaginary and even	Imaginary and even
Imaginary and odd	Real and odd

Reversal & Shift in Time

- **Reversal** in the time domain corresponds to **reversal also** in the frequency domain. Specifically, if the DTFT of a signal $x[n]$ is $X(e^{j\omega})$, then:

$$x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$$

- A **shift** in the time domain corresponds to **a phase shift** in the frequency domain, while the magnitude spectrum (meter) remains the **same.** Specifically, if the DTFT of a signal $x[n]$ is $X(e^{j\omega})$, then:

$$x[n-n_0] \xleftrightarrow{DTFT} e^{-jn_0\omega} X(e^{j\omega})$$

- From the property it becomes clear that the frequency content of a signal depends only on its form and not on its position.

Example 9

Calculate the DTFT of the signals and plot the magnitude and phase spectra:

(a) $x_1[n] = u[n+2] - u[n-2]$ (b) $x_2[n] = x_1[n-2]$

Answer: (a) The given signal can be written:

$$x_1[n] = \delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1]$$

We know that:

$$\delta[n] \xleftrightarrow{DTFT} \Delta(e^{j\omega}) = 1$$

Based on the time shift property it follows that:

$$\delta[n-n_0] \xleftrightarrow{DTFT} e^{-jn_0\omega}\Delta(e^{j\omega})$$

Therefore for the time-shifted versions of $\delta[n]$, we have:

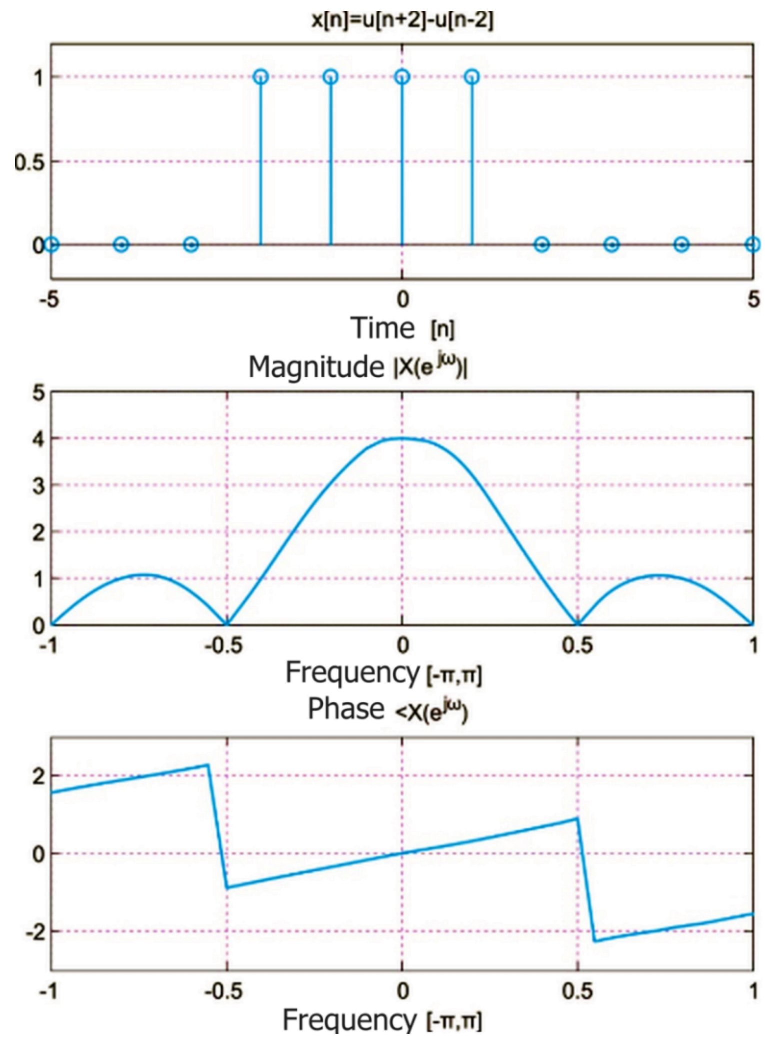
$$\delta[n+2] \xleftrightarrow{DTFT} e^{j2\omega} \qquad \delta[n+1] \xleftrightarrow{DTFT} e^{j\omega} \qquad \delta[n-1] \xleftrightarrow{DTFT} e^{-j\omega}$$

Based on the linearity property of the DTFT it follows:

$$X_1(e^{j\omega}) = e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega}$$

Example 9 (continued)

- In the figure are given the graphical representations of $x_1[n]$ and a full period in the frequency interval $[-\pi, \pi]$ of the magnitude and phase spectra of the function $X_1(e^{j\omega})$.
- We notice that the maximum value of the spectrum is equal to the number of pulses (in our example it is 4).
- Also, the number of spectrum waves in a period depends on the number of pulses that make up the signal $x_1[n]$.



Example 9 (continued)

(b) The given mark may be written: $x_2[n] = x_1[n-2] = u[n] - u[n-4]$.

Based on the time shift property of the DTFT, the DTFT of $x_2[n]$ is:

$$X_2(e^{j\omega})$$
$$= e^{-j2\omega} X_1(e^{j\omega})$$
$$= e^{-j2\omega}\left[e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega}\right]$$
$$= 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} \qquad (1)$$

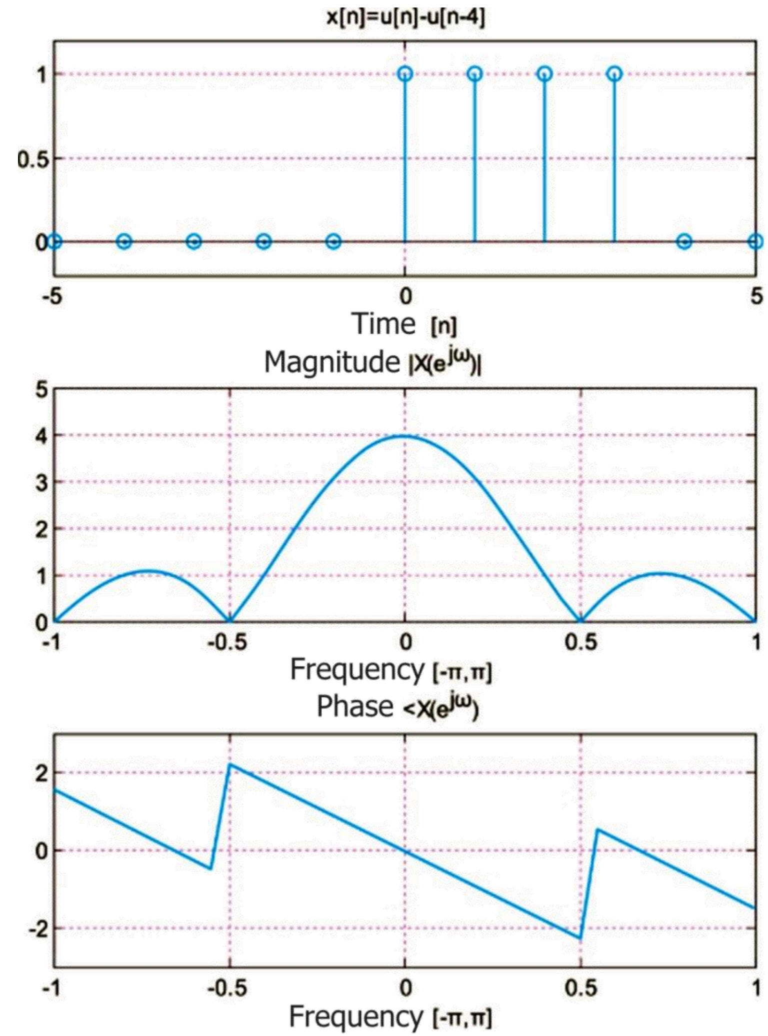
Regarding the magnitude and the phase of the function $X_2(e^{j\omega})$:

$$\left|X_2(e^{j\omega})\right| = \left|X_1(e^{j\omega})\right| \text{and} \measuredangle X_2(e^{j\omega}) = \measuredangle X_1(e^{j\omega}) + 2\omega$$

Example 9 (continued)

- From the adjacent figure, we observe that the magnitude spectrum of the time-shifted signal $x_2[n]$ remains the same as that of the original signal $x_1[n]$, while the phase spectrum is shifted by 2ω.

- <u>Note:</u> The spectra were designed in the frequency range $[-\pi, \pi)$.



Shift & Differentiation in Frequency

- Multiplying a discrete-time signal by a complex exponential term $e^{jn_0\omega}$ causes the spectrum $X(e^{j\omega})$ **to shift** in the frequency domain by ω_0. Specifically, if the DTFT of a signal $x[n]$ is $X(e^{j\omega})$, then:

$$e^{jn_0\omega}x[n] \xleftrightarrow{DTFT} X\left(e^{j(\omega-\omega_0)}\right)$$

- If the DTFT of a signal $x[n]$ is $X(e^{j\omega})$, then the **derivative** of the spectrum holds the equation:

$$-jnx[n] \xleftrightarrow{DTFT} \frac{dX(e^{j\omega})}{d\omega}$$

Example 10

To calculate the DTFT of the signal $y[n] = x[n]\cos[n\omega_0]$, when the DTFT of $x[n]$ is known.

<u>Answer</u>: Based on the Euler equation, the cosine is written:

$$\cos[n\omega_0] = \frac{1}{2}\left\{e^{jn\omega_0} + e^{-jn\omega_0}\right\}$$

Therefore, the signal $y[n]$ is:

$$y[n] = x[n]\cos[n\omega_0] = x[n]\frac{1}{2}\left\{e^{jn\omega_0} + e^{-jn\omega_0}\right\} =$$

$$= \frac{1}{2}e^{jn\omega_0}x[n] + \frac{1}{2}e^{-jn\omega_0}x[n]$$

Applying the shift property to the frequency of the DTFT, we have:

$$x[n]\cos[n\omega_0] \xleftrightarrow{DTFT} \frac{1}{2}X\left(e^{j(\omega-\omega_0)}\right) + \frac{1}{2}X\left(e^{j(\omega+\omega_0)}\right)$$

Time Scale Change – Frequency Derivation

1. Multiplying the time variable n by a rational number k corresponds to dividing the frequency variable by the same number k, i.e. if $x[n] \overset{DTFT}{\longleftrightarrow} X\left(e^{j\omega}\right)$, then:

$$x[kn] \overset{DTFT}{\longleftrightarrow} X\left(e^{j\omega/k}\right)$$

2. If $x[n] \overset{DTFT}{\longleftrightarrow} X\left(e^{j\omega}\right)$, then:

$$nx[n] \overset{DTFT}{\longleftrightarrow} j\,\frac{dX\left(e^{j\omega}\right)}{d\omega}$$

$$(-jn)^k x[n] \overset{DTFT}{\longleftrightarrow} j\,\frac{d^k X\left(e^{j\omega}\right)}{d\omega^k}$$

Convolution Theorem

- If the DTFTs of the sequences $x[n]$ and $h[n]$ are $X\left(e^{j\omega}\right)$ and $H\left(e^{j\omega}\right)$, then the **convolution** of the two sequences holds:

$$h[n] * x[n] \stackrel{DTFT}{\longleftrightarrow} H\left(e^{j\omega}\right) X\left(e^{j\omega}\right)$$

- This property simplifies the analysis of discrete-time systems, as it transforms the computationally difficult operation of convolution into the computationally simple operation of multiplication.

- The computation in the time-domain of the output of an LSI system is given by the convolution:

$$y[n] = h[n] * x[n]$$

using the convolution property, the DTFT of the output is:

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right)$$

- Writing the above relationship in magnitude and phase, yields:

$$\left|Y\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right| \left|X\left(e^{j\omega}\right)\right|$$

$$\measuredangle Y\left(e^{j\omega}\right) = \measuredangle H\left(e^{j\omega}\right) + \measuredangle X\left(e^{j\omega}\right)$$

Example 11

Calculate the output $y[n]$ of a discrete-time LSI system with impulse response $h[n] = 0.2^n u[n]$ when the input is $x[n] = 0.5^n\, u[n]$.

<u>Answer</u>: We calculate the DTFT of the sequences $h[n]$ και $x[n]$. Is:

$$H\left(e^{j\omega}\right) = \frac{1}{1-0.2\, e^{-j\omega}} \quad \text{and} \quad X\left(e^{j\omega}\right) = \frac{1}{1-0.5\, e^{-j\omega}}$$

The output results from convolution $y[n] = h[n] * x[n]$. Using the convolution property of the DTFT we have:

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right) = \frac{1}{1-0.2\, e^{-j\omega}} \frac{1}{1-0.5\, e^{-j\omega}}$$
$$= \frac{1}{(1-0.2e^{-j\omega})(1-0.5e^{-j\omega})}$$

To obtain the sequence $y[n]$ from the function $Y\left(e^{j\omega}\right)$ we will need to calculate the inverse DTFT of $Y\left(e^{j\omega}\right)$.

Example 11 (continued)

We break it down $Y\left(e^{j\omega}\right)$ into a sum of several fractions:

$$Y\left(e^{j\omega}\right) = \frac{1}{(1-0.2e^{-j\omega})(1-0.5e^{-j\omega})} = \frac{C_1}{1-0.2e^{-j\omega}} + \frac{C_2}{1-0.5e^{-j\omega}} \qquad (1)$$

where the coefficients C_1 και $C_2$are constants to be determined. We perform the operations on the right member of equation (1) and obtain:

$$Y\left(e^{j\omega}\right) = \frac{C_1 - 0.5C_1e^{-j\omega} + C_2 - 0.2C_2e^{-j\omega}}{(1-0.2e^{-j\omega})(1-0.5e^{-j\omega})} \qquad (2)$$

Equating the numerators of the left-hand member of equation (1) and equation (2), we have:

$$1 = C_1 - 0.5C_1e^{-j\omega} + C_2 - 0.2C_2e^{-j\omega} = (C_1+C_2) - (0.5C_1 + 0.2C_2)e^{-j\omega}$$

Example 11 (continued)

We calculate the constants C_1 και C_2 by solving the system:

$$C_1 + C_2 = 1$$

$$0.5C_1 + 0.2C_2 = 0$$

The result is:

$$C_1 = -2/3 \text{ and } C_2 = 5/3$$

so equation (1) is written:

$$Y(e^{j\omega}) = \frac{-2/3}{1-0.2e^{-j\omega}} + \frac{5/3}{1-0.5e^{-j\omega}} \qquad (1)$$

Therefore based on the table of useful DTFT pairs, the inverse DTFT is:

$$y[n] = -\frac{2}{3}(0.2)^n u[n] + \frac{5}{3}(0.5)^n u[n] = \left[-\frac{2}{3}(0.2)^n + \frac{5}{3}(0.5)^n\right] u[n]$$

Signal Multiplication – Periodic Convolution

1. If the DTFTs of the sequences $x[n]$ and $y[n]$ are $X(e^{j\omega})$and $Y(e^{j\omega})$ respectively, then the DTFT of the product of the signals $x[n]$ and $y[n]$ is:

$$x[n]\, y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\, X(e^{j\omega}) * Y(e^{j\omega})$$

2. The DTFT of the product of two periodic sequences $x[n]$ and $y[n]$ with the same period N, is the periodic convolution of the individual DTFTs $X(e^{j\omega})$ and $Y(e^{j\omega})$ signals.

$$x[n]\, y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})\, Y(e^{j(\omega-\theta)}) d\theta = X(e^{j\omega}) \circledast Y(e^{j\omega})$$

Time and Frequency Symmetry

The duality (symmetry) property of the Fourier transform between the time and frequency domains for continuous time signals also applies to the DTFT.

Is particularly useful for calculating the DTFT on signals that are not fully summable.

The pairs of equations apply:

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}$$

$$\sum_{k=-\infty}^{\infty} X[k]e^{-j\omega_k n} \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} 2\pi\, X[k]\, \delta(\omega+\omega_k)$$

Example 12

Calculate the DTFT of the signal $x[n] = \cos(\omega_0 n)$ using the duality (symmetry) property.

Answer: From Example 8 we know that the DTFT of $\cos(\omega_0 n)$ in range $[-\pi, \pi)$, is:

$$X\left(e^{j\omega}\right) = \pi\{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\}$$

We will verify the result with the duality (symmetry) property.
From the Euler equation we have $x[n] = \cos(\omega_0 n) = 0.5\left\{e^{jn\omega_0} + e^{-jn\omega_0}\right\}$.

Because $e^{jn\omega_0} \longleftrightarrow \delta(\omega - \omega_0)$ from the relationship:

$$\sum_{k=-\infty}^{\infty} X[k]e^{-j\omega_k n} \overset{DTFT}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi\, X[k]\, \delta(\omega + \omega_k)$$

resulting:

$$\cos(\omega_0 n) = \frac{1}{2}\left\{e^{jn\omega_0} + e^{-jn\omega_0}\right\} \longleftrightarrow \pi\{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\}$$

Correlation

If the DTFTs of the sequences $x[n]$ and $y[n]$ are $X\left(e^{j\omega}\right)$ and $Y\left(e^{j\omega}\right)$, respectively, then for the correlation $R_{xy}[n]$ of the two sequences defined by the equation:

$$R_{xy}[n] = \sum_{n=-\infty}^{\infty} x[n]\, y[n-k]$$

apply:

$$R_{xy}[n] \overset{DTFT}{\longleftrightarrow} X\left(e^{j\omega}\right) Y\left(e^{-j\omega}\right)$$

Parseval's Theorem

- The energy of a discrete-time signal $x[n]$ can be calculated from the equation:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$$

- The DTFT conserves the total energy during the transition from the time domain to the frequency domain, which is why it is also called the conservation of energy theorem.

- The term $\left|X(e^{j\omega})\right|^2$ is called **energy-density spectrum** and expresses the energy of the signal per frequency unit.

- If the sign $x[n]$ is real, the equation is written:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{0}^{\pi} \frac{\left|X\left(e^{j\omega}\right)\right|^2}{\pi} d\omega$$

Example 13

Calculate the energy of the sequence $x[n] = 0.5^n u[n]$ from its spectrum.

Answer: The DTFT of the sequence $x[n] = 0.5^n u[n]$ is:

$$X(e^{j\omega}) = \frac{1}{1 - 0.5\, e^{-j\omega}}$$

Its width is:

$$\left|X(e^{j\omega})\right|^2 = X(e^{j\omega})\, X^*(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}} \frac{1}{1 - 0.5e^{j\omega}} = \frac{1}{1.25 - \cos\omega}$$

Since the sequence is real, the energy is given by the Parseval equation:

$$E_x = \int_0^{\pi} \frac{\left|X(e^{j\omega})\right|^2}{\pi} d\omega = \int_0^{\pi} \frac{1}{\pi(1.25 - cos\omega)} d\omega = \cdots = \frac{4}{3}$$

Relationship of DTFT with other Transforms

- With the Fourier transform
- With the Z-transform

DTFT relationship with Fourier

- A discrete-time signal $x[n]$derived from sampling a continous-time signal $x(t)$ with sampling period T_s, i.e. $x[n] = x(nT_s) = x(t)|_{t=nT_s}$ has a DTFT:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-jn\omega}$$

- Continuous-time Fourier transform $X_s(\Omega)$ Fourier of the sequence $x[n]$ calculated for $\omega = \Omega T_s$

$$X_s(\Omega) = X_s(e^{j\omega})\Big|_{\omega=\Omega T_s} = X_s(e^{j\Omega T_s}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-jn\Omega T_s}$$

- From the time shift property of the $\delta(t - nT_s) \overset{F}{\longleftrightarrow} e^{-jn\Omega T_s}$ of Z-Transform, the inverse Fourier transform of the function is found $X_s(\Omega)$. Is:

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x[n]\, \delta(t - nT_s)$$

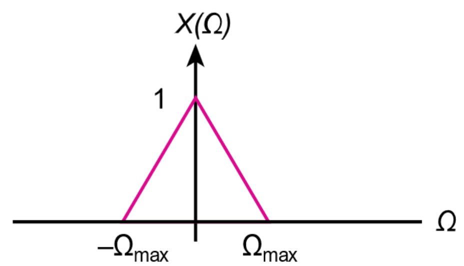
- Therefore, its $x[n]$ DTFT is identical to the continuous-time $X_s(\Omega)$ Fourier transform of the sampled signal $x_s(t)$, since:

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x[n]\, \delta(t - nT_s) \overset{F}{\longleftrightarrow} X_s(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-jn\Omega T_s}$$

DTFT relationship with Fourier

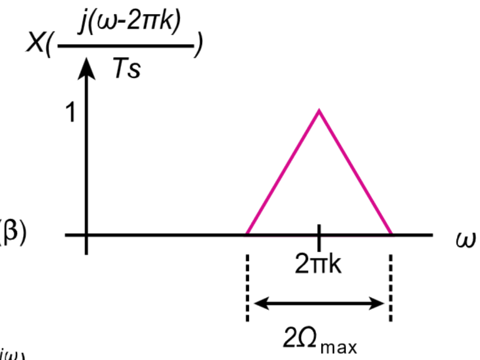
(a) Signal spectrum $x(t)$,

(α)



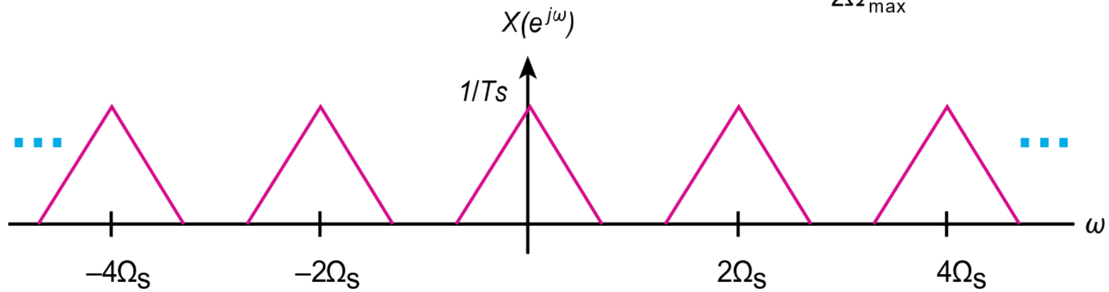
(b) One of the spectrum terms $X_s(\Omega)$ of the sampled signal $x_s(t)$,

(β)



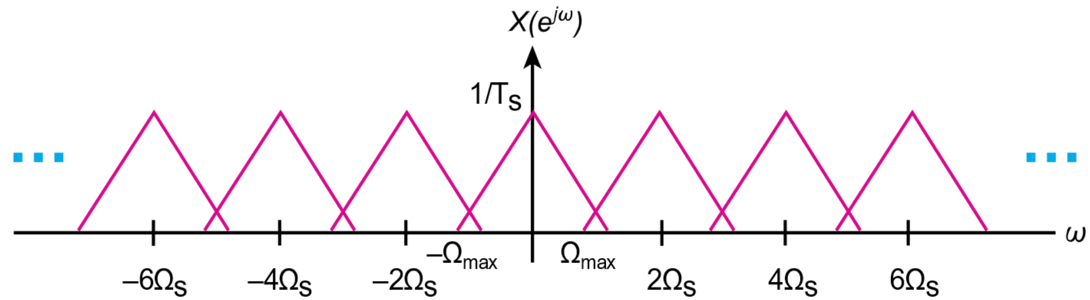
(c) Periodic $X\left(e^{j\omega}\right)$ sequence spectrum $x[n]$ for $f_s > 2f_{max}$

(γ)



(d) Periodic $X\left(e^{j\omega}\right)$ sequence spectrum $x[n]$ for $f_s < 2f_{max}$

(δ)



DTFT relationship with Fourier Transform

- Between the DTFT of a sequence $x[n] = x(nT_s) = x(t)|_{t=nT_s}$, which has resulted from sampling (sampling period T_s) of a continuous time signal $x(t)$, the relationship applies:

$$X_s\left(e^{j\Omega T_s}\right) = X_s\left(e^{j\omega}\right)\Big|_{\omega=\Omega T_s} = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X\left(\frac{\omega}{T_s} - \frac{2\pi k}{T_s}\right)$$

Therefore, the spectrum of the sequence $x[n]$ consists of periodic repetitions with a period $2\pi/T_s$ of the continuous-time signal spectrum $x(t)$, with magnitude multiplied by $1/T_s$.

- In order to be able to reconstruct the continuous-time signal $x(t)$ from the sequence $x[n]$, the periodic repetitions of the spectrum must $X\left(e^{j\omega}\right)$ not overlap. This condition is met when:

$$\Omega_{max} T_s < \pi \;\Rightarrow\; 2\pi f_{max}(1/f_s) < \pi \;\Rightarrow\; f_s > 2 f_{max}$$

which is known as **the Nyquist condition or criterion.**

DTFT relationship with Z -transform

- We know that the DTFT results from the Z-transform calculated on the unit circle, since it is within the convergence region, i.e. by setting $z = e^{j\omega}$:

$$DTFT\{x[n]\} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = X(z)\Big|_{z=e^{j\omega}}$$

- Therefore, the DTFT can be considered as a subcase of the Z-transform for $|z| = 1$.

- However, there are discrete-time signals for which it is not possible to calculate the DTFT from Z, because the latter does not converge.

Relationship of DTFT to discrete Fourier Series

For a periodic discrete-time signal with period and fundamental frequency $\omega_0 = 2\pi/N$, the coefficients of the exponential $X[k]$ Fourier series are calculated from the equation:

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\omega_0 kn}$$

and the signal is decomposed into an exponential Fourier Series as: $x[n] = \sum_{k=0}^{N-1} X[k]\, e^{j\omega_0 kn}$

Calculating the DTFT for the above signal, we have:

$$X\left(e^{j\omega}\right) = F\{x[n]\} = F\left\{\sum_{k=0}^{N-1} X[k]e^{j\omega_0 kn}\right\} = \sum_{k=0}^{N-1} X[k]\, F\left\{e^{j\omega_0 kn}\right\} = 2\pi\sum_{k=0}^{N-1} X[k]\, \delta(\omega - k\omega_0)$$

For the same signal (period N and fundamental frequency $\omega_0 = 2\pi/N$), the DTFT is:

$$X\left(e^{j\omega}\right) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty} X_N\left(e^{jk\omega_0}\right)\delta(\omega - k\omega_0)$$

From the previous two relations, it follows:

$$X[k] = \frac{1}{N} X_N\left(e^{jk\omega_0}\right)\Big|_{\omega = k\omega_0}$$

Fourier series are identical to the coefficients of the discrete spectrum of the DTFT for periodic signals at integer multiples of the fundamental frequency.

Sample Rate Conversion

- Down-sampling
- Up-sampling
- Real number sample rate conversion

Down-sampling

- It turns out that the DTFT transform of the subsampled signal $x_d[n] = x[nM\ ]$, $(M > 1)$, is:

$$X_d(e^{j\omega}) = \frac{1}{M} X(e^{j\omega/M})$$

- Shrinking the signal in time by a factor M causes the spectrum to expand by the same factor.

- Attention: In the subsampling of a DX signal with a coefficient M, in order not to result in an overlap of the frequencies, the signal must be spectrally **limited** up to the frequency π/M.

- If the condition holds, then the spectrum of the subsampled signal is an extended version of the spectrum of the original signal.

- If the condition does not hold, then before subsampling we have to filter the signal with a deep-pass filter with a cut-off frequency $\omega_c = \pi/M$ (filter gain M).

Example 14

A discrete pulse is given by the equation $x[n] = u[n+2] - u[n-2]$. The pulse is subsampled with $M = 2$and the sequence is obtained $x_d[n]$. Find the DTFTs of the sequences $x[n]$ and $x_d[n]$.

Answer: The sequence $x[n]$ is written:

$$x[n] = \delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1]$$

and has a Z-transform:

$$X(z) = z^2 + z^1 + z^0 + z^{-1} = z^2 + z + 1 + z^{-1}$$

with region of convergence the whole field Z. Since the unit circle is included in the region of convergence of the Z-transform, the DTFT is:

$$X\left(e^{j\omega}\right) = X(z)\Big|_{z=e^{j\omega}} = e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega}$$

$$= e^{j\frac{\omega}{2}}\left(e^{j\frac{3\omega}{2}} + e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}} + e^{-j\frac{3\omega}{2}}\right) = e^{j\frac{\omega}{2}}\left(e^{j\frac{3\omega}{2}} + e^{-j\frac{3\omega}{2}} + e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right)$$

$$= 2e^{j\frac{\omega}{2}}\left[\cos\left(\frac{3\omega}{2}\right) + \cos\left(\frac{\omega}{2}\right)\right]$$

Example 14 (continued)

The subsampled sequence is given by the equation:

$$x_d[n] = x[2n] = u[2n+2] - u[2n-2] = u[n+1] - u[n-1] \Rightarrow x_d[n] = \delta[n+1] + \delta[n]$$

and has a Z-transform:

$$X(z) = z^1 + z^0 = z + 1$$

with region of convergence the whole field Z. Since the unit circle is included in the region of convergence, the DTFT is:

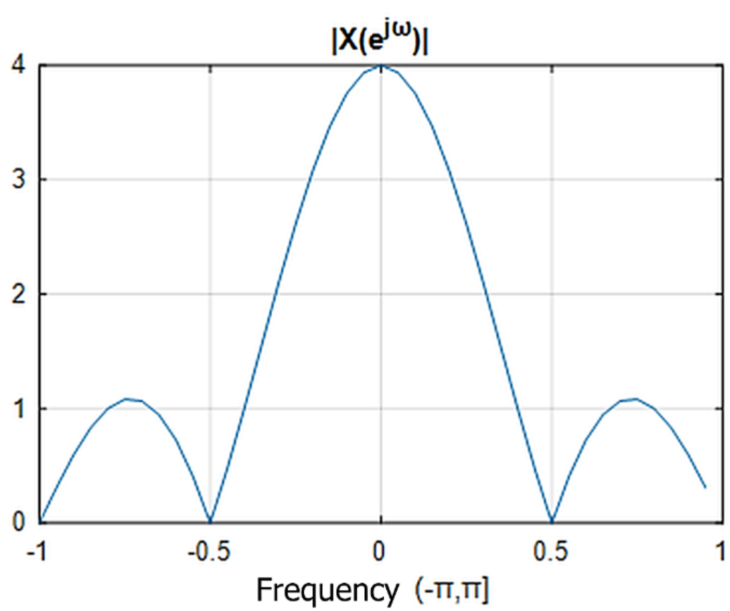
$$X_d\left(e^{j\omega}\right) = X(z)\Big|_{z=e^{j\omega}} = e^{j\omega} + 1 = e^{j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right) = 2e^{j\frac{\omega}{2}}\cos\left(\frac{\omega}{2}\right)$$

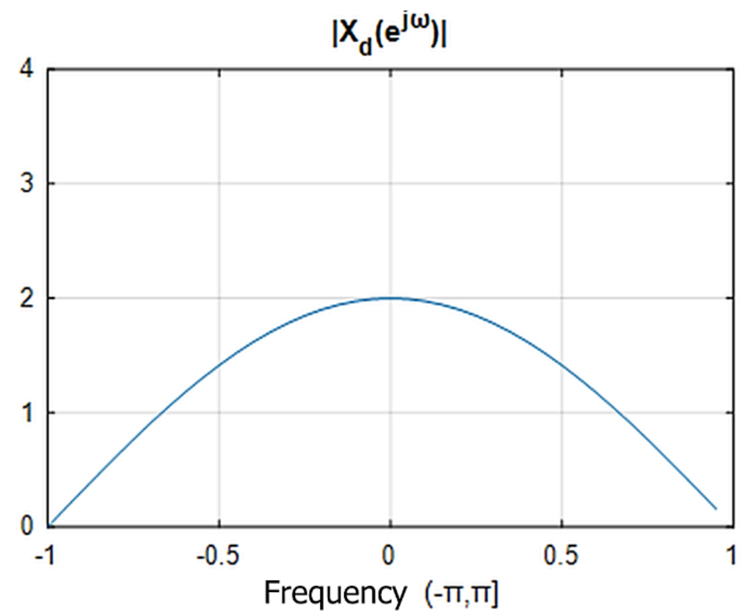
Comparing the DTFTs we notice that they do not satisfy the equation (10.63), that is:

$$X_d\left(e^{j\omega}\right) \neq \frac{1}{2}X\left(e^{j\omega/2}\right)$$

Example 14 (continued)

The same observation results from the comparison of the following spectra. The reason this happens is the aliasing effect, because as can be seen from the figure, the original signal is not of finite bandwidth and therefore its maximum frequency exceeds the frequency $\pi/2$. Therefore, during the sampling we performed, the Nyquist criterion was violated.





(a) Magnitude spectrum of the sequence$x[n] = u[n+2] - u[n-2]$
(b) Magnitude spectrum of the sequence$x_d[n] = u[n+1] - u[n-1]$

Up-sampling

- In **up-sampling,** the new signal $x_u[n]$is formed by interpolating a number of $N-1$ zero values between two consecutive samples of it $x[n]$:

$$x_u[n] = \begin{cases} x[n/N], & n = 0, \pm 1, \pm 2, \dots \\ 0, & \text{αλλού} \end{cases}$$

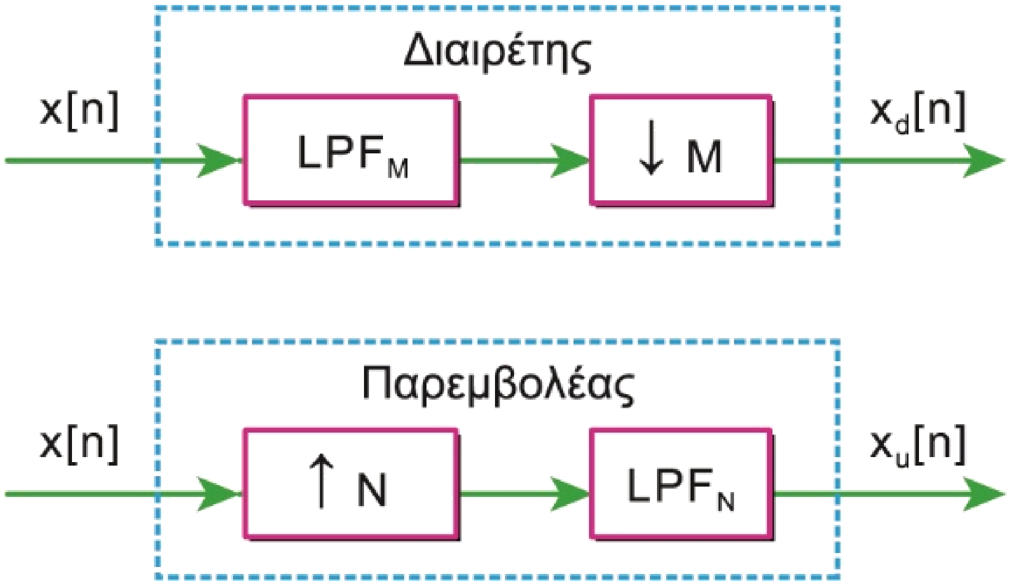
- If $X(e^{j\omega})$ is the DTFT of the signal $x[n]$, then the DTFT of the downsampled signal $x_u[n]$, is:

$$X_u(e^{j\omega}) = \sum_{n=0,\pm N,\dots} x[n/N]\, e^{-jn\omega} = \sum_{m=-\infty}^{\infty} x[m]\, e^{-jNm\omega} = X(e^{j\omega N})$$

- Stretching the signal in time by a factor N causes the spectrum to shrink by the same factor.

- Up-sampling does not lead to a violation of the Nyquist criterion. But after frequency multiplication, its images must be removed $X(e^{j\omega})$ except those that are in integer multiples of 2π. This is done by filtering the up-sampled signal $x_u[n]$ with a low-pass filter with cut-off frequency $\omega_c = \pi/N$ and gain N.

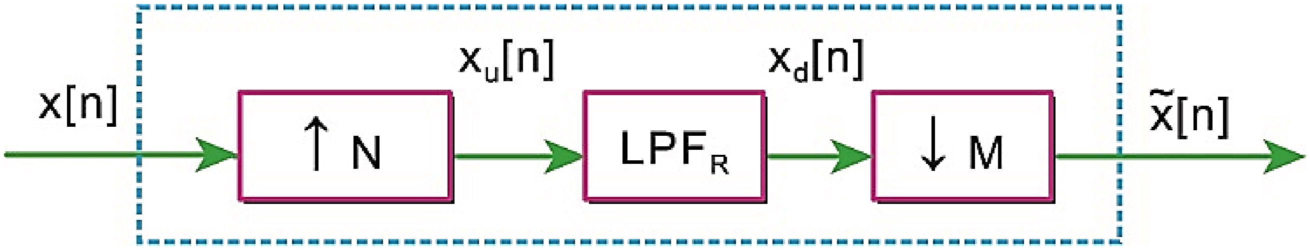
Divisor and Interceptor



- Decimator: the series connection of the frequency divider with the low- pass filter.
- **Interpolator:** the series connection of the frequency multiplier with the low- pass

Real number sample rate conversion

- To convert the sampling rate by an real factor $R = N/M$, we connect in series a divider (decimator), which decreases the sampling rate by a factor M, with an interpolator (interpolator), which increases the sampling rate by a factor N.



Explicit sample rate converter

Example 15

Storing digital audio on a CD disc uses $f_s = 44.1\ kHz$ while on a magnetic tape (DAT) it uses $f_s = 48\ kHz$. Find the filter in the explicit sample rate converter to enable direct transfer of digitized music from CD to DAT.

Answer: We factor the sampling rates into prime factors and have: $f_{sDAT} = 2^7 3\ 5^3$ and $f_{sCD} = 2^2 3^2 5^2 7^2$.

$$R = \frac{M}{N} = \frac{2^7 3\ 5^3}{2^2\ 3^2\ 5^2\ 7^2} = \frac{2^5\ 5}{3\ 7^2} = \frac{160}{147}$$

To convert the sampling rate from 44.1 kHz to 48 kHz, frequency multiplication with M=160 and then frequency division with N=147.

The low-pass filter between the multiplier and the frequency divider has a cut-off frequency:

$$\omega_c = \min\left\{\frac{\pi}{M}, \frac{\pi}{N}\right\} = \min\left\{\frac{\pi}{160}, \frac{\pi}{147}\right\} = \frac{\pi}{160}$$

and gain $R = 160$.

Example 16

We sample the analog signal $x_\alpha(t) = 1 + \cos(15\pi t)$ with a sampling period $T_s = 0{,}1\ sec$ and pass it through a low-pass filter with a cut-off frequency $f_c = 2{,}5\ Hz$. What is the signal produced at the output of the filter?

<u>Answer</u>: The analog signal contains a DC component with zero frequency and a cosine component with frequency $2\pi F = 15\pi \Rightarrow F = 7{,}5\ Hz$, which is also the maximum frequency (F_{max}) of the analog signal.

The sampling frequency is $f_s = 1/T_s = 1/0{,}1\ sec\ = 10\ Hz$. Since, it follows that the $f_s = 7{,}5 < 10 = 2F_{max}$ Nyquist criterion is not satisfied, so frequency folding will appear for those frequencies that are outside the spectral range defined based on the sampling frequency, i.e. the spectral range $[-f_s/2, f_s/2] = [-5Hz, 5Hz]$.

Therefore, the frequency $F = F_{max} = 7{,}5\ Hz$ of the signal will be folded and show the aliasing frequency $F' = F - kf_s = 7{,}5 - k10 = 7{,}5 - 1x10 = -2{,}5\ Hz$. So the $\cos(15\pi t)$7.5 Hz cosine component of the analog signal, when sampled will look like a 2.5 Hz cosine.

The DC component is not affected by sampling.

Example 16 (continued)

Based on the above, the discrete-time signal resulting from the sampling is:

$$x_s[n] = x_a(t)\Big|_{t=nTs=n/10} = 1 + \cos\left(\frac{15\pi}{10}n\right)$$

$$= 1 + \cos\left(\frac{3\pi}{2}n\right) = 1 + \cos\left(2\pi - \frac{\pi}{2}\right)n = 1 + \cos\left(\frac{\pi n}{2}\right)$$

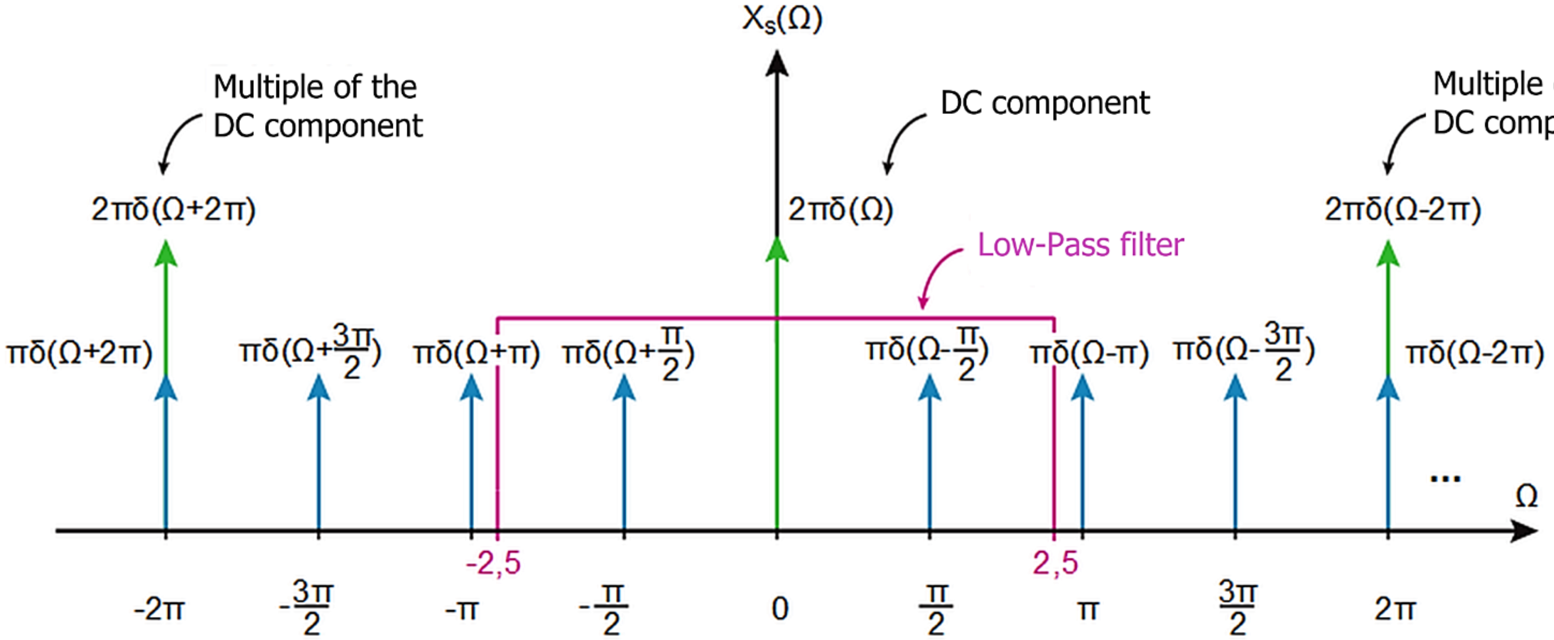
To calculate the signal at the output of the low-pass filter, we need to obtain the spectral form of the discrete-time signal. The DTFT of the sampled signal $x_s[n]$ is:

$$X_s(\Omega) = X_s\left(e^{j\omega}\right)\Big|_{\omega=\Omega T_s} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\,(\Omega - k\Omega_s)$$

$$= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - k\Omega_s) + \pi\left[\delta\left(\Omega - k\Omega_s - \frac{\pi}{2}\right) + \delta\left(\Omega - k\Omega_s + \frac{\pi}{2}\right)\right]$$

where $\Omega_s = 2\pi/T_s$.

Example 16 (continued)



Spectrum of a discrete-time signal and spectrum of the low-pass filter.

Example 16 (continued)

The only components of the signal spectrum that exit the low-pass filter are:

$$\hat{X}_s(\Omega) = 2\pi\delta(\Omega) + \pi\left[\delta\left(\Omega - \frac{\pi}{2}\right) + \delta\left(\Omega + \frac{\pi}{2}\right)\right]$$

Fourier transform we find that the analog signal produced at the output of the low-pass filter is:

$$\hat{x}_s(t) = 1 + \cos\left(\frac{\pi t}{2}\right)$$