University of the Peloponnese
Electrical and Computer
Engineering Department

Digital Signal Processing

Unit 05: Z-Transform

Dr. Michael Paraskevas
Professor

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Lecture Contents

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Z-Transform
and Region of Convergence

Z- Transform

For a signal $x[n]$**the bilateral, two-sided Z-Transform (ZT)** is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

where $z = re^{j\omega}$complex variable ($z \in C$) called **the complex frequency.**

Valid: $z = |z|\, e^{j\omega}$, where $|z|$ is the **damping** and ω is the **digital frequency.**

The values of z for which the sum converges define in the z-plane the **Region of Convergence (ROC)**, denoted by R_x.

The bilateral Z-Transform Transforms the sequence $x[n]$ into a polynomial $X(z)$ with (possibly) infinite terms of positive and negative powers of z, where each sample $x[n_0]$corresponds to the monomial z^{-n_0}.

The two-sided Z-Transform is not useful in solving difference equations with initial conditions because it cannot include initial conditions in its calculation. For this reason we prefer the **one-sided Transform Z,** which we will study next.

Inverse Z Transform

If $X(z)$ a complex function with region of convergence R_x, is the Z-Transform of a signal $x[n]$, then $x[n]$ can be recovered by the **inverse Z-Transform:**

$$x[n] = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

The term C describes any closed simple curve in $z \; = \; 0$, that encloses the origin of the axes of the complex plane, lies within the region of convergence, encloses all poles, and deletes counterclockwise.

In practice, to calculate the inverse Z-Transform we use techniques that we will study in next slides.

The notations for the Z-Transform are:

- $x[n] \xleftrightarrow{\mathrm{Z}} X(z)$
- $X(z) = Z\{[x[n]\}$ $\qquad\qquad$ $x[n] = Z^{-1}\{X(z)\}$

Area of Convergence of Z-Transform

The values of z for which the sum converges, define at the z- level **the Region of Convergence (**ROC) denoted by R_x:

$$R_x = \left\{ z \in C : \sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| < \infty \right\}$$

or equivalent:

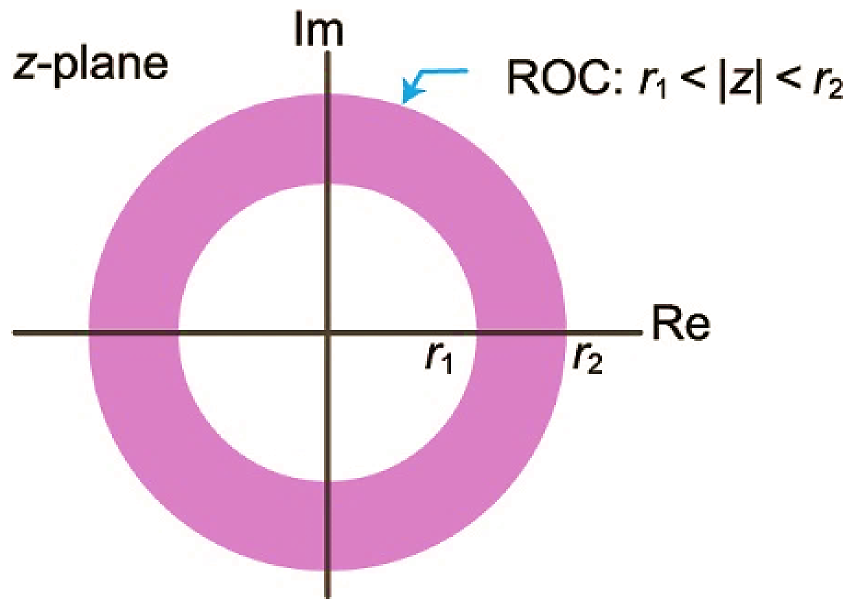
$$R_{x-} < |z| < R_{x+}$$

Writing the function $X(z)$in fractional form, we have:

$$X(z) = \frac{B(z)}{A(z)}$$

- The values of z for which the function $X(z)$goes to infinity are called **poles** $\{p_k\}$ and are always outside the convergence region. Apply $X(p_k) \to \infty$. The poles are calculated from the equation: $A(z) = 0$.
- The values of z for which the function $X(z)$is zero are called **zeros** $\{z_k\}$. Apply $X(z_k) = 0$. Zeros are calculated from: $B(z) = 0$.

Area of Convergence of Z-Transform



General form of Z-Transform region of convergence (R_x or ROC).

Area of Convergence of Z-Transform

- The convergence region is determined solely by the **duration** and **form** of the discrete-time signal:
  - If the duration of the signal is **finite** then the region of convergence is **the entire z-plane,** possibly excluding 0 and ∞, depending on the duration of the signal.
  - If the duration of the signal is **infinite** then the region of convergence depends on the form of the signal, i.e. whether the signal is **right-sided** (causal), **left-sided** (anti-causal) or **double-sided** (non-causal).
- The region of convergence is always defined by a **circle,** because it is defined with respect to $|z|$. It is always a single area and cannot be made up of individual sections.
- The function $|z| = 1$ or $z = e^{j\omega}$ is a circle with unit radius and is called **the unit circle.**
- The region of convergence does not contain poles, because poles are points in the complex frequency plane z, at which the function $X(z)$ goes to infinity.
- It R_{x-} can be zero and/or it R_{x+} can go to infinity.
- If $R_{x+} < R_{x-}$ then the region of convergence is the empty space and the Z-Transform does not exist.

Example 1

Calculate the Z-Transform of the signals:

(a) $x_1[n] = a^n u[n], \quad 0 < |a| < \infty$

(b) $x_2[n] = -b^n u[-n-1], \;\; 0 < |b| < \infty$

(c) $x[n] = x_1[n] + x_2[n]$

Answer: (a) The signal $x_1[n]$ is causal and has values only for positive-time sequence. The bilateral Z-Transform is:

$$X_1(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^n \, z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

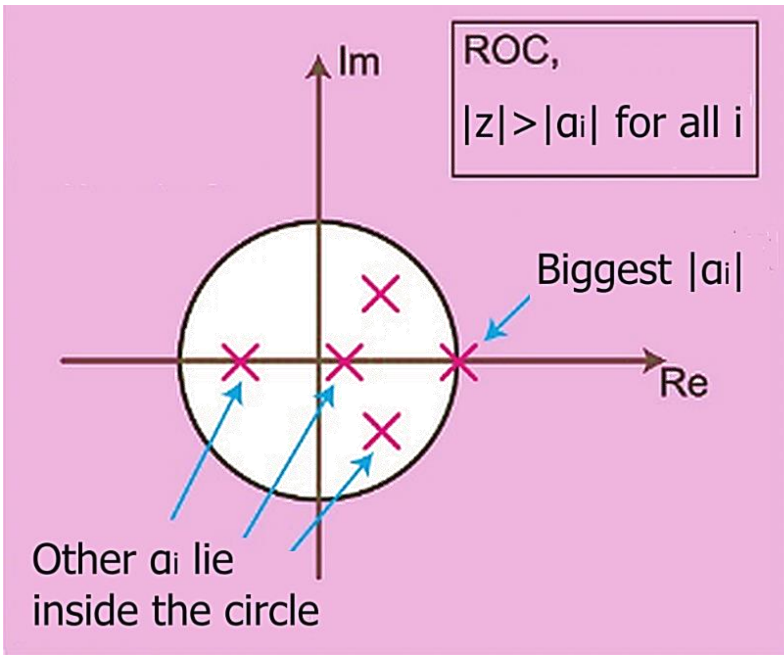
The function $X_1(z)$ converges when $|az^{-1}| < 1 \Rightarrow |z| > |a|$. So the ROC is the **outer surface** of a circle defined by the set of points for which $R_{x1}\text{:} \;\; |z| > |a|$.

That is:

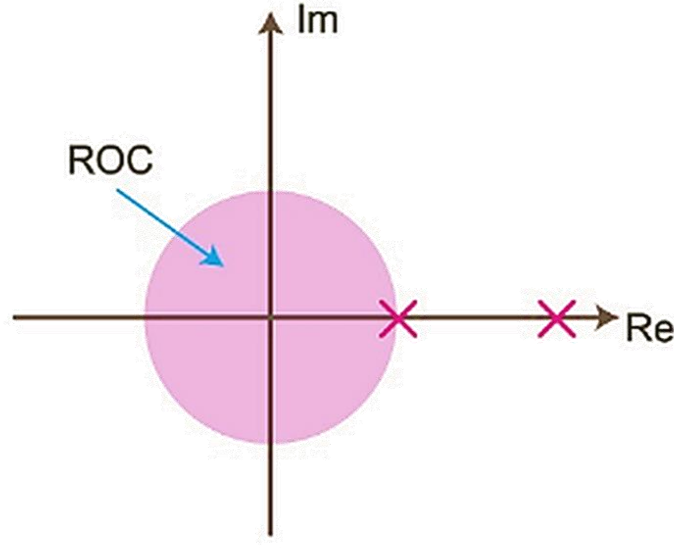
$$R_{x1}\text{:} \quad |\alpha| < |z| < \infty$$

Also, there is a pole for $z = a$ and a zero for $z = 0$ (figure (a)).

Example 1 (continued)



(a)



(b)

Regions of Convergence (ROC) of the sequences:
(a) $x_1[n]$ and (b) $x_2[n]$

Example 1 (continued)

(b) The signal $x_2[n]$ is anti-causal and has values only for negative-time sequence. The bilateral Z-Transform is:

$$X_2(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} -b^n\, u[-n-1]\, z^{-n} = -\sum_{n=-\infty}^{-1} (bz^{-1})^n$$

We set $m = -n$ and have:

$$X_2(z) = -\sum_{m=1}^{\infty} (b^{-1}z)^m = 1 - \sum_{m=1}^{\infty} (b^{-1}z)^m = 1 - \frac{1}{1 - bz^{-1}} = \frac{z}{z-b}$$

The function $X_2(z)$ converges when $|b^{-1}z| < 1 \Rightarrow |z| < |b|$. So the ROC is the **inner surface** of a circle defined by the set of points for which $R_{x2}\colon\ |z| < |b|$.

That is:

$$R_{x2}\colon\ 0 < |z| < |b|$$

Also, there is a pole for $z = b$ and a zero for $z = 0$. (figure (b)).

Example 1 (continued)

- If in the above sequences we put $a = b$, then while the sequences will be different $x_1[n] \neq x_2[n]$, the functions of the Z-Transform will be the same, that is $X_1(z) = X_2(z)$, but with different areas of convergence ($R_{x1} \neq R_{x2}$).
- So, the calculation of the Z-Transform requires, not only the calculation of $X(z)$, but also the determination of the region of convergence.

Example 1 (continued)

(c) $x[n]$ is the sum $x_1[n] + x_2[n] = a^n u[n] - b^n u[-n-1]$and is called a two-side sequence. The Z-Transform is:

$$X(z) = \sum_{n=0}^{\infty} a^n\, z^{-n} - \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= \left\{\frac{z}{z-a}, R_{x1}\!:\ |z| > |\alpha|\right\} + \left\{\frac{z}{z-b}, R_{x2}\!:\ |z| < |b|\right\} = \frac{z}{z-a} + \frac{z}{z-b}$$

$$R_x = R_{x1} \cap R_{x2}$$

- If $|b| < |a|$, then the region of convergence R_x <u>does not exist,</u> because the intersection of the regions of convergence R_{x1} and R_{x2} is the empty set.

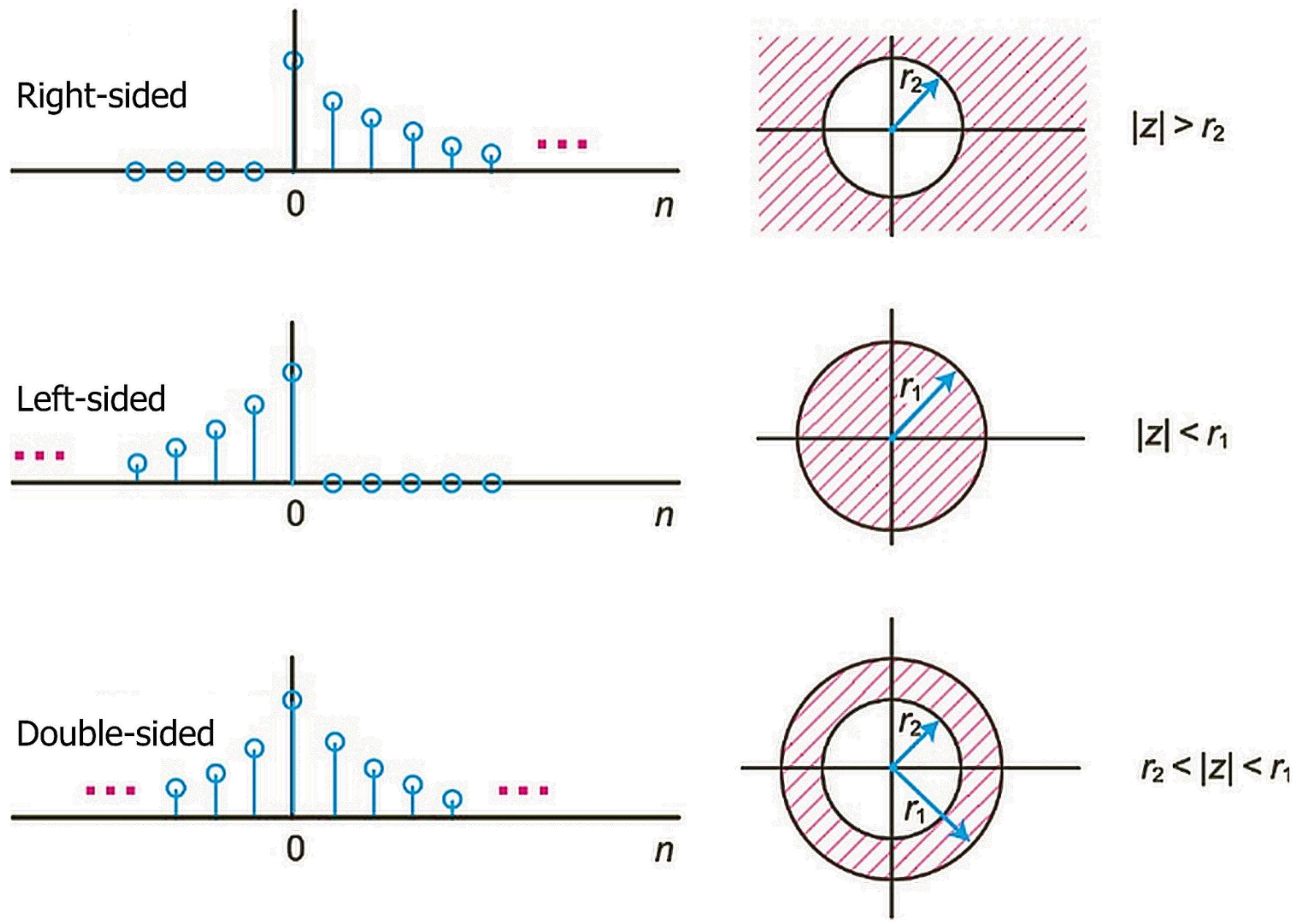
- If $|a| < |b|$, then the region of convergence is R_x: $|\alpha| < |z| < |b|$

Conclusions on the Area of Convergence

From the solution of Example 1 it follows that for signals **of infinite duration** the convergence zone is distinguished in the following cases:

- **Right-sided signals (causal):** the area of convergence is the **exterior** of a circle with radius R_{x-} the maximum radius of its poles $X(z)$ or $|z| > R_{x-}$
- **Left-sided signals (anti-causal):** The region of convergence is the **inner** circle with radius R_{x+} the minimum radius of the poles of $X(z)$ or $|z| < R_{x+}$
- **Double-sided signals (non-causal):** The area of convergence is the **interior** of a ring with inner radius R_{x-} and outer radius R_{x+}, which correspond to the maximum and minimum radius of its poles $X(z)$, i.e. holds $R_{x-} < |z| < R_{x+}$

Region of convergence of sequences <u>of infinite duration</u>



Example 2

Find the Z-Transform of the signal $x[n] = 3\delta[n] + \delta[n-2] + \delta[n+2]$.

Answer: Z-Transform of the finite duration signal $x[n]$ is:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} \{3\delta[n] + \delta[n-2] + \delta[+2]\}\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 3\delta[n]\, z^{-n} + \sum_{n=-\infty}^{\infty} \delta[n-2]\, z^{-n} + \sum_{n=-\infty}^{\infty} \delta[n+2]\, z^{-n}$$

Therefore:

$$X(z) = 3 + z^{-2} + z^{2}$$

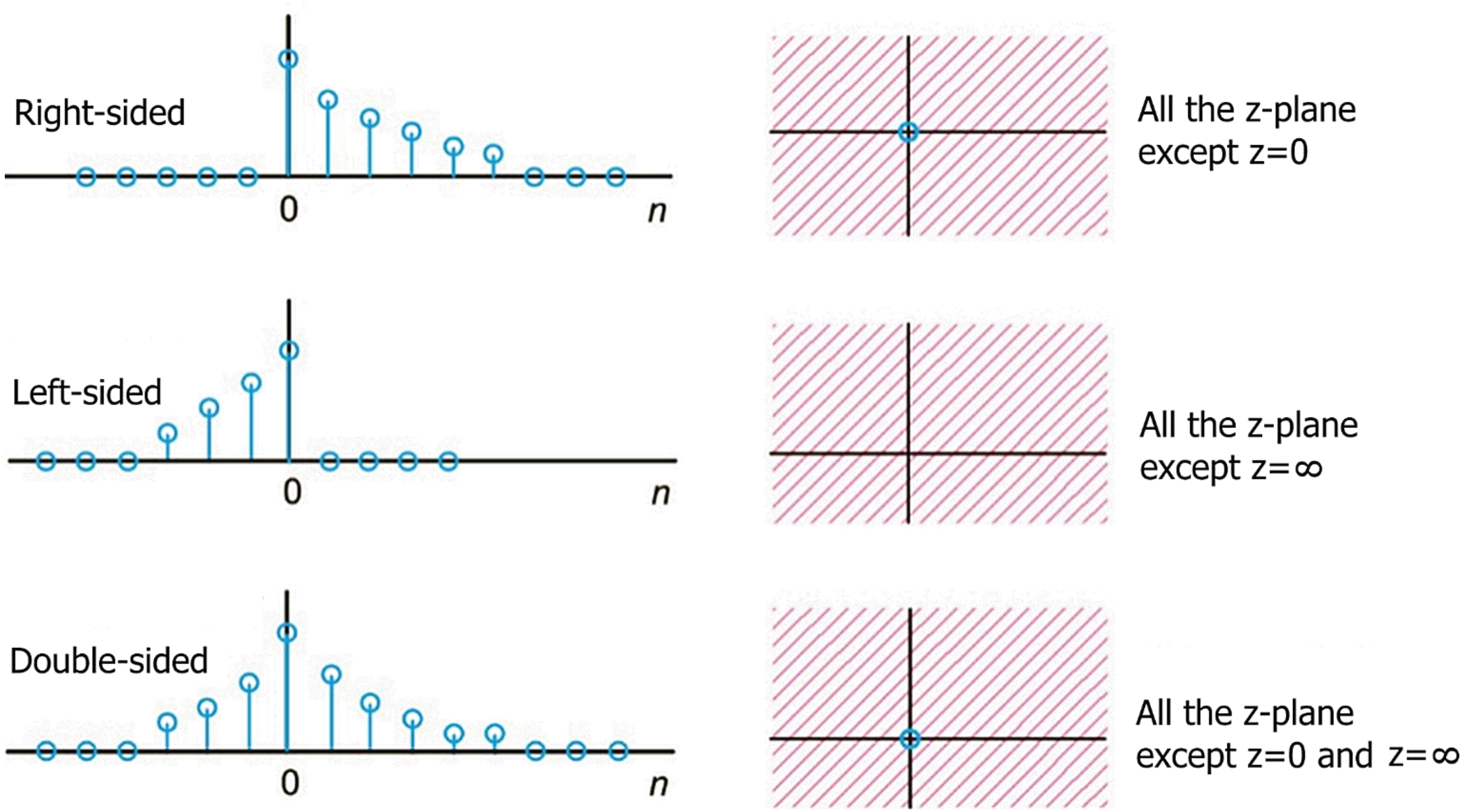
The area of convergence is:

$$R_x\colon\ 0 < |z| < \infty$$

- Because $x[n] \neq 0$for $n < 0$, the ROC does not include the points with $|z| = \infty$.
- Because $x[n] \neq 0$for $n > 0$, the ROC does not include the point $z = 0$.

The region of convergence of finite duration sequences is illustrated in the next figure.

Region of convergence of finite-duration sequences



Comment on ROC of finite-duration signals

- For signals **of finite duration** $[N_1,\, N_2]$ where $(-\infty < N_1 \leq n \leq N_2 < \infty)$, the region of convergence is the entire complex field z, possibly excluding 0 and/or $\pm\infty$*.* These points are excluded because it z^n $(n > 0)$ tends to infinity for $z = \infty$ and z^{-n} tends to infinity for $z = 0$. Specifically:
  - If $N_1 < 0$, then is $z = \pm\infty$ not included in the region of convergence.
  - If $N_2 > 0$, then is $z = 0$ not included in the region of convergence.
- From a mathematical point of view, the Z-Transform is an alternative way of representing a signal, since its coefficient z^{-n} is the value of the signal at time n.
- Its exponent z provides that information about time, which is necessary to determine the samples of a signal.

Example 3

Find the Z-Transform of the signal $x[n] = u[n] - u[n-5]$.

Answer: The badge is of finite duration. The Z-Transform is:

$$X(z) = \sum_{n=0}^{5} z^{-n} = \frac{1 - z^{-5}}{1 - z^{-1}} = \frac{z^5 - 1}{z^4(z-1)}$$

and the ROC is: R_x: $|z| > 0$. The zeros are the solutions of $z^5 - 1 = 0$. These roots are$z = e^{j2\pi k/5}, k = 0,1, \dots, 4$ and are placed at 5 equidistant points on the unit circle.

From its form $X(z)$ it appears that there is a pole at $z = 1$. But this pole is neutralized by zero at $z = 1$and specifically for $k = 0$. Indeed, factoring the numerator, we have:

$$X(z) = \frac{(z-1)(z^4 + z^3 + z^2 + z + 1)}{z^4(z-1)} = \frac{z^4 + z^3 + z^2 + z + 1}{z^4}$$

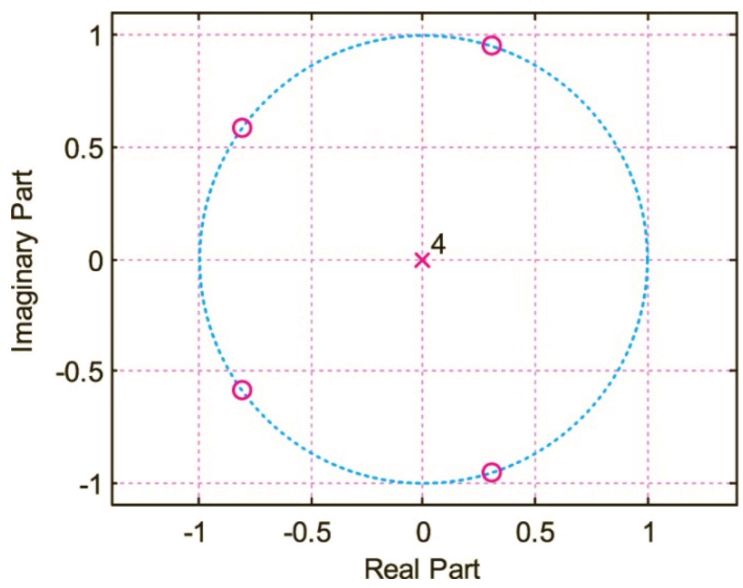
Thus, there is no longer any need to exclude from the convergence region $z = 1$.
This is confirmed by computing $X(1)$, which is $X(1) = 5$.

Example 3 (continued)

Finally, the $Z-$Transform can be expressed in the form:

$$X(z) = \frac{\prod_{k=1}^{4}\left(z - e^{jk\pi/5}\right)}{z^4}$$

and has four zeros placed on the unit circle in addition to the position $z = 1$, as well as four poles placed at the point $z = 0$.



Pole - zero diagram

Example 4

Without directly calculating of $X(z)$, find the region of convergence of the Z-Transform of each of the following signals:

(a) $x[n] = \left[\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right] u[n-10]$

(b) $x[n] = 2^n u[-n]$

Answer: (a) Because $x[n]$ is a right-hand sequence, the region of convergence covers the <u>outer </u>surface of a circle. With a pole at $z = 1/2$ derived from the term $(1/2)^n$ and a pole at$z = 3/4$ derived from the term $(3/4)^n$, it follows that the region of convergence includes the points for which is $|z| > 3/4$.

(b) Because $x[n]$ is a left-sided sequence, the region of convergence covers the <u>inner</u> surface of a circle. With a pole at $z = 2$, it follows that the region of convergence covers the points $|z| < 2$.

Relationship of Z-Transform to other Transforms

- With the Fourier Transform
- With the Laplace Transform

Relationship between Z and Fourier Transform

- The Z-Transform is more general than the **Discrete-Time Fourier Transform** (DTFT), which we will study in the next lecture. If we set $z = re^{j\omega}$ the Z-Transform is:

$$X(z)\Big|_{z=re^{j\omega}} = \sum_{n=-\infty}^{+\infty} x[n]\, r^{-n}\, e^{-j\omega n} = DTFT\{x[n]\, r^{-n}\}$$

- When DTFT cannot be calculated, the calculation of the Z-Transform may be possible.

- If the region of convergence of the Z-Transform includes the unit circle ($r = |z| = 1$), then we can obtain the DTFT by calculating the Z-Transform on the unit circle:

$$X(z)\Big|_{z=e^{j\omega}} = X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = DTFT\{x[n]\}$$

  i.e. for $r = 1$ we end up to Discrete-Time Fourier Transform.

- Therefore, the DTFT can be considered as a <u>subcase</u> of Z-Transform to calculate on the unit circle ($|z| = 1$), as long as the unit circle <u>is included</u> in the convergence region.

Example 5

The Z-Transform of a sequence $x[n]$ is:

$$X(z) = \frac{z + 2z^{-2} + z^{-3}}{1 - 3z^{-4} + z^{-5}}$$

If the region of convergence includes the unit circle, find the DTFT of $x[n]$ for $\omega = \pi$.

Answer: If $X(z)$ is the Z -Transform of $x[n]$ and the unit circle lies within the region of convergence, the DTFT of $x[n]$ can be found by calculating $X(z)$ on the unit circle, i.e.:

$$X(e^{j\omega}) = X(z)\Big|_{z=-1}$$

Therefore, the DTFT at the point $\omega = \pi$, is:

$$X(e^{j\omega})\Big|_{\omega=\pi} = X(z)\Big|_{z=e^{j\pi}} = X(z)\Big|_{z=-1}$$

and we have:

$$X(e^{j\omega})\Big|_{\omega=\pi} = \frac{z + 2z^{-2} + z^{-3}}{1 - 3z^{-4} + z^{-5}}\Bigg|_{z=e^{j\pi}} = \frac{-1 + 2 - 1}{1 - 3 - 1} = 0$$

Relationship between Z and Laplace Transform

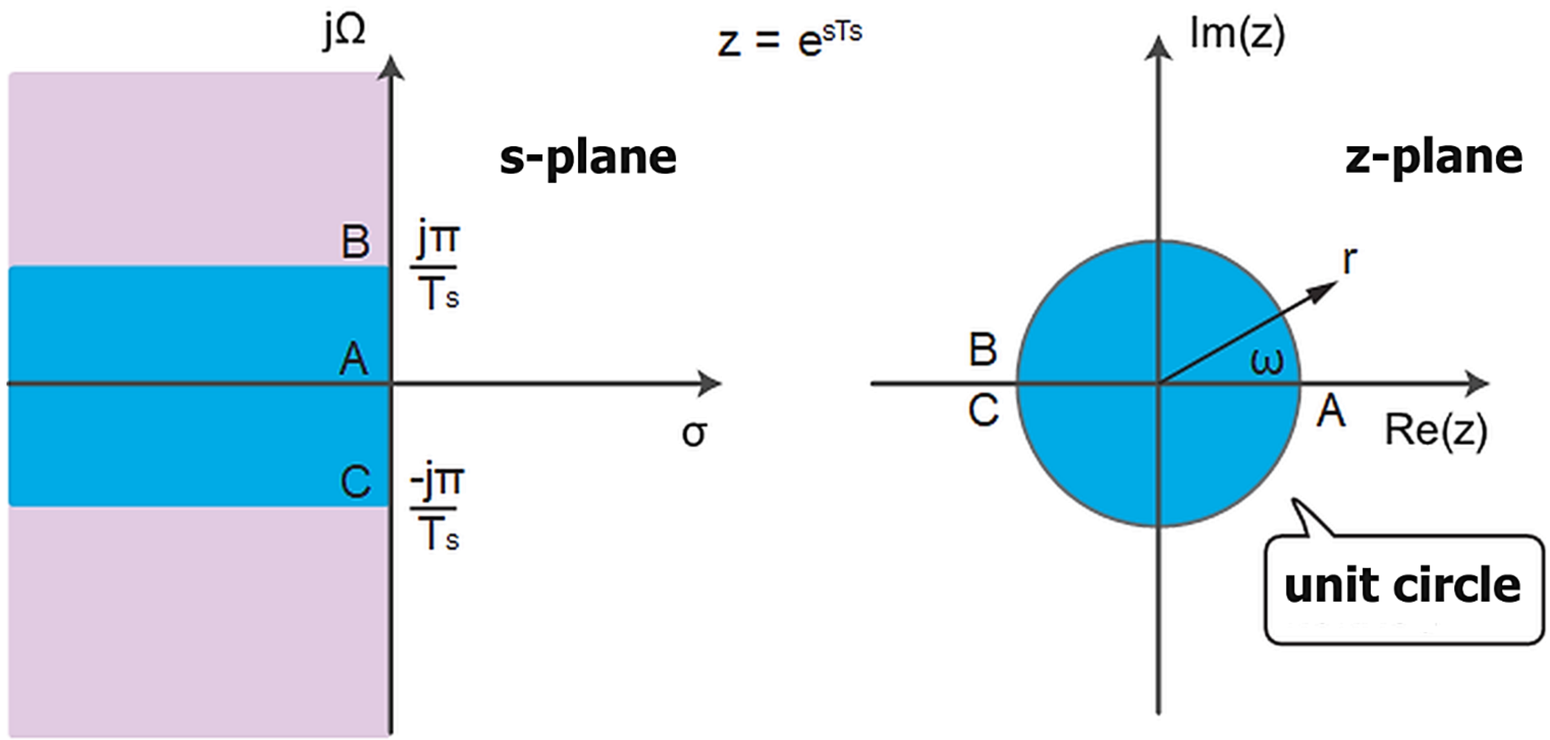
- **Z-Transform:** produces a description of a DX signal in the **digital complex frequency domain** z.
- **Laplace Transform:** describes continuous time signals in the **analog complex frequency domain** s.
- Analog complex frequency: $s = \sigma + j\Omega$, where σ is the damping factor and Ω is the analog real frequency.
- The relation between the complex variables frequency z and s is:

$$z = e^{s\,T_s} = e^{(\sigma + j\Omega)\,T_s} = e^{\sigma\,T_s}\;e^{j\Omega T_s}$$

  where T_s is the sampling period.

- Therefore, the digital complex frequency z is obtained by **sampling** the analog complex frequency s.
- Setting $r = e^{\sigma\,T_s}$ and $\omega = \Omega T_s$ yields $\boldsymbol{z = r\;e^{j\omega}}$, where r is the damping factor and ω is the digital (real) frequency.
- The complex level z corresponds to circles of radius r and angle $-\pi \leq \omega < \pi$.

Relationship between Z and Laplace Transform



Relationship between analog complex frequency fields s
and digital complex frequency z

Relationship between Z and Laplace Transform

The relationship $z = e^{s\,T_s}$ is fundamental to converting analog signals to digital. Based on this and the figure, the following observations emerge:

- The equation $z = e^{s\,T_s}$ depicts the real part of $s = \sigma + j\Omega$,

i.e. the $Re(s) = \sigma$ circle of radius $r = e^{\sigma\,T_s} \geq 0$.

- Analog frequencies $-\pi/T_s \leq \Omega < \pi/T_s$ are mapped to digital frequencies $-\pi \leq \omega < \pi$ according to the relationship $\omega = \Omega T_s$.

- If $\sigma = 0$ then $s = j\Omega$, therefore $|z| = 1$ and $\measuredangle z = \omega = \Omega T_s$. That is, the axis $j\Omega$ of the analog complex frequency plane $s$is mapped onto the circumference of the unit circle of the digital complex frequency plane z.

- For $\sigma < 0$ the left half-plane of the plane s is depicted inside the unit circle of the plane z, with radius $r = e^{0\,T_s} = 1$. Conversely, for $\sigma > 0$ the right half-plane of the plane s it is depicted on the outside of the unit circle of the plane z.

Relationship between Z and Laplace Transform

- The frequency band width $2\pi/T_s$ in the plane $s$corresponds to the Nyquist criterion, which determines the maximum frequency $\Omega_{max} = \Omega_s/2$ of the analog signal that can be sampled with a sampling frequency Ω_s. As decreases T_s the frequency band width increases.
- If we have an analog signal $x(t)$ with a finite frequency range [- $\pi/T_s, \pi/T_s$] and a maximum frequency π/T_s and we sample it with a sampling period T_s, then according to the equation $z = e^{s\,T_s}$ the spectrum of the sampled signal ranges at a digital frequency [- π, π].
- Moving iteratively over the unit circle, it follows that the spectrum of the sampled signal is periodic.
- Points A, B, C in the frequency band of the analog complex frequency domain correspond to points A, B, C in the complex frequency domain.

Relationship between Z and Laplace Transform

The Laplace Transform of a sampled signal $x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$ is:

$$X(s) = \sum_{n=-\infty}^{\infty} x(nT_s) L\{\delta(t - nT_s)\} = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-nsT_s}$$

Putting $z = e^{s\,T_s}$ the above equation is written:

$$X(s) = \sum_{n=-\infty}^{\infty} x(nT_s)\, z^n$$

equivalents:

$$Z\{x(nT_s)\} = L\{x_s(t)\}\Big|_{z=e^{s\,T_s}}$$

This equation connects the two Transforms and shows that the Z-Transform is equivalent to the Laplace, for discrete-time signals.

Relationship between Z and Laplace Transform

- Z and Laplace Transforms have common properties. Just as the Laplace Transform Transforms a convolutional integral into a product, so the Z-Transform Transforms a convolutional sum into a product.
- The one-sided Z-Transform is used to solve difference equations resulting from the discretization of differential equations.
- The method of partial fractions for computing the inverse Laplace is also used to compute the inverse Z-Transform.
- Finally, the use of the Z-Transform to design digital filters, analogous to the use of the Laplace Transform to design analog filters, is widespread, and there are many effective methods of designing FIR and IIR digital filters.

Unilateral Z-Transform

In practical cases of digital systems the system is **causal,** i.e. $h[n] = 0$ για $n < 0$ and the input signal is also causal, i.e. $x[n] = 0$ για $n < 0$. For the causal signal case the definition of the Z-Transform is written:

$$X^+(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

- with a convergence area R_x and defines the **unilateral** (unilateral, one sided Z^+ Transform) Transformation Z^+.
- We can define the one-sided Transform for any signal $x[n]$, which we transform into a causal one by multiplying it by the unit step $u[n]$:

$$X^+(z) = Z\{x[n]\, u[n]\} = \sum_{n=0}^{\infty} x[n]\, u[n]\, z^{-n}$$

with area of convergence R_{x^+}

Relationship between Unilateral Z^+ and Bilateral Z −Transform

The two-sided Z-Transform can be expressed in terms of a one-sided Transform Z^+ by the equation:

$$X(z) = Z^+\{x[n]\, u[n]\} + Z^+\{x[-n]\, u[n]\} - x[0]$$

with region of convergence $R_x = R_{x1} \cap R_{x2}$, where R_{x1} the region of convergence of $Z^+\{x[n]\, u[n]\}$ and R_{x2} the region of convergence of $Z^+\{x[-n]\, u[n]\}$.

- The Z^+**coincides** with Z when the signal $x[n]$ is **causal.**

- The Z^+ is very useful in studying LSI responses of systems to causal input signals, as well as in solving LDECC because it can **include the initial conditions of** $y[-1], y[-2], \ldots, y[N-1]$ the system output.

- Because of its utility, reference to a Z-Transform is usually equated with Z^+, unless the bisector is explicitly mentioned.

Example 6

Find the one-sided (unilateral) Z-Transform of the signal:

$$x[n] = \delta[n-5] + \delta[n] + 2^{n-1}u[-n]$$

Answer: We consider the notation $x_+[n]$ for the sequence formed by $x[n]$, setting $x[n] = 0$ for $n < 0$, namely:

$$x_+[n] = \begin{cases} x[n] & n \geq 0 \\ 0 & n < 0 \end{cases}$$

For this sequence, because it is:

$$x_+[n] = \delta[n-5] + \delta[n] + 2^{-1}\delta[n]$$

The one-sided (unilateral) Z-Transform is:

$$X_1(z) = z^{-5} + 1 + 0.5 = 1.5 + z^{-5}$$

Useful Pairs of Z-Transforms and Regions of Convergence

Signal $x[n]$	Z-Transform $X(z)$	Region of Convergence (ROC)
$\delta[n]$	1	The entire z field
$\delta[n-n_0]$	z^{-n_0}	The entire field z, except 0 if $n_0 > 0$ and ∞ if $n_0 < 0$
$u[n]$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert < 1$
$a^n\, u[n]$	$\frac{1}{1-az^{-1}} = \frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n\, u[-n-1]$	$\frac{1}{1-az^{-1}} = \frac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
$n\, a^n\, u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2} = \frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$

Useful Pairs of Z-Transforms and Regions of Convergence

Signal $x[n]$	Z-Transform $X(z)$	Region of Convergence (ROC)
$\cos(\omega_0 n)\, u[n]$	$\dfrac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)\, u[n]$	$\dfrac{z^{-1}\sin\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}$	$\lvert z\rvert > 1$
$a^n \cos(\omega_0 n)\, u[n], \quad \lvert\alpha\rvert < 1$	$\dfrac{1 - az^{-1}\cos\omega_0}{1 - 2az^{-1}\cos\omega_0 + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^n \sin(\omega_0 n)\, u[n], \quad \lvert\alpha\rvert < 1$	$\dfrac{az^{-1}\sin\omega_0}{1 - 2az^{-1}\cos\omega_0 + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Properties of Z-Transform

- Linearity
- Shift in Time
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- Initial-Value Theorem
- Final-Value Theorem

Linearity

If for $x_2[n]$ to the signs $x_1[n]$ apply:

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \qquad R_{x_1}$$

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \qquad R_{x_2}$$

then the Z-Transform of the linear combination $a_1x_1[n] + \alpha_2x_2[n]$ will be:

$$a_1x_1[n] + \alpha_2x_2[n] \overset{z}{\longleftrightarrow} a_1X_1(z) + \alpha_2X_2(z)$$

$$R_x \ = R_{x_1} \cap R_{x_2}$$

Time Shift (Sample Shift)

If the Z-Transform of the signal $x[n]$ is:

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad R_x$$

then the Z-Transform of the time-shifted signal $x[n-n_0]$ is:

$$x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0}X(z)$$

with the region of convergence the same, except for the point $z = 0$ if $n_0 > 0$ and the point $z = \infty$ if $n_0 < 0$.

If the signal $x[n]$ is causal, for the one-sided Z^+- Transform:

$$x[n-n_0]\,u[n] \overset{Z^+}{\longleftrightarrow} z^{-n_0}X^+(z) + \sum_{n=1}^{n_0} x[-n]\,z^{n-n_0}$$

Example 7

Find the Z-Transform of the signal $y[n] = \sum_{k=-\infty}^{n} x[k]$ as a function of Z-Transform $x[n]$.

Answer: The relationship $y[n] = \sum_{k=-\infty}^{n} x[k]$ can be written $y[n] = y[n-1] + x[n]$. Therefore:

$$x[n] = y[n] - y[n-1]$$

If we transform both members of the equation and use the time-shift property of the Z-Transform, we find:

$$X(z) = Y(z) - z^{-1}Y(z) \Rightarrow X(z) = Y(z)[1 - z^{-1}]$$

We solve in terms of $Y(z)$:

$$Y(z) = \frac{1}{1 - z^{-1}} X(z)$$

Therefore:

$$y[n] = \sum_{k=-\infty}^{n} x[k] \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}} X(z)$$

Time Reversal and Scaling

Time Reversal (convolution): If the Z-Transform shift of a $x[n]$signal is:

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad R_x : r_1 < |z| < r_2$$

then the Z-Transform of the reflection $x[-n]$ is:

$$y[n] = x[-n] \overset{Z}{\longleftrightarrow} X(z^{-1}), \qquad R_y: \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

Time Scaling: If the Z-Transform of a signal $x[n]$ is:

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad R_x: r_1 < |z| < r_2$$

then the Z-Transform of the oversampled signal $x[n/N]$, $(N > 1)$ is:

$$y[n] = x[n/N] \overset{Z}{\longleftrightarrow} X(z^N), \qquad R_y : R = R_x^{1/k}$$

Complex Frequency Scaling

If the Z-Transform of a signal $x[n]$ is:

$$x[n] \xleftrightarrow{Z} X(z), \qquad R_x\colon\ r_1 < |z| < r_2$$

then the Z-Transform of the product $a^n x[n]$, $a \in C$, is:

$$y[n] = a^n x[n] \xleftrightarrow{Z} X\left(\frac{z}{\alpha}\right), \qquad R_y\colon |a| r_1 < |z| < |a| r_2$$

Convolution Theorem

If the Z-Transforms of two signals $x[n]$ and $h[n]$ are:

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad R_x$$

$$h[n] \overset{Z}{\longleftrightarrow} H(z), \qquad R_h$$

then the Z-Transform of the convolution $y[n] = h[n] * x[n]$ is the **product** of the individual Transforms:

$$y[n] = h[n] * x[n] \overset{Z}{\longleftrightarrow} Y(z) = H(z)\, X(z)$$

The area of convergence is the **intersection** of the individual areas of convergence, namely:

$$R_y = R_h \cap R_x$$

The convolution property is extremely useful for studying LSI systems because it gives an alternative and simpler calculation of convolution compared to time-domain calculations.

Example 8

To calculate the convolution between of sequences $x[n] = \{\hat{1}, -2, 0, 3, -1\}$ and $h[n] = \{2, \hat{3}, 0, 1\}$.

Answer: We calculate the Z-Transform of each sequence using the time shift property and we have:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n} = \sum_{n=0}^{4} x[n]\, z^{-n} = 1 - 2^{-1} + 3z^{-3} - z^{-4}$$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n} = \sum_{n=-1}^{2} h[n]\, z^{-n} = 2z + 3 + z^{-2}$$

Based on the convolution property we have:

$$\begin{aligned} Y(z) &= X(z)\, H(z) = (1 - 2z^{-1} + 3z^{-3} - z^{-4})(2z + 3 + z^{-2}) \\ &= 2z + 3 + z^{-2} - 4 - 6z^{-1} - 2z^{-3} + 6z^{-2} + 9z^{-3} + 3z^{-5} - 2z^{-3} - 3z^{-4} - z^{-6} \\ &= 2z - 1 + 6z^{-1} + 7z^{-2} + 5z^{-3} - 3z^{-4} + 3z^{-5} - z^{-6} = \sum_{n=-1}^{6} y[n]\, z^{-n} \end{aligned}$$

From the time shift property, we get the result:

$$y[n] = \{2, -\hat{1}, -6, 7, 5, -3, 3, -1\}$$

Derivation and Complex Conjugation

Derivation in the z -Field: If the Z-Transform of a signal $x[n]$ is:

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \qquad R_x\colon\ r_1 < |z| < r_2$$

then the following applies:

$$y[n] = n\, x[n] \stackrel{z}{\longleftrightarrow} \frac{dX(z)}{dz}, \qquad R_y = R_x$$

Complex Conjugate: If the Z-Transform of a complex signal $x[n]$ is:

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \qquad R_x$$

then the following applies:

$$y[n] = x^*[n] \stackrel{z}{\longleftrightarrow} X^*(z^*), \qquad R_y = R_x$$

Multiplication of Signals

If the Z-Transforms of two signals $x_1[n]$ and $x_2[n]$ are:

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \qquad R_{x_1}$$

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \qquad R_{x_2}$$

then for the Z-Transform of the product of the signals:

$$y[n] = x_1[n]\, x_2[n] = \frac{1}{2\pi j} \oint_c X_1(v) X_2(z/v)\, v^{-1}\, dv$$

where C is a closed curve, located within the region of convergence, and sweeps clockwise.

The area of convergence is:

$$R_y\!: \; R_{x_1} \cap \bar{R}_{x_2}$$

where $\bar{R}_{x_2}$is its complementary region R_{x_2}.

Initial and Final Value Theorems

Initial-Value Theorem: If $X(z)$ is the Z-Transform of a causal signal $x[n](x[n] = 0$ for $n < 0))$, then the limit of the function $X(z)$when it $z$tends to infinity is equal to the value of the signal at $n = 0$:

$$x[0] = \lim_{z \to \infty} X(z)$$

Final-Value Theorem: If $X(z)$ is the Z-Transform of a signal $x[n]$, then the limit of the sequence $x[n]$, when it $n$tends to infinity, is given by the equation:

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z - 1) X(z)$$

Example 9

A discrete-time causal signal has a Z-Transform given by the equation:

$$X(z) = \frac{1}{1 - az^{-1}}$$

Calculate the value of the signal $x[n]$ at position $n = 0$.

Answer: For $x[0]$ from the initial value theorem it follows:

$$x[0] = \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \frac{1}{1 - az^{-1}} = 1$$

Example 10

A discrete-time causal signal has a Z-Transform given by the equation:

$$X(z) = \frac{4z^2 + 3z + 1}{(z-1)(z+2)^2}$$

Calculate the signal value $x[n]$ for $n \to \infty$.

<u>Answer:</u> For $X[n], n \to \infty$ from the final value theorem it follows:

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1}(z-1)\frac{4z^2 + 3z + 1}{(z-1)(z+2)^2}$$

$$= \lim_{z\to 1}\frac{4z^2 + 3z + 2}{(z+2)^2} = \frac{4+3+1}{3^2} = \frac{8}{9}$$

Properties of Z-Transform

Property	Discrete time signal	Z-Transform	Area of Convergence
Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(s) + a_2X_2(s)$	$R_{x_1} \cap R_{x_2}$
Time shift	$x[n-k]$	$z^{-k}X(s)$	R_x
Frequency shift	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
Convolution	$x_1[n] * x_2[n]$	$X_1(z)\, X_2(z)$	$R_{x_1} \cap R_{x_2}$
Conjugate in time	$x^*[n]$	$X^*(z^*)$	R_x
Time scaling	$x[n/k]$	$X\left(z^k\right)$	$R_x^{1/k}$
Reflection	$x[-n]$	$X(1/z)$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Sum in time	$\sum_{k=-\infty}^{+\infty} x[k]$	$\frac{1}{1-z^{-1}} X(z)$	$R_x \cap \{\|z\| > 1\}$
Difference in time	$x[n] - x[n-1]$	$(1-z^{-1})X(z)$	$R_x \cap \{\|z\| > 0\}$
Derivation in frequency	$n\, x[n]$	$-\frac{dX(z)}{dz}$	R_x
Initial-value theorem	$x[n]$ causal	$x[0] = \lim_{z\to\infty} X(z)$	
Final-value theorem	$x[n]$causal	$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$	

Useful pairs of Z-Transform

Discrete time signal	Z-Transform	Area of Convergence
$\delta[n]$	1	z
$\delta[n-n_0]$	z^{-n_0}	All z-plane
$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 0$
$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$\cos(\omega_0 n)\, u[n]$	$\dfrac{1-z^{-1}\cos(\omega_0 n)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)\, u[n]$	$\dfrac{z^{-1}\sin(\omega_0 n)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
$a^n\cos(\omega_0 n)u[n]$	$\dfrac{1-az^{-1}\cos(\omega_0 n)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^n sin(\omega_0 n)u[n]$	$\dfrac{az^{-1}\sin(\omega_0 n)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Poles and Zeros of Z-Transform

Poles and Zeros of Z-Transform

The function $X(z)$ can be expressed in fractional form as a quotient of polynomial terms of z^{-1} or z:

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{m=0}^{M} b_m \, z^{-m}}{1 + \sum_{k=1}^{N} a_k \, z^{-k}}$$

After factoring numerator and denominator, the function $X(z)$ is written:

$$H(z) = b_0 \, z^{N-M} \frac{\prod_{m=1}^{M}(z - z_m)}{\prod_{k=1}^{N}(z - p_k)}$$

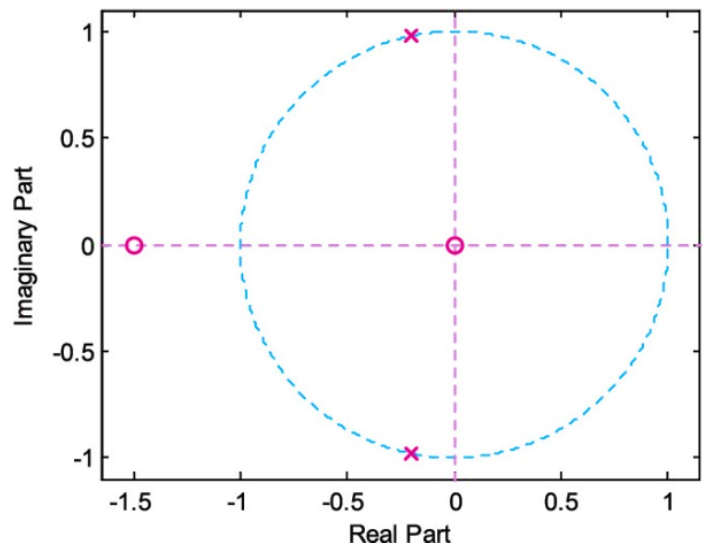
- The roots $\boldsymbol{z_k}$ of the numerator are called zeros **and** the roots $\boldsymbol{p_k}$ of the denominator are called **poles.**
- The poles and zeros provide a graphical representation of $X(z)$, which is called **a pole–zero diagram.**
- Pole-zero diagrams are a very useful tool for the study of discrete-time LSI systems, as they are easily calculated and provide qualitative information about the behavior of the system, such as causality and stability.

Example 11

Draw the pole-zero diagram of the function:

$$X(z) = \frac{2z^2 + 3z}{z^2 + 0.4z + 1}$$

Answer: By factoring the numerator and denominator we find the zeros and poles respectively. Is:$p_{1,2} = -0.20 \pm j0.9798$ and $z_1 = 0, \quad z_2 = -1.5$



Pole - zero diagram

Methods for Computing the Inverse Z-Transform

- Using Residual Theorems
- Expanding in Power Trains
- Expanding in Partial Sums

Development in Power-Train

When the function $X(.)$ is given in fractional form $X(z) = B(z)/A(z)$ and has a region of convergence outside the circle of radius R, (i.e. it $x[n]$ is causal), then it can be expressed in polynomial form, dividing $B(z)$ by $A(z)$:

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

In this case the sequence $x[n]$can be produced by the equation:

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

The method is also called the "long - division method".

To calculate the inverse Z-Transform it is enough to perform the division $B(z)/A(z)$, which gives us a (possibly infinite order) polynomial, whose coefficients arranged in descending order of the powers of z^{-1} or z are the values of the sequence $x[n]$.

The disadvantage of the method is that it does not lead to a closed mathematical form of the sequence $x[n]$.

Example 12

Find the inverse Z- Transform of the function:

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad R_x: |z| > \frac{1}{2}$$

Answer: The shape of the region of convergence suggests that the signal $x[n]$ is right-sided (causal). We perform the division $B(z)/A(z)$ in order to render the function as $X(z)$a power series with respect to $z^{-1}$and find:

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} = 1 + \frac{1}{2}\ z^{-1} + \frac{1}{4}\ z^{-2} + \frac{1}{8}\ z^{-3} + \frac{1}{16}\ z^{-4} + \cdots = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n}$$

Therefore, the discrete-time signal sequence is:

$$x[n] = \left(\frac{1}{2}\right)^{n} u[n]$$

Example 12 (continued)

The division $B(z)/A(z)$ we performed is shown below:

$$
\begin{array}{l|l}
1 & 1-\frac{1}{2}z^{-1} \\
\hline
-1+\frac{1}{2}z^{-1} & 1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}+\frac{1}{8}z^{-3}+\frac{1}{16}z^{-4}+\cdots \\
\qquad -\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2} & \\
\qquad\qquad -\frac{1}{4}z^{-2}+\frac{1}{8}z^{-3} & \\
\qquad\qquad\qquad -\frac{1}{8}z^{-3} & \\
\qquad\qquad\qquad +\frac{1}{16}z^{-4} & \\
\qquad\qquad\qquad\quad \ldots &
\end{array}
$$

Expansion into the Sum of Partial Fractions

When the function $X(z)$ is expressed in the fractional form of terms z^{-1} or z, it can be expressed as a sum of simple (first-order) fractions, where each fraction has a known Z-Transform (is the most common).

We consider the function $X(z)$ written in fractional form as:

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{B(z)}{A(z)}$$

and has a region of convergence $R_{x-} < |z| < R_{x+}$.

The method is applied if the degree of the polynomial of the numerator is **lower** than the degree of the polynomial of the denominator ($M < N$).

Expansion into the Sum of Partial Fractions

If this is not the case, that is, if $M \geq N$,then we first perform polynomial division and convert the function $X(z)$to the form:

$$X(z) = \frac{\tilde{b}_0 + \tilde{b}_1 z^{-1} + \cdots + \tilde{b}_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} + \sum_{k=0}^{M-N} C_k z^{-k}$$

In the event that the degree of the numerator is equal to or greater than the degree of the denominator ($M \geq N$), then terms will appear in the result $\delta[n]$ and their derivatives.

However, this does not pose a problem, as the appearance of functions $\delta(t)$ on continuous-time signals did, because the $\delta[n]$ discrete-time sequence is well defined.

Expansion into the Sum of Partial Fractions

The sum (straight form) exists only if $M \geq N$, otherwise it is zero.

Factoring $A(z)$, the fractional part of the sum is written:

$$X(z) = \sum_{k=1}^{N} \frac{R_k}{1 - p_k z^{-1}}$$

where p_k is the k pole of $X(z)$ and R_k is the **remainder** of dividing the fractional part by the pole of p_k.

If the poles are **simple and distinct,** the remainder R_k is given by the equation:

$$R_k = \left. \frac{\tilde{b}_0 + \tilde{b}_1 z^{-1} + \cdots + \tilde{b}_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} (1 - p_k z^{-1}) \right|_{z=p_k}$$

If there is **a multiple pole,** even the one p_k with multiplicity r, then it is analyzed:

$$\sum_{l=1}^{r} \frac{R_{k,l\, z^{-(l-1)}}}{(1 - p_k z^{-1})^l} = \frac{R_{k,1}}{1 - p_k z^{-1}} + \frac{R_{k,2} z^{-1}}{(1 - p_k z^{-1})^2} + \cdots + \frac{R_{k,r} z^{-(r-1)}}{(1 - p_k z^{-1})^r}$$

Expansion into the Sum of Partial Fractions

Finally, the sequence $x[n]$ is given by the equation:

$$x[n] = \sum_{k=1}^{N} R_k Z^{-1}\left\{\frac{1}{1 - p_k z^{-1}}\right\} + \sum_{k=0}^{M-N} C_k\, \delta[n-k]$$

If the first sum exists, then its inverse Z-Transform is calculated with the time-shift property. The function $X(z)$ can be described as a sum of simple fractions:

$$\frac{1}{1 - p_k z^{-1}} = \frac{z}{z - p_k}$$

which have an inverse Z-Transform:

$$Z^{-1}\left\{\frac{z}{z - p_k}\right\} = \begin{cases} p_k^n\, u[n], & \alpha\nu\ |z_k| \leq R_{x-} \\ -p_k^n\, u[-n-1], & \alpha\nu\ |z_k| \geq R_{x+} \end{cases}$$

The upper branch describes a **right-hand sided sequence** and the lower branch describes a **left-hand sided sequence.**

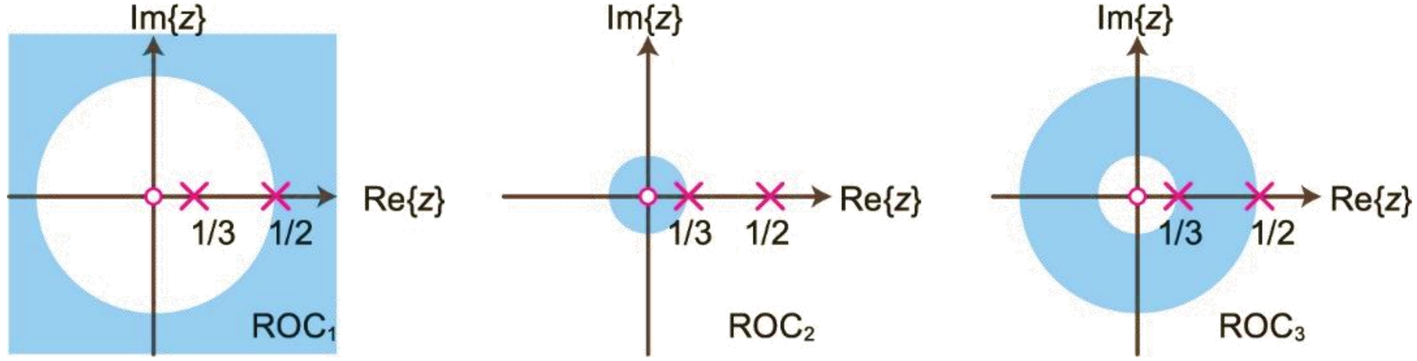
If the function $X(z)$ is given as a ratio of polynomials expressed in terms of z (and not of z^{-1}), then we apply the same procedure and develop in some fractions the expression $X(z)/z$.

Example 13

Find the inverse Z- Transform of the function:

$$X(z) = \frac{3}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

Answer: The given Z-Transform is the sum of two exponential functions of the first degree, that is, it is already in the form of a sum of simple fractions. The poles of the Z-Transformw are $z_1 = 1/2$ and $z_2 = 1/3$. Because the Region of Convergence is not defined, there are three possible cases of regions of convergence, as shown in the figure:



Areas of Convergence

Example 13 (continued)

(a) Region of Convergence R_{x_1}: $1/2 < |z| < \infty$

Because the ROC of $X(z)$ is the outer surface of a circle and its poles lie on the <u>inner </u>side of the circle, it follows that the sequence $x[n]$ is <u>right-handed</u> (causal signal). Using the matching pair for right-hand side exponential sequences, we find:

$$x[n] = 3\left(\frac{1}{2}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n] = \left\{3\left(\frac{1}{2}\right)^n + 2\left(\frac{1}{3}\right)^n\right\} u[n]$$

(b) Region of Convergence R_{x_2}: $0 < |z| < 1/3$

Because the ROC of $X(z)$ is the inner surface of a circle and its poles are on the <u>outer </u>side of the circle, the sequence $x[n]$is <u>left-handed</u> (anti-causal signal). Using the matching pair for left-hand exponential sequences, we find:

$$x[n] = -3\left(\frac{1}{2}\right)^n u[-n-1] - 2\left(\frac{1}{3}\right)^n u[-n-1] = -\left\{3\left(\frac{1}{2}\right)^n + 2\left(\frac{1}{3}\right)^n\right\} u[-n-1]$$

Example 13 (continued)

(c) Region of Convergence R_{x_3}: $1/3 < |z| < 1/2$

Because the ROC of $X(z)$ is the inner surface of a circular ring, the pole z_1 is on the outer side of the great circle while the pole $z_2$is on the inner side of the small circle, the sequence $x[n]$ is double-sided formed by the sum of a left-side sequence and a right side.

Similarly to above, we find:

$$x[n] = -3\left(\frac{1}{2}\right)^n u[-n-1] + 2\left(\frac{1}{3}\right)^n u[n]$$

Example 14

Find the inverse Z-Transform of the function:

$$X(z) = \frac{1}{1 + 3z^{-1} + 2z^{-2}}, \qquad |z| > 2$$

Answer: $X(z)$ is an exponential function of z^{-1} whose denominator is quadratic with respect to the term z. The degree of the numerator is less than the degree of the denominator, so we proceed directly to factoring the denominator and expanding it $X(z)$ into some fractions. Is:

$$X(z) = \frac{1}{1 + 3z^{-1} + 2z^{-2}} = \frac{1}{(1 + 2z^{-1})(1 + z^{-1})}$$

$$= \frac{2}{1 + 2z^{-1}} - \frac{1}{1 + z^{-1}}$$

Since $|z| > 2$, the sequence $x[n]$ is right-sided and the inverse Z- Transform is:

$$x[n] = 2(-2)^n\, u[n] - (-1)^n\, u[n] = \{2(-2)^n - (-1)^n\}u[n]$$

Example 15

Calculate the convolution of the signals:

$$h[n] = \left(\frac{1}{2}\right)^n u[n] \text{ and} x[n] = 3^n u[-n]$$

<u>Answer:</u> The sequence $h[n]$ is right-sided (causative) and has Z-Transform:

$$H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}, \qquad R_h\colon\ |z| > \frac{1}{2}$$

The sequence $x[n]$ is left -sided (anti -causal) and the Z-Transform can be found using the time-shift and time-reversal properties:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{0} 3^n z^{-n} = \sum_{n=0}^{+\infty} \left(\frac{1}{3} z\right)^n = \frac{1}{1 - \frac{1}{3} z} = -\frac{3z^{-1}}{1 - 3z^{-1}},$$

$$R_x\colon\ |z| < 3$$

Example 15 (continued)

So, the Z- Transform of the convolution $y[n] = h[n] * x[n]$, is:

$$Y(z) = -\frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{3z^{-1}}{1-3z^{-1}}$$

The area of convergence is $R_y = R_x \cap R_h$, for which |z| $> 1/2$ and $|z| < 3$. Therefore it is R_y: $1/2 < |z| < 3$.

Because of the shape of the region of convergence we expect the sequence $y[n]$ to be the sum of a right-hand side sequence and a left-hand side sequence.

The reverse Z-Transform of $Y(z)$ comes out by expanding into some fractions:

$$Y(z) = \frac{R_1}{1-\frac{1}{2}z^{-1}} + \frac{R_2}{1-3z^{-1}}$$

where the remainders R_1 and R_2 of the polynomial division for the respective poles are given by the following equations:

Example 15 (continued)

$$R_1 = \left[\left(1-\frac{1}{2}z^{-1}\right)Y(z)\right]_{z=\frac{1}{2}} = \left[\left(1-\frac{1}{2}z^{-1}\right)\frac{1}{1-\frac{1}{2}z^{-1}}\cdot\frac{3z^{-1}}{1-3z^{-1}}\right]_{z=\frac{1}{2}}$$

$$= \left[\frac{3z^{-1}}{1-3z^{-1}}\right]_{z=\frac{1}{2}} \Rightarrow R_1 = \frac{6}{5}$$

and

$$R_2 = \left[(1-3z^{-1})\,Y(z)\right]_{z=3} = \left[(1-3z^{-1})\frac{1}{1-\frac{1}{2}z^{-1}}\cdot\frac{3z^{-1}}{1-3z^{-1}}\right]_{z=3}$$

$$= \left[\frac{3z^{-1}}{1-\frac{1}{2}z^{-1}}\right]_{z=3} \Rightarrow R_2 = -\frac{6}{5}$$

Considering that the first fraction $Y(z)$ is on the left-hand side, while the second is on the right-hand side, it follows:

$$y[n] = \left(\frac{6}{5}\right)\left(\frac{1}{2}\right)^n u[n] + \left(\frac{6}{5}\right)3^n u[-n-1]$$

Example 16

Calculate the inverse Z-Transform of the function:

$$X(z) = \frac{2z(z-0.5)}{(z-1)(z+1)}, \qquad R_x\colon\ |z| > 1$$

Answer: The degree of the numerator is $M = 2$ also the degree of the denominator $N = 2$.

Since the polynomials are expressed in positive powers of z, we will calculate the expansion of the expression $X(z)/z$.

$$\frac{X(z)}{z} = \tilde{X}(z) = \frac{2(z-0.5)}{(z-1)(z+1)} = \frac{R_1}{z-1} + \frac{R_2}{z+1}$$

To calculate the inverse Z-Transform of the function $\tilde{X}(z)$ we apply the method of partial fractions. Is:

$$\tilde{X}(z) = \sum_{k=1}^{N} \frac{R_k}{1 - p_k z^{-1}} = \frac{R_1}{z-1} + \frac{R_2}{z+1}$$

Example 16 (continued)

where the remainders R_1 and R_2 of the polynomial division for the corresponding poles are given by the following equations:

$$R_1 = [(z-1)\tilde{X}(z)]_{z=1} = \left[(z-1)\frac{2(z-0.5)}{(z-1)(z+1)}\right]_{z=1} = \left[\frac{2(z-0.5)}{(z+1)}\right]_{z=1} = 0.5$$

$$R_2 = [(z+1)\tilde{X}(z)]_{z=-1} = \left[(z+1)\frac{2(z-0.5)}{(z-1)(z+1)}\right]_{z=-1} = \left[\frac{2(z-0.5)}{(z-1)}\right]_{z=-1} = 1.5$$

Therefore, the expansion of the function $X(z)$ is:

$$X(z) = \frac{0.5\,z}{z-1} + \frac{1.5\,z}{z+1}$$

from which it follows:

$$x[n] = 0.5\,u[n] + 1.5(-1)^n\,u[n]$$

Example 16 (continued)

To verify the correctness of the solution we apply the initial and final value theorems. We write the function $X(z)$in terms of z^{-1}. Is:

$$X(z) = \frac{2z(z-0.5)}{(z-1)(z+1)} = \frac{2(1-0.5z^{-1})}{(1-z^{-1})(1+z^{-1})}$$

so it $\lim_{z\to\infty} X(z)$ is:

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{2(1-0.5z^{-1})}{(1-z^{-1})(1+z^{-1})} = 2$$

We calculate the value $x[0]$ from solving:

$$x[0] = 0.5\,u[0] + 1.5(-1)^0\,u[0] = 0.5 + 1.5 = 2$$

and we find that $x[0] = \lim_{z\to\infty} X(z)$, then the initial value theorem is satisfied. We also have:

$$\lim_{n\to\infty} x[n] = \lim_{n\to\infty} 0.5\,u[n] + 1.5(-1)^n\,u[n] = 0.5$$

and:

$$\lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1}(z-1)\frac{2z(z-0.5)}{(z-1)(z+1)} = \lim_{z\to 1}\frac{2z(z-0.5)}{(z+1)} = 0.5$$

Because $\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$ we find that the final value theorem is also satisfied.