



University of the Peloponnese

Electrical and Computer  
Engineering Department

# Digital Signal Processing

**Unit 04: Discrete Time Systems  
study using Difference Equations**

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# Lecture Contents

Study of Discrete Time Systems using Difference Equations:

- Difference Equations
- Solving Differential Equations with Linear Coefficients
- Classification of Systems according to the type of Impulse Response
- Asymptotic Stability LSI Discrete Time Systems

# System Description using Difference Equations

We have seen that in Discrete Time (DT) systems the input-output equationship can take the general form (difference equation):

$$y[n] = \sum_{m=0}^M b_m x[n-m] - \sum_{k=1}^N a_k y[n-k], \quad n \geq 0$$

- $a_k, b_m$  constants that define the system
  - $y[-k], k = 1, \dots, N$  initial state of the output
  - $x[-m], m = 1, \dots, M$  initial state of the input
- 
- The above difference equation describes the **structure** of the system.
  - If the system is causal, that is  $x[-m] = 0 \forall m = 1, \dots, M$ , to calculate the output, only the initial conditions of the output are required.
  - If  $y[-k] = 0, \forall k = 1, \dots, N$ , then the system is **initially relaxed**.

# System Description using Difference Equations

- The Difference Equation fully describes the DT system and is the counterpart of the Differential Equation for continuous-time systems.
- As in continuous-time systems, if the Difference Equation is linear with Constant Coefficients, zero initial conditions, and the input is zero for  $n < 0$ , then it represents a Linear and Shift Invariant (LSI) system.
- In this case the above equationship is called a **Linear Difference Equation with Constant Coefficients** (LDECC) and constitutes an iterative way of calculating the output of the system.
- The Difference Equation provides the ability to describe a system with an infinite impulse response using a finite number of coefficients. This simplifies how to describe an LSI system.
- The LDECC can be resolved in both time domain and frequency domains using the Z-transform.
- The use of the Z-transform simplifies the solution of LDECC describing LSI systems.

# Linear Difference Equation with Constant Coefficients (LDECC)

**Linear Difference Equation with Constant Coefficients (LDECC):** for an LSI system, the Difference Equation is linear, with constant coefficients, with zero initial conditions and the system input is zero for  $n < 0$ .

Examples:

- **LDECC first class** ( $M = N = 1$ ):

$$y[n] = -a_1y[n - 1] + b_0x[n] + b_1x[n - 1]$$

- **LDECC second order** ( $M = N = 2$ ):

$$y[n] = -a_2y[n - 2] - a_1y[n - 1] + b_0x[n] + b_1x[n - 1] + b_2x[n - 2]$$

# Example 1

Let the following discrete-time system, described by the input-output equationship and called as accumulator:

$$y[n] = \sum_{k=-\infty}^n x[k] = \sum_{k=-\infty}^{n-1} x[k] + x[n] = y[n-1] + x[n]$$

From the initial part of the above equation we observe that the output  $y[n]$  of the system at the present time results from the summation of all samples of the input signal, from the beginning of the input signal up to the current value  $x[n]$ .

From the last part of the equation we notice that the calculation of the signal value  $y[n]$  for the time instants  $n \geq n_0$  requires the knowledge of the input  $x[n]$  and the initial condition  $y[n_0 - 1]$ .

If,  $y[n_0 - 1] = 0$ , then the system is initially relaxed.

# **Classification of Systems based on the duration of the Impulse Response**

- Recursive or Infinite Impulse Response (IIR)
- Non-Recursive or Finite Impulse Response (FIR)

# Recursive Systems (IIR) (1/2)

- If even one term  $\alpha(k)$  of the LDECC is non-zero, i.e.  $a(k) \neq 0$ ,  $k = 1, \dots, N$  the system is called **recursive** and is described by the LDECC:

$$y[n] = \sum_{m=0}^M b_m x[n-m] - \sum_{k=1}^N a_k y[n-k], \quad n \geq 0$$

- The output  $y[n]$  is calculated, not only based on the samples of the input  $x[n]$ , but also on the previous samples of the output  $\{y[n-k], k = 1, \dots, N\}$ , (feedback of the output).



# Recursive Systems (IIR) (2/2)

- Recursive systems are also called **Infinite Impulse Response (IIR)**, because the  $h[n]$  has infinite duration:

$$h[n] = \sum_{k=0}^{\infty} b_m \delta[n - k]$$

- Also called **Autoregressive Moving Average (ARMA)** filters of class  $(N, M)$ .
- At LSI recursive systems, the output  $y[n]$  is computed by the convolution:

$$y[n] = \sum_{k=0}^{\infty} x[k] h[n - k]$$

## Example 2

To calculate the impulse response of the IIR (ARMA) system, which is initially relaxed and is described by the LDECC:

$$y[n] = -0.5y[n - 1] + x[n] - x[n - 1]$$

**Answer:** Since the impulse response is requested,  $h[n]$  we consider it an input  $x[n] = \delta[n]$ . We iteratively solve the LDECC for values of  $n = 0, 1, 2, \dots$

- $n = 0, h[0] = -0.5 h[-1] + \delta[0] - \delta[-1] = 0 + 1 - 0 \Rightarrow h[0] = 1$
- $n = 1, h[1] = -0.5 h[0] + \delta[1] - \delta[0] = -0.5 + 0 - 1 \Rightarrow h[1] = -1.5$
- $n = 2, h[2] = -0.5 h[1] + \delta[2] - \delta[1] = 0.75 + 0 - 0 \Rightarrow h[2] = 0.75$
- $n = 3, h[3] = -0.5 h[2] + \delta[3] - \delta[2] = -0.375 + 0 - 0 \Rightarrow h[3] = -0.375$
- $n = 4, h[4] = -0.5 h[3] + \delta[4] - \delta[3] = 0.1875 + 0 - 0 \Rightarrow h[4] = 0.1875$

# Non-Recursive Systems (FIR) (1/2)

- In **non-recursive** systems, the input-output equationship results from the LDECC for  $a(k) = 0, k = 1, \dots, N$ :

$$y[n] = \sum_{m=0}^M b_m x[n - m]$$

- The current sample of the output  $y[n]$  is calculated based on weighted and time-shifted values of the input only:  $\{b_m x[n - m], m = 0, \dots, M\}$ .
- Their impulse response is given by the equation:

$$h[n] = \sum_{m=0}^M b_m \delta[n - m]$$

# Non-Recursive Systems (FIR) (2/2)

- Non-recursive systems are also called **Finite Impulse Response (FIR)** or **Moving Average (MA)** systems.
- The coefficients of the convolution sum of a Finite Impulse Response (FIR) filter are the same as the coefficients of  $h[n]$  of the FIR filter:

$$y[n] = \sum_{k=0}^M h[k] x[n - k]$$

- Important Note: The output can be computed from convolution, but with finite duration  $M+1$  points:

$$y[n] = \sum_{m=0}^M b_m x[n - m]$$

- The impulse response of an FIR filter is a finite duration signal. But when the input to the filter is of infinite duration, then the output of the filter is also of infinite duration.

# Example 3

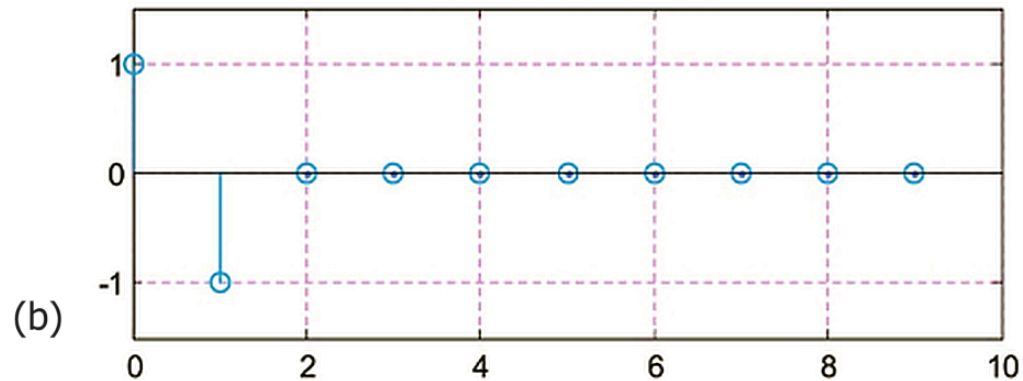
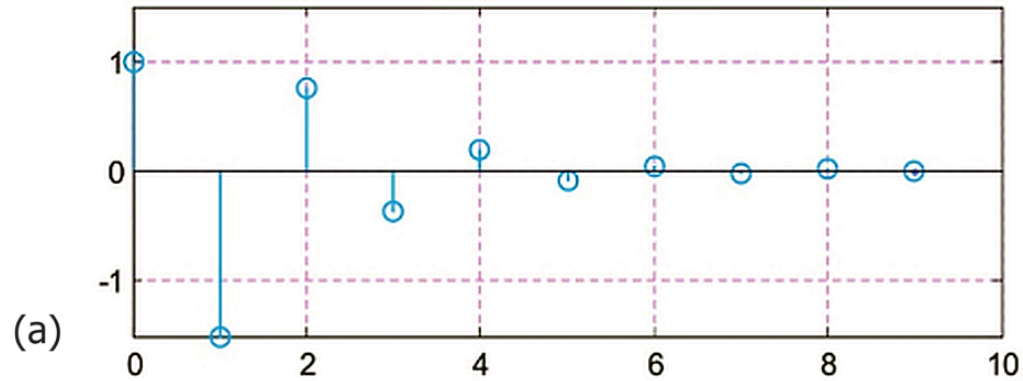
Calculate the impulse response of the FIR system, which is initially relaxed and is described by the LDECC:

$$y[n] = x[n] - x[n - 1]$$

**Answer:** As long as the impulse response is requested,  $h[n]$  we consider it an input  $x[n] = \delta[n]$ . We iteratively solve the LDECC S for values of  $n = 0, 1, 2, \dots$

- $n = 0, h[0] = \delta[0] - \delta[-1] = 1 - 0 \Rightarrow h[0] = 1$
- $n = 1, h[1] = \delta[1] - \delta[0] = 0 - 1 \Rightarrow h[1] = -1$
- $n = 2, h[2] = \delta[2] - \delta[1] = 0 - 0 \Rightarrow h[2] = 0$
- $n = 3, h[3] = \delta[3] - \delta[2] = 0 - 0 \Rightarrow h[3] = 0$
- $n = 4, h[4] = \delta[4] - \delta[3] = 0 - 0 \Rightarrow h[4] = 0$

# Impulse system response



Impulse response for: (a) Example 2, (b) Example 3

# Solving Difference Equations

(in the time-domain)

- Finding the partial solution
- Finding the homogeneous solution

# Solving Differential Equations

The general solution of LDECC:

$$y[n] = \sum_{m=0}^M b_m x[n-m] - \sum_{k=1}^N a_k y[n-k], \quad n \geq 0$$

can be broken-down into the following two parts:

$$y[n] = y_h[n] + y_p[n]$$

- $y_h[n]$  the **homogeneous solution**: is the response of the system for zero input and given initial conditions (also called **zero input response** or **natural response**) (also denoted as  $y_{zi}[n]$ ).
- $y_p[n]$  the **partial solution**: the response of the system for the given input  $x[n]$ , assuming zero initial conditions (also called **zero-state response** or **forced response**) (also denoted as  $y_{zs}[n]$ ).



# Finding the Homogeneous Solution (1/2)

The **homogeneous solution**  $y_h[n]$  is the response of the system for zero input and given the initial conditions. The LDECC is written:

$$y[n] + \sum_{k=1}^N a_k y[n-k] = 0, \quad n \geq 0$$

and is called a homogeneous LDECC, which has the solution (homogeneous solution):

$$y_h[n] = \lambda^n$$

Substituting the homogeneous solution into the homogeneous LDECC we have:

$$\lambda^n + \sum_{k=1}^N a_k \lambda^{n-k} = 0 \Rightarrow \lambda^{n-N} \{ \lambda^N + a_1 \lambda^{N-1} + \dots + a_{N-1} \lambda + a_N \} = 0$$

The expression inside the brackets is called a **characteristic polynomial**  $Q(\lambda)$  and has  $N$  roots:

$$Q(\lambda) = \lambda^N + a_1 \lambda^{N-1} + \dots + a_{N-1} \lambda + a_N$$

# Finding the Homogeneous Solution (2/2)

If the coefficients  $a[k]$  are real numbers, then the roots  $\lambda_k, k = 1, \dots, N$  are complex conjugated. If the roots are **simple**, then the homogeneous solution is given by:

$$y_h[n] = \sum_{k=1}^N A_k \lambda_k^n$$

If there is any **multiple root**, even  $\lambda_1$  with multiplicity  $m$ , then the homogeneous solution is given by the equation:

$$y_h[n] = [A_1 + A_2 n + \dots + A_m n^{m-1}] \lambda_1^n + \sum_{k=m+1}^N A_k \lambda_k^n$$

In both cases, the coefficients  $A_k, k = 1, \dots, N$ , result from solving a system of equations with unknown coefficients  $A_k$ , which is formed from the original LDECC with the general solution, substituting in it the initial conditions for  $n = 0, 1, \dots, N - 1$ .

# Finding the Partial Solution

The **partial solution**  $y_p[n]$  is the response of the system for the given input  $x[n]$ , assuming zero initial conditions.

Finding the partial solution is not a simple process, however for many common inputs the output has the same form as the input.

Some solutions for simple input signals:

Term in $x[n]$ :	$C$	Partial solution:	$C_1$
	$C n$		$C_1 n + C_2$
	$C a^n$		$C_1 a^n$
	$C \cos(n\omega_0)$		$C_1 \cos(n\omega_0) + C_2 \sin(n\omega_0)$
	$C \sin(n\omega_0)$		$C_1 \cos(n\omega_0) + C_2 \sin(n\omega_0)$
	$C a^n \cos(n\omega_0)$		$C_1 a^n \cos(n\omega_0) + C_2 a^n \sin(n\omega_0)$
	$C \delta(n)$		None

# Impulse Response calculation of LSI Systems

- Since we are looking for the impulse response  $h[n]$ , we put in LDECC as input  $x[n] = \delta[n]$  and zero initial conditions, because the system is initially relaxed.
- But the appearance at the entrance of such a strong sequence as Delta creates new initial conditions in the system.
- The output in this case, i.e. the impulse response, will result from the solution of the homogeneous equation (i.e. zero input response) for the new initial conditions.
- Therefore, the problem of finding the impulse response from the LDECC is reduced to finding the new initial conditions due to the Delta sequence and then solving the homogeneous solution to find the zero-input response, i.e. the impulse response.
- In the following examples we will deal with the calculation of the impulse response when the LDECC is known and the system is initially relaxed, i.e. it is an LSI system.

# Example 3

To find the impulse response of the recursive system described by LDECC:

$$y[n] + \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

**Answer:** We put as input  $x[n] = \delta[n]$  and zero initial conditions because the system is initially relaxed. We have:

$$h[n] + \frac{3}{4}h[n-1] + \frac{1}{8}h[n-2] = \delta[n]$$

The homogeneous equation is:

$$h[n] + \frac{3}{4}h[n-1] + \frac{1}{8}h[n-2] = 0$$

The characteristic polynomial is:

$$\lambda^2 + \frac{3}{4}\lambda + \frac{1}{8}$$

and has roots  $\lambda_1 = -1/2$  and  $\lambda_2 = -1/4$ . Therefore, the solution of the homogeneous equation (zero input response) is:

$$h[n] = A_1\lambda_1^n + A_2\lambda_2^n = A_1\left(-\frac{1}{2}\right)^n + A_2\left(-\frac{1}{4}\right)^n, \quad n \geq 0$$

## Example 3 (continued)

We will calculate the values of the constants  $A_1$  and  $A_2$  using the initial conditions. For the given LDECC apply:

$$h[0] + \frac{3}{4}h[-1] + \frac{1}{8}h[-2] = \delta[0] \Rightarrow h[0] + 0 + 0 = 1 \Rightarrow h[0] = 1$$

and

$$h[1] + \frac{3}{4}h[0] + \frac{1}{8}h[-1] = \delta[1] \Rightarrow h[1] + \frac{3}{4} + 0 = 0 \Rightarrow h[1] = -\frac{3}{4}$$

Applying these initial conditions to the homogeneous solution so far, we have:

$$h[0] = A_1 \left(-\frac{1}{2}\right)^0 + A_2 \left(-\frac{1}{4}\right)^0 \Rightarrow A_1 + A_2 = 1$$

and

$$h[1] = A_1 \left(-\frac{1}{2}\right)^1 + A_2 \left(-\frac{1}{4}\right)^1 \Rightarrow -\frac{1}{2}A_1 - \frac{1}{4}A_2 = -\frac{3}{4}$$

## Example 3 (continued)

We write the pair of equations with unknowns  $A_1$  and  $A_2$  in tabular form:

$$\begin{bmatrix} 1 & 1 \\ -1/2 & -1/4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3/4 \end{bmatrix}$$

We solve the system and find:  $A_1 = 2$  and  $A_2 = -1$ .

Since the right-hand member of the LDECC is  $x[n]$  it follows that the impulse response is equal to the homogeneous solution, that is:

$$h_g[n] = h[n] = \left[ 2 \left( -\frac{1}{2} \right)^n - \left( -\frac{1}{4} \right)^n \right] u[n]$$

# Example 4

Find the impulse response of the system with initial conditions given by  $y[-1] = y[-2] = 1/8$  and described by the LDECC:

$$y[n] - 8y[n - 1] + 16y[n - 2] = x[n] - 2x[n - 1]$$

Answer: Finding the partial solution: For input  $x[n] = \delta[n]$ , the partial solution/zero-state solution according to the table in slide 19 is:

$$y_p[n] = 0, \quad n \geq 0$$

Finding the homogeneous solution:

- Characteristic polynomial:  $\lambda^2 - 8\lambda + 16$
- Natural frequencies:  $\lambda_{1,2} = \frac{1}{2} (1 \pm j\sqrt{3}) = e^{\pm j\pi/3}$   
(imaginary conjugate)
- Homogeneous solution/  
zero input:  $y_h[n] = A_1 e^{jn\pi/3} + A_2 e^{-jn\pi/3}$
- General solution:  $y[n] = A_1 e^{jn\pi/3} + A_2 e^{-jn\pi/3}, n \geq 0$  (1)



## Example 4 (continued)

To calculate the constants  $A_1$  and  $A_2$ , we solve the LDECC for  $n = 0$  and  $n = 1$  and we have:

$$y[0] = 8y[-1] - 16y[-2] + x[0] - 2x[-1] = 1 - 2 + 1 - 0 = 0 \Rightarrow y[0] = 0$$

$$y[1] = 8y[0] - 16y[-1] + x[1] - 2x[0] = 0 - 2 + 0 - 2 \Rightarrow y[1] = 4$$

We substitute the new initial conditions into equation (1) and find:

$$y[0] = y_h[0] = A_1 e^0 + A_2 e^0 = A_1 + A_2 = 0$$

$$y[1] = y_h[1] = A_1 e^{j\pi/3} + A_2 e^{-j\pi/3} = 4$$

We write the pair of equations with unknowns  $A_1$  and  $A_2$  in tabular form:

$$\begin{bmatrix} 1 & 1 \\ e^{j\pi/3} & e^{-j\pi/3} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Solving the system, we find:

$$A_1 = 0 - 2.31j = 2.31 e^{-j\pi/2}$$

$$A_2 = 0 + 2.31j = 2.31 e^{j\pi/2}$$

## Example 4 (continued)

Substituting the constants  $A_1$  και  $A_2$  in (1) and using the Euler equation we find:

$$\begin{aligned} h[n] &= 2.31 (e^{jn\pi/3} e^{-j\pi/2} + 2.31 e^{-jn\pi/3} e^{j\pi/2}) u[n] \\ &= 2.31 (e^{j\pi(\frac{n}{3}-\frac{1}{2})} + e^{-j\pi(\frac{n}{3}-\frac{1}{2})}) u[n] = 4,62 \sin\left(\frac{\pi n}{3}\right) u[n] \end{aligned}$$

Since the right-hand member of the LDECC is  $x[n] + 2x[n - 1]$ , the impulse response is equal to the sum  $h[n] + 2h[n - 1]$ , i.e. it is:

$$h_g[n] = h[n] + 2h[n - 1] = 4,62 \left[ \sin\left(\frac{\pi n}{3}\right) + 2 \sin\left(\frac{\pi(n - 1)}{3}\right) \right] u[n]$$

# Example 5

Calculate the homogeneous solution (zero input response), the partial solution (zero state response), and the total output for input  $x[n] = (-0.6)^n u[n]$  for a recursive system with initial conditions  $y[-1] = 1$  and  $y[-2] = 1$  and LDECC given by:

$$y[n] - \frac{3}{2}y[n-1] - y[n-2] = x[n]$$

**Answer:** Finding the partial solution: For input  $x[n] = (-0.6)^n u[n]$ , the partial solution (zero state response) according to the table in slide 19, is:

$$y_p[n] = C_1(-0.6)^n u[n]$$

The partial solution satisfies the LDECC, namely:

$$C_1(-0.6)^n u[n] - \frac{3}{2}C_1(-0.6)^{n-1} u[n-1] - C_1(-0.6)^{n-2} u[n-2] = (-0.6)^n u[n]$$

## Example 5 (continued)

We find its value  $C_1$  by putting  $n = 2$  in the above equation, so we have:

$$C_1(-0.6)^2 u[2] - \frac{3}{2} C_1(-0.6)^1 u[1] - C_1(-0.6)^0 u[0] = (-0.6)^2 u[2]$$

from which we find:

$$C_1 = \frac{18}{13}$$

So the partial solution is:

$$y_p[n] = \frac{18}{13} (-0.6)^n u[n]$$

## Example 5 (continued)

Finding the homogeneous solution: The characteristic polynomial is:

$$\lambda^2 - \frac{3}{2}\lambda - 1$$

and the natural frequencies are:

$$\lambda_1 = -\frac{1}{2} \quad \lambda_2 = 2$$

Therefore, the homogeneous solution (i.e. zero input response) is:

$$y_h[n] = A_1 \left(-\frac{1}{2}\right)^n + A_2(2)^n$$

The total solution is the sum of the partial and homogeneous solutions. i.e.:

$$y[n] = y_p[n] + y_h[n] = \frac{18}{13}(-0.6)^n u[n] + A_1 \left(-\frac{1}{2}\right)^n + A_2(2)^n$$

## Example 5 (continued)

To calculate the constants  $A_1$  and  $A_2$ , we solve the LDECC for  $n = 0$  and  $n = 1$  and we have:

$$y[0] = \frac{3}{2}y[-1] - y[-2] + x[0] = \frac{3}{2} - 1 + 1 \Rightarrow y[0] = \frac{3}{2}$$
$$y[1] = \frac{3}{2}y[0] - y[-1] + x[1] = \frac{9}{4} - 1 + (-0.6) \Rightarrow y[1] = \frac{13}{20}$$

We substitute the new initial conditions into the complete solution and we find:

$$y[0] = y_p[0] + y_h[0] = \frac{3}{2} \Rightarrow \dots \Rightarrow A_1 + A_2 = \frac{3}{26}$$
$$y[1] = y_p[1] + y_h[1] = \frac{13}{20} \Rightarrow \dots \Rightarrow -\frac{1}{2}A_1 + 2A_2 = -\frac{47}{260}$$

We solve the system and find  $A_1 = 0,1646$  and  $A_2 = -0.0492$ .

$$y[n] = \frac{18}{13}(-0.6)^n u[n] + 0,1646 \left(-\frac{1}{2}\right)^n - 0,0492(2)^n$$

# Example 6

To find the impulse response of the recursive system described by the LDECC  $y[n] + 0.1y[n - 1] - 0.72y[n - 2] = x[n]$  and which is initially relaxed.

Answer: We apply the same solution methodology. We also set zero initial conditions as input  $x[n] = \delta[n]$  because the system is initially relaxed.

- Characteristic polynomial:  $\lambda^2 + 0.1\lambda - 0.72 = (\lambda - 0.8)(\lambda + 0.9)$  (1)
- Natural frequencies:  $\lambda_1 = 0.8$  and  $\lambda_2 = -0.9$  (simple roots)
- Homogeneous solution  
/zero input:  $y_h[n] = A_1(0.8)^n + A_2(-0.9)^n, n \geq 0$  (2)
- Partial  
/zero-state:  $y_p[n] = 0$ , because  $x[n] = \delta[n]$
- Complete solution:  $y[n] = y_h[n] + y_p[n] = y_h[n]$  (3)

## Example 6 (continued)

Finding a homogeneous solution: Since the system is initially relaxed, the initial conditions are zero, that is:  $y[-1] = y[-2] = 0$ . Solving the LDECC for  $n = 0$  and  $n = 1$ , we have for the complete solution the new initial conditions due to the application of the impulse input:

$$y[0] = -0.1y[-1] + 0.72y[-2] + x[0] = 1 \Rightarrow y[0] = 1$$

$$y[1] = -0.1y[0] + 0.72y[-1] + x[1] = -0.1 \Rightarrow y[1] = -0.1$$

To calculate the constants  $A_1$  and  $A_2$ , we solve the homogeneous solution (2) for  $n = 0$  and  $n = 1$ , consider equation (3) and find:

$$y[0] = y_h[0] = A_1(0.8)^0 + A_2(-0.9)^0 \Rightarrow A_1 + A_2 = 1 \quad (4)$$

$$y[1] = y_h[1] = A_1(0.8)^1 + A_2(-0.9)^1 \Rightarrow 0.8A_1 - 0.9A_2 = -0.1 \quad (5)$$

We solve the above system of equations (4) and (5) in terms of  $A_1, A_2$  and find:

$$A_1 = \frac{8}{17} \quad \text{και} \quad A_2 = \frac{9}{17}$$



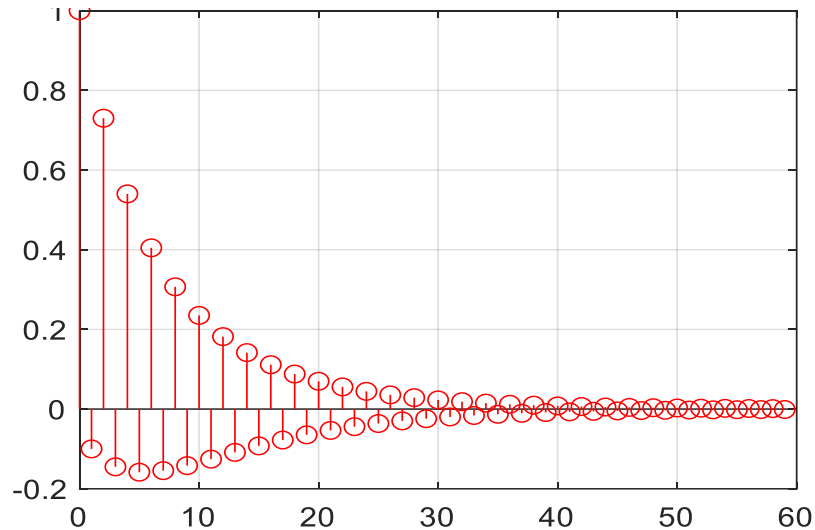
# Example 6 (continued)

Therefore, the complete solution of the LDECC is:

$$y[n] = y_h[n] = \left[ \frac{8}{17} (0.8)^n + \frac{9}{17} (-0.9)^n \right] u[n]$$

Based on the right-hand member of the LDECC, the impulse response is equal to the homogeneous solution and is:

$$h[n] = y_h[n] = \left[ \frac{8}{17} (0.8)^n + \frac{9}{17} (-0.9)^n \right] u[n]$$



Impulse response

# Example 7

Given a system described by the following LDECC:

$$y[n] = \frac{1}{M} \sum_{m=0}^{M-1} x[n - m]$$

To  $M = 4$  calculate the system output for the input:

$$x[n] = s[n] + e[n] = 2 \cos\left(\frac{\pi n}{4}\right) + \textit{noise}$$

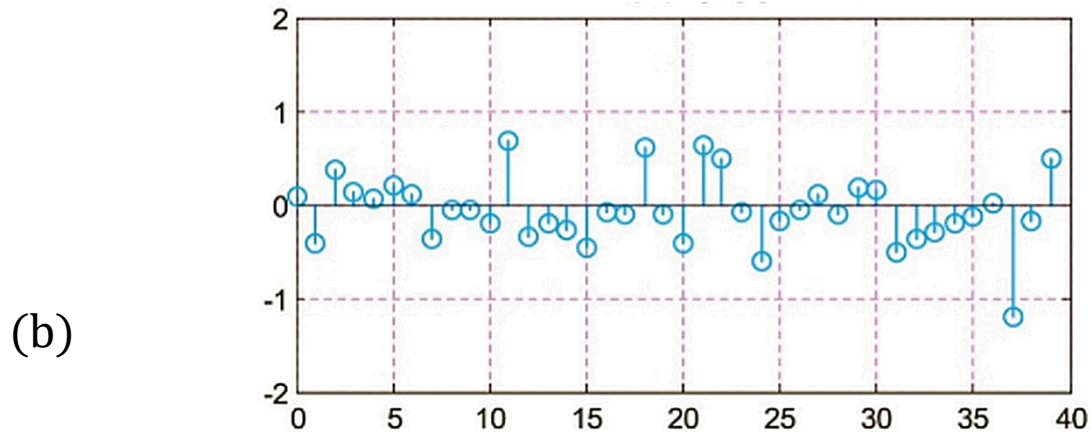
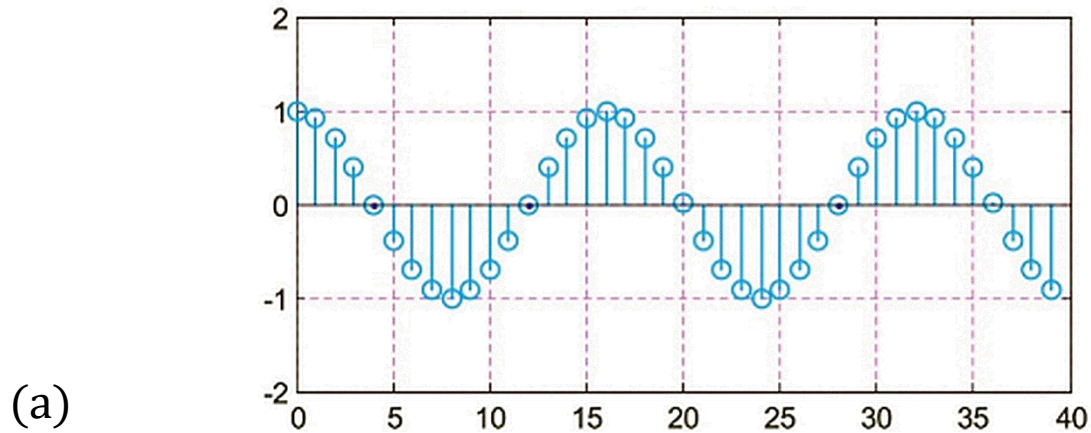
**Answer:** From the LDECC we observe that the output of the system is the average of the  $M$  previous values of the input. This system is called an **moving average (MA)** filter of class  $M$  and the coefficients  $b[m]$  are equal to each other, i.e.:

$$b[0] = b[1] = \dots = b[m - 1] = 1/M$$

Moving Average filters are used to remove noise from signals, because computing the output from the average of some previous values of the input tends to compensate for the randomness of the noise values added to the useful signal.

Solving in Matlab we get the following results:

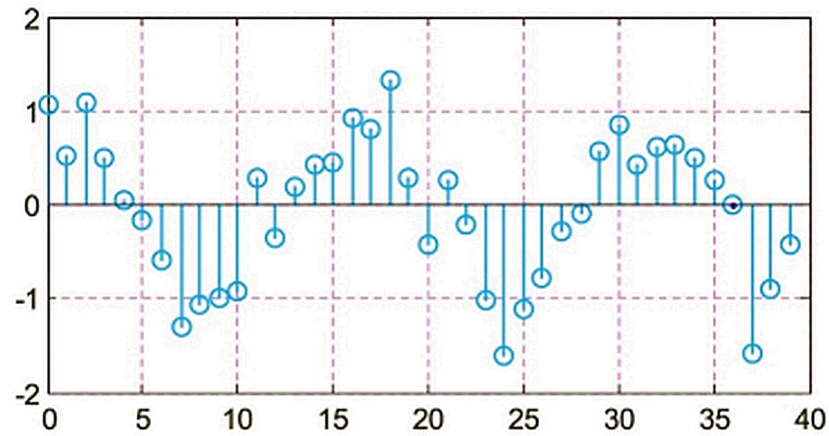
# Example 7 (continued)



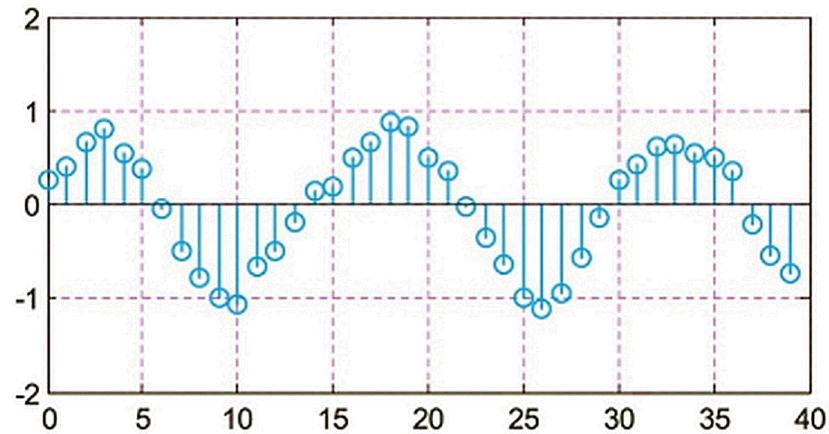
(a) Clean signal  $s[n]$ , (b) Noise  $e[n]$

# Example 7 (continued)

(c)

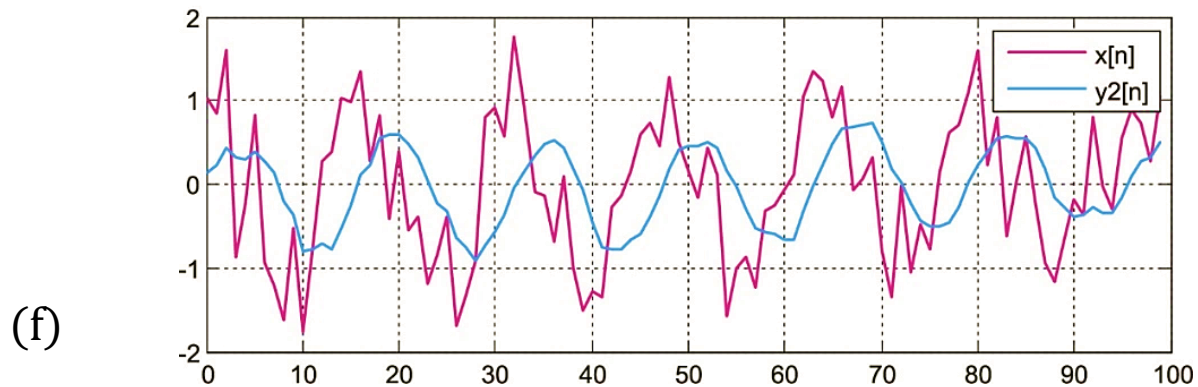
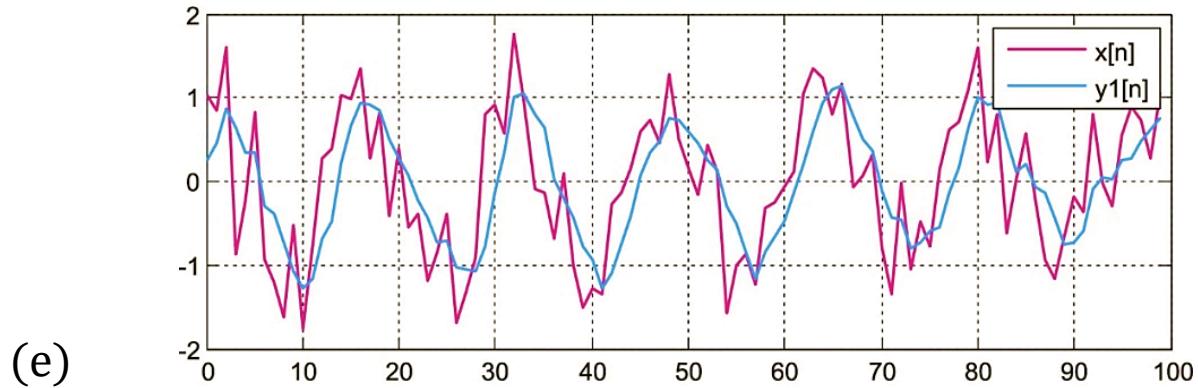


(d)



(a) Signal with noise  $x[n]$ , (d) Processed signal  $y[n]$

# Example 7 (continued)



Signal smoothing for: (e)  $M = 4$ , (f)  $M = 8$

# Example 7 (continued)

From figures (e) and (f) we observe that the output  $y[n]$  of the mean value filter:

- It looks more like the original "clean" signal  $s[n]$  than the noisy signal  $x[n]$ .
- This similarity improves as the coefficient increases  $M$ , because a larger number of prior values of the input better compensates for the randomness of the noise values. However, this improvement comes at the expense of output calculation speed, as a  $M/2$  point delay is added relative to the input.
- The above filtering technique is done in the time-domain and is a **linear** technique (Linear Filtering). It is suitable for dealing with noise with a Gaussian distribution, but unsuitable for dealing with impact noises.
- The most sophisticated noise removal techniques are found in the frequency domain. We will study some of these in next lectures.

# Stability of a Discrete Time System

- In the case of asymptotic stability, we calculate the roots of the characteristic polynomial of the LDECC that describes the system and distinguish between the following cases:
  - When all the roots of the characteristic polynomial (possibly some have multiplicity  $p > 1$ ) are inside the unit circle, i.e. are  $|\lambda_i| < 1$ ,  $i = 1, 2, \dots, N$ , then the system is **asymptotically stable**.
  - When the characteristic polynomial has roots of multiplicity  $p > 1$ , some of which lie on the unit circle, that is, is  $|\lambda_i| = 1$ , then the system is **asymptotically unstable**.
  - If at least one root is found outside the unit circle, i.e. is  $|\lambda_i| > 1$ , then the system is **asymptotically unstable**.
  - If a simple root is on the unit circle ie is  $|\lambda_i| = 1$ ,  $i = 1, 2, \dots, N$ , then the system is **marginally stable**.
  - If a multiple root lies on the unit circle, then the system is **unstable**.
- We will complete the study of the stability of LSI systems in the next lecture using the Z-transform.

# Example 8

Study the asymptotic stability of the discrete-time system with LDECC:

$$9y[n] + 13y[n - 1] + 5y[n - 2] + y[n - 3] = x[n] + 2x[n - 1]$$

Answer: The expatriate LDECC is:

$$9y[n] + 13y[n - 1] + 5y[n - 2] + y[n - 3] = 0$$

from which it follows that the characteristic equation is:

$$\begin{aligned}\lambda^3 + 4\lambda^2 + 9\lambda + \lambda^2 + 4\lambda + 9 &= (\lambda + 1)(\lambda^2 + 4\lambda + 9) \\ &= (\lambda + 1)(\lambda + 2 - 3j)(\lambda + 2 + 3j) = 0\end{aligned}$$

Therefore, the roots (natural frequencies - eigenvalues) are  $\lambda_1 = -1$ ,  $\lambda_2 = -2 + 3j$ ,  $\lambda_3 = -2 - 3j$ . The measures of the roots are:  $|\lambda_1| = 1$ ,  $|\lambda_2| = |\lambda_3| = \sqrt{13}$ . Since there are roots outside the unit circle, we conclude that the system is asymptotically unstable.