



University of the Peloponnese

Electrical and Computer  
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# Digital Signal Processing

## Unit 03: Discrete-Time Systems Convolutional Sum

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# Lecture Contents

- Introduction to Discrete Time Systems
- Categorization of Discrete Time Systems
  - Causal Systems
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  - Time Invariant Systems
  - Homogeneous and Summative Systems
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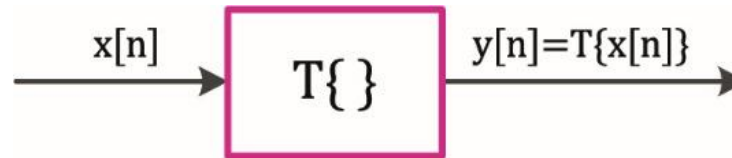
# Lecture Contents

- Ways of Describing Discrete-Time Systems
  - With Stage Diagrams
  - With Convolutional Sum and Impulse Response
  - With Difference Equations (Recursive and Nonrecursive Systems)
- Study of Systems with Convolutional Sum (Convolution)
- Properties of Convolution

# **Introduction to Discrete Time Systems**

# Discrete Time Systems

A discrete-time system is considered as a mathematical operator  $T[\cdot]$  with which an input signal [excitation]  $x[n]$  is transformed into an output signal [response]  $y[n]$ .



The usual way of describing a system is through the **input-output relationship**  
 $y[n] = T[x[n]]$ .

# Advantages of Discrete-Time Systems

Discrete-time systems, or digital systems for short, have significant advantages over analog systems, such as:

- Greater resistance to noise and the ability to more easily remove noise from useful signals using sophisticated denoising techniques.
- Increased operational reliability due to zero effect from component temperature changes or material aging.
- Easier design, lower power consumption and a significant reduction in manufacturing costs, as digital systems are mainly implemented in the form of software, which runs on a general purpose computing system, such as PCs, DSPs, FPGAs, ASICs, thus low cost.
- Easily add features like data storage, data compression, and data encryption.

Analog systems excel in implementing very high frequency systems.

# Example 1

To calculate the impulse response of the system with input-output relationship:

$$y[n] = x[n] + x[n - 1] + 3x[n - 2].$$

**Answer:** We put input  $x[n] = \delta[n]$  in input - output relationship, so the impulse response is  $h[n] = y[n] = \delta[n] + \delta[n - 1] + 3\delta[n - 2]$ . We set values to  $n$  and find:

- $n = -1, h[-1] = \delta[-1] + \delta[-2] + 3\delta[-3] = 1.0 + 1.0 + 3.0 = 0$
- $n = 0, h[0] = \delta[0] + \delta[-1] + 3\delta[-2] = 1.1 + 1.0 + 3.0 = 1$
- $n = 1, h[1] = \delta[1] + \delta[0] + 3\delta[-1] = 1.0 + 1.1 + 3.0 = 1$
- $n = 2, h[2] = \delta[2] + \delta[1] + 3\delta[0] = 1.0 + 1.0 + 3.1 = 3$
- $n = 3, h[3] = \delta[3] + \delta[2] + 3\delta[1] = 1.0 + 1.0 + 3.0 = 0$
- $n \geq 4, h[n] = 0$

Therefore, the impulse response is  $h[n] = \{\hat{1}, 1, 1, 3, 0, 0, \dots\}, 0 \leq n \leq 2$

## Example 2

The entry signal is given  $x[n] = \{1, -1, 2, -1, \hat{0}, 1, 2, -3\}$ . Calculate the output signals  $y_1[n] = x[n - 3]$  and  $y_2[n] = 2x[n] - x[n - 3]$ .

**Answer:** From its input-output equation  $y_1[n]$  we observe that the current sample of the output  $y_1[n]$  is produced if in  $x[n]$  we apply a delay of three units of time to the input signal. Therefore the output signal is given by:

$$y_1[n] = \{1, -\hat{1}, 2, -1, 0, 1, 2, -3\}$$

For  $y_2[n]$  we will create the sequences  $2x[n]$  and  $x[n - 3]$ , then remove the corresponding samples. To "align" the two sequences in time we will **add zero** samples so that the element  $x[0]$  corresponds to the same position in both sequences, e.g. in 5<sup>th</sup> place. Is:

$$2x[n] = \{2, -2, 4, -2, \hat{0}, 2, 4, -6, 0, 0, 0\}$$
$$x[n - 3] = \{0, 0, 0, 1, -\hat{1}, 2, -1, 0, 1, 2, -3\}$$

Therefore after subtraction the sequence results:

$$y_2[n] = 2x[n] - x[n - 3] = \{2, -2, 4, -3, \hat{1}, 0, 5, -6, -1, -2, 3\}$$



# Categorization of Discrete Time Systems

- Causal systems
- Static and Dynamic systems
- Shift Invariant systems
- Homogeneous and Cumulative systems
- Invertible systems
- Linear systems

# Causal Discrete-Time Systems

A system with zero initial output, can be:

- **Causal or causal**, when its output at each moment in time  $n = n_0$  depends only on the input **up to** that moment  $n = n_0$ .

Mathematical formulation:  $x[n] = 0, n < n_0$  then  $y[n] = 0, n < n_0$ .

- **Non-causal** is the system in which the output at a time  $n = n_0$  can also depend on future samples of the input.

Example: the system with an input-output relationship  $y[n] = ax[n] + bx[n - 1]$  is causal, while the system  $y[n] = ax[n] + bx[n + 1]$  is non-causal.

- A non-causal discrete-time system can be implemented by storing the output samples in a memory and delaying the reading of the input samples relative to the output.
- If real-time signal processing is desired then the system must be causal.

# Causal Discrete-Time Systems

- Non-causal discrete-time systems are the upper performance bound of causal discrete-time systems.
- If the system is in a state of **initial rest**, i.e. the initial conditions are zero and the system is linear and shift invariant, then the sufficient and necessary condition of causality is given by the equation:

$$h[n] = 0, \quad n < 0$$

where  $h[n]$  is the impulse response of the system.

- The exit  $y[n]$  of Causal and Linear and Time-Invariant (LTI) systems for causative entry  $x[n]$  is:

$$y[n] = \sum_{k=0}^n x[k] h[n - k], \quad n \geq 0$$

The lower limit of the sum depends on the causality of the input  $x[k] = 0 \forall k < 0$ , while the upper limit depends on the causality of the system

$$h[n - k] = 0 \text{ for } n - k < 0 \text{ or } k > n.$$

# Static and Dynamic Discrete-Time Systems

A system at rest (i.e. zero initial output), can be:

- **Static** or **without memory** (memory less), when its output at each time  $n = n_0$  depends only on the input at the **same** time moment  $n = n_0$ . A static system does not include delay stages.
- In the opposite case, the system is called **dynamic**, and the number of previous samples that affect the current sample is referred to as the **system's memory**."

A static system is always also causal. The reverse is not always true.

# Example 3

Consider whether the following systems are: (a) with memory or without memory and (b) causal or non-causal.

$$(a)y[n] = x^3[n]$$

$$(b)y[n] = x[n] + x[n - 2]$$

$$(c)y[n] = \sum_{k=-\infty}^{+\infty} x[n - k]$$

**Answer:** (a) We observe that the current ( $n$ ) sample of the output  $y[n]$  is affected only by the current ( $n$ ) sample of the input  $x[n]$ . Therefore the system is memoryless and causal.

(b) Regarding causality, we observe that the current ( $n$ ) sample of the output  $y[n]$  is not affected by future samples of the input  $x[n]$ . Therefore the system is causal. In terms of memory, we observe that the current ( $n$ ) sample of the output  $y[n]$  is affected by the current ( $n$ ) and previous samples of the input  $x[n]$ . Therefore the system is with memory or potential.

## Example 3 (continued)

(c) The given input-output relationship can be written:

$$y[n] = \sum_{n=k}^{+\infty} x[n-k] = \sum_{m=-\infty}^0 x[m]$$

from which we observe that to determine the output  $y[n]$  at time  $n$ , the input must be known for all  $n \leq 0$ . E.g. to find the value  $y[-5]$  we need to know future values of the input, such as  $x[0], x[-1], x[-2], \dots$ . Therefore the system is not causal.

The system is not static either, because the present value of its output depends not only on the present value of the input but also on previous ones. System memory is  $k$ .

# Shift-Invariant Discrete-Time Systems

A DT system is called **shift-invariant** when, for any amount of time  $n_0$ , the output of the system for input  $x[n - n_0]$  is equal to  $y[n - n_0]$ .

In other words, time shifts by  $n_0$  in the input  $x[n]$  cause an **identical shift** by  $n_0$  in the output  $y[n]$ .

## Verification of time invariance:

- We calculate the output  $y[n, k] = T\{x[n - k]\}$ , i.e. the output that occurs for an input that is time-shifted by an amount of time  $k$ .
- We compute the time-shifted output  $y[n - k]$  over the same amount of time  $k$ .
- If we find that the equality holds  $y[n, k] = y[n - k], \forall k$ , then the system is time invariant.

# Example 4

To examine whether the following systems are time-invariant.

(a)  $y[n] = x[n] + x[n - 1] + x[n - 2]$

(b)  $y[n] = nx[n]$

**Answer:** (a) To test for time-invariance, we compare the shifted response  $y[n - n_0]$ , with the response for a time-shifted input  $x[n - n_0]$ .

The shifted output (response) is:

$$y[n - n_0] = x[n - n_0] + x[n - n_0 - 1] + x[n - n_0 - 2]$$

The response  $y'[n]$  for shifted input  $x'[n] = x[n - n_0]$ , is:

$$\begin{aligned} y'[n] &= x'[n] + x'[n - 1] + x'[n - 2] = \\ &= x[n - n_0] + x[n - n_0 - 1] + x[n - n_0 - 2] \end{aligned}$$

Because  $y'[n] = y[n - n_0]$  the system is valid it is invariant when shifting.



## Example 4 (continued)

(b) The shifted response  $y[n - n_0]$  is:

$$y[n - n_0] = (n - n_0) x[n - n_0]$$

The response  $y'[n]$  for shifted input  $x'[n] = x[n - n_0]$ , is:

$$y'[n] = nx'[n] = n x[n - n_0]$$

Since  $y'[n] \neq y[n - n_0]$ , the system is time-invariant.

# Homogeneous and Summative DTS

- A Discrete Time System is called **homogeneous** if multiplying the input by a constant leads to multiplying the output by exactly the same constant. The mathematical formulation of homogeneity is given by the equation:

$$T\{c x[n]\} = c T\{x[n]\}$$

for any complex constant  $c$  and for any discrete-time signal  $x[n]$ .

- A Discrete Time System is called **summative** if a sum of signals at the input leads to an output that is the sum of the outputs for each input signal separately. The mathematical formulation of additivity is given by the equation:

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}$$

for any discrete-time signals  $x_1[n]$  and  $x_2[n]$ .

# Invertible Discrete-Time Systems

- A Discrete Time System is called **invertible** if its input can be determined from its output in a unique way.
- For a system to be reversible, separate input signals must produce separate output signals, i.e. if:

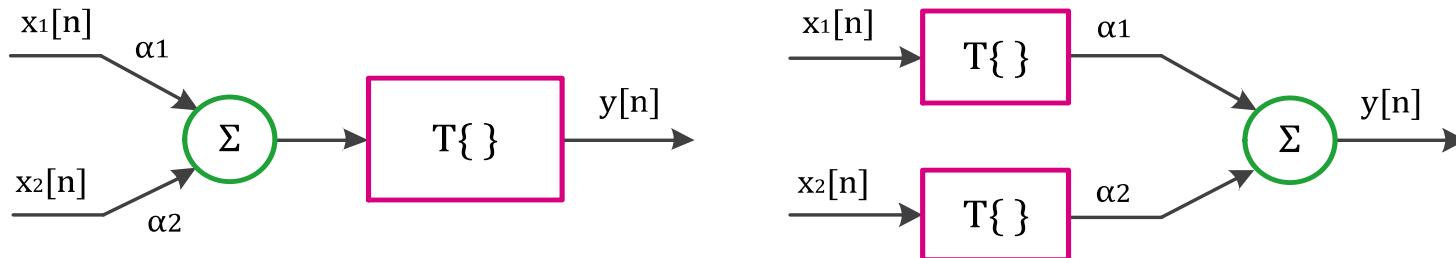
$$x_1[n] \neq x_2[n] \text{ then and } y_1[n] \neq y_2[n]$$

# Linear Discrete-Time Systems

A discrete-time system is called linear **when** the input-output relationship that describes it satisfies the equality:

$$T\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} = \alpha_1 T\{x_1[n]\} + \alpha_2 T\{x_2[n]\}, \forall \alpha_1, \alpha_2 \in \mathcal{C}$$

- In a linear system the effect of each input on the output of the system is independent of the effect of the other inputs.
- The property of linearity in a discrete-time system is very important because it simplifies the calculation of the system's response to a given input.



Discrete-time system linearity representation

# Linear and Shift-Invariant Discrete-Time Systems

- A system is **Linear and Shift Invariant** (LSI) when it simultaneously satisfies Linearity and Shift Invariance.
- If  $h[n]$  it is the **impulse response** of a LSI system, [i.e. the output of the system for input equal to  $\delta[n]$  ], then due to invariance the response to input  $\delta[n - k]$  will be  $h[n - k]$ , ie  $h_k[n] = h[n - k]$ . From the sum of the superposition, we find that the output is:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n - k]$$

- The above relationship is called **Sum of Convolution** and is briefly written:

$$y[n] = x[n] * h[n]$$

We will study convolution in more detail next.

# Example 5

To examine whether the system is linear or not with an input-output relationship:

$$y[n] = \sum_{k=-\infty}^n x[k]$$

**Answer:** For inputs  $x_1[n]$  and  $x_2[n]$  the corresponding outputs of the system are  $y_1[n] = \sum_{k=-\infty}^n x_1[k]$  and  $y_2[n] = \sum_{k=-\infty}^n x_2[k]$ .

For input equal to linear combination  $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$ , the output is:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^n x[k] = \sum_{k=-\infty}^n \{\alpha_1 x_1[k] + \alpha_2 x_2[k]\} = \sum_{k=-\infty}^n \alpha_1 x_1[k] + \sum_{k=-\infty}^n \alpha_2 x_2[k] \\ &= \alpha_1 \sum_{k=-\infty}^n x_1[k] + \alpha_2 \sum_{k=-\infty}^n x_2[k] = \alpha_1 y_1[n] + \alpha_2 y_2[n] \end{aligned}$$

So the system is **linear**.

# Example 6

To examine whether the system with an input-output relationship  $y[n] = x[-n]$  is:  
(a) linear and (b) shift invariant.

**Answer:** (a) For inputs  $x_1[n]$  and  $x_2[n]$ , the corresponding outputs of the system are  $y_1[n] = x_1[-n]$  and  $y_2[n] = x_2[-n]$ . If we set input equal to the linear combination  $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$ , the output is:

$$y[n] = x[-n] = \alpha_1 x_1[-n] + \alpha_2 x_2[-n] = \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

So the system is **linear**.

(b) For a time-shifted  $n_0$  input  $x'[n] = x[n - n_0]$ , the output is:

$$y[n, n_0] = x'[-n] = x[-n - n_0]$$

The time-shifted  $n_0$  output is:

$$y[n - n_0] = x[-(n - n_0)] = x[-n + n_0]$$

Because  $y[n, n_0] \neq y[n - n_0]$  the system is **time-varying**.

# Stable Discrete-Time Systems

A discrete-time system is called **stable** or **bounded input – bounded output** (BIBO stable), when for every bounded input  $|x[n]| \leq A < \infty$ , the output is also bounded, i.e.  $|y[n]| \leq B < \infty$ . If for a bounded input the system produces an output that tends to infinity, then the system is called **unstable**.

## Remarks:

- A LSI system is always **stable** when its impulse response  $h[n]$  is absolutely convergent, i.e. it satisfies the equation:

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$$

- A finite impulse response (FIR) filter is always stable because its impulse response always satisfies the above equation.
- An infinite impulse response (IIR) filter is not a priori stable, but only if it proves to satisfy the above equation.



# Asymptotic Stability of Discrete-Time Systems

In the next lecture we will study the asymptotic stability of systems. According to her:

- A causal system is stable when **all** the roots of the characteristic polynomial lie **inside** the unit circle.
- If a root lies **on** the unit circle then the system is **marginally stable**.
- If more than one root lies on the unit circle or even one root lies outside the unit circle, then the system is **unstable**.

We will study in detail the stability of systems with the help of the Z-transform.

# Example 7

To determine whether the LSI system with impulse response  $h[n] = a^n u[n]$ ,  $a < 1$  is it stable or not.

**Answer:** To study the stability of the system, we need to determine whether the impulse response  $h[n]$  is perfectly convergent. We have:

$$\sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=-\infty}^{+\infty} |a^n u[n]| = \sum_{n=0}^{+\infty} |a^n| = \frac{1}{1 - |a|} < \infty$$

Therefore,  $h[n]$  is absolutely convergent, so the system is **stable**.

# Example 8

To examine whether the system described by the following difference equation is stable or not:

$$-0.08y[n] + 0.2y[n - 1] + y[n - 2] = x[n] + 2x[n - 1]$$

**Answer:** The characteristic polynomial is  $-0.08 + 0.2\lambda + \lambda^2$ , which is factored:

$$\lambda^2 + 0.2\lambda - 0.08 = \lambda^2 - 0.2\lambda + 0.4\lambda - (0.2)(0.4) = (\lambda - 0.2)(\lambda + 0.4)$$

The roots of the polynomial are:  $\lambda_1 = 0.2$  and  $\lambda_2 = -0.4$

Because **all** the roots lie inside the unit circle, the system is **stable**.

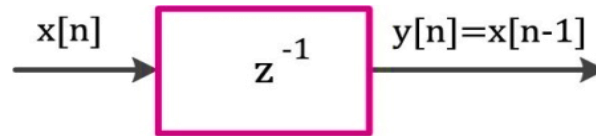
# Methods to Describe LSI Systems

- Stage Diagrams
- Difference Equations
- Convolutional Sum (Linear Convolution)

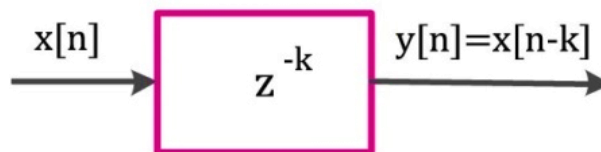
# **System Description using Stage Diagrams**

# System Description using Stage Diagrams

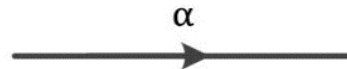
**Delay step of one time unit:** Satisfies the equation:  $y[n] = x[n - 1]$ . For the advance  $y[n] = x[n + 1]$ .



**Delay stage of several units of time:** produces an output signal  $y[n]$  with a delay of several ( $k$ ) units of time with respect to the input signal  $x[n]$ , Satisfies the equation:  $y[n] = x[n - k]$ .

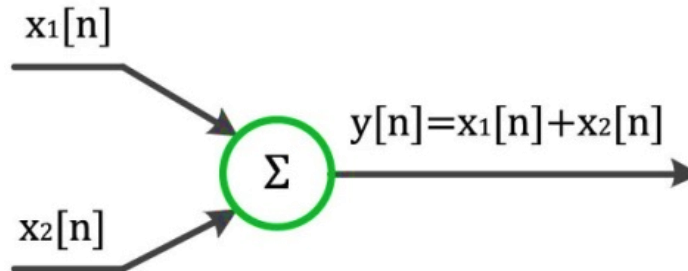


**Multiplier stage with a constant:** produces an output signal  $y[n]$  that is the product of the input  $x[n]$  with a constant  $\alpha$ , that is:  $y[n] = \alpha x[n]$ .

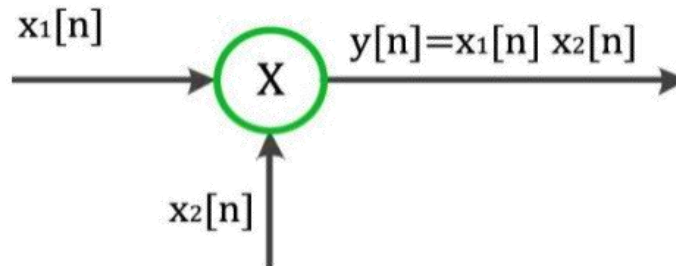


# System Description using Stage Diagrams

**Adder stage:** accepts two or more input signals and produces an output  $y[n]$  that is the sum of the input signals, i.e. it satisfies the equation:  $y[n] = x_1[n] + x_2[n]$ .



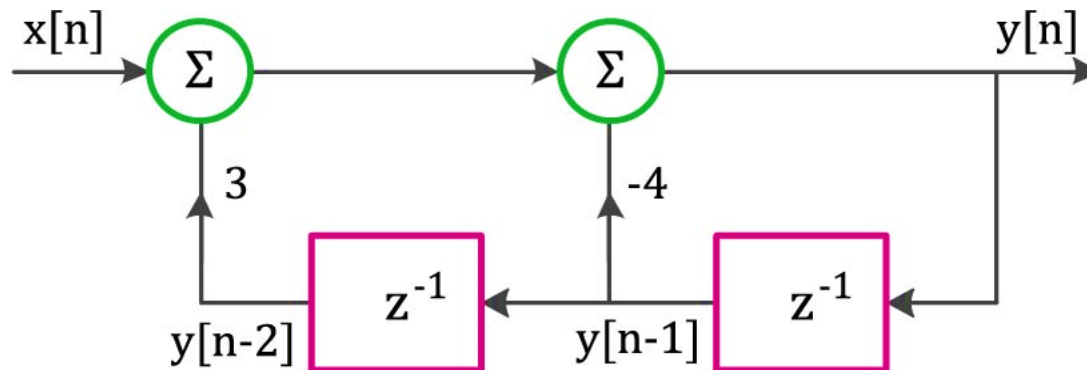
**Multiplier stage:** accepts two or more input signals and produces an output  $y[n]$  that is the product of the input signals, i.e. it satisfies the equation:  $y[n] = x_1[n] x_2[n]$ .



# Example 9

Render the system described by the difference equation in the form of a stage diagram  $y[n] = 3y[n - 2] - 4y[n - 1] + x[n]$ .

**Answer:** Based on the available stages, we draw the figure:





# **System Description using Convolution Sum**

# System Description using Convolution Sum

For a LSI system the output  $y[n]$  is given as  $y[n] = T\{x[n]\}$ . Because a signal e.g. can be written as the weighted sum of unit impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]$$

the output is written:

$$y[n] = T \left\{ \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \right\}$$

If the system is linear, then:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] T\{\delta[n - k]\}$$

Therefore, the output  $T\{\delta[n - k]\}$  of a linear system can be thought of as the output of the system at time  $n$  due to a unit impulse  $\delta[n - k]$  applied to its input at time  $k$ . This is called **the impulse response** and is denoted  $h(n, k)$ .

**Impulse response** LSI system:  $h(n - k) = T[\delta(n - k)]$

# Description of a Convolutional Sum System

Therefore, the output of the linear system is written:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n, k]$$

If the system is Linear and Shift Invariant (LSI), then the output at the impulse input is:

$$h[n, k] = h[n - k]$$

Substituting the above equation into the previous one, we find that for input  $x[n]$ , the output of a LSI system with impulse response  $h[n]$ , is given by the sum:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n - k] = \sum_{k=-\infty}^{+\infty} x[n - k] h[k] = x[n] * h[n]$$

- The \* operator represents the **convolution sum** between  $x[n]$  and  $h[n]$ .
- Convolution offers a way to calculate for LSI systems through  $h[n]$ , which describes the **behavior** of the system under impulse input. The  $h[n]$  does not carry information about the structure of the system.
- Convolution is a powerful tool for studying LSI systems.

# Description of a Convolutional Sum System

When the sequences  $x[n]$  and  $h[n]$  are:

- of infinite duration, then the convolution is also **of infinite duration**.
- of finite duration, then the convolution is also **of finite duration**.

More specifically, if the non-zero values of the sequence  $x[n]$  are contained in an interval  $[M_x, N_x]$  of length  $L_x = N_x - M_x + 1$  then:

- and the **non-zero values** of the sequence  $y[n]$  are bounded by the interval:

$$[M_x + M_h, N_x + N_h]$$

- and the **length** of the sequence  $y[n]$  is:

$$L_y = L_x + L_h - 1$$

Proof:

$$\begin{aligned} L_y &= (N_x + N_h) - (M_x + M_h) + 1 \\ &= (N_x - M_x + 1) + (N_h - M_h + 1) - 1 = \\ &= L_x + L_h - 1 \end{aligned}$$

# **System Description using Difference Equations**

# System Description using Difference Equations

In discrete-time systems, the input-output relationship can take the form:

$$y[n] = \sum_{m=0}^M b_m x[n-m] - \sum_{k=1}^N a_k y[n-k], n \geq 0$$

- $a_k$   $b_m$  constants that define the system
- $y[-k], k = 1, \dots, N$  initial state of the output
- $x[-m], m = 1, \dots, M$  initial state of the input
- If the system is causal, that is  $x[-m] = 0 \forall m = 1, \dots, M$ , to calculate the output only the initial conditions of the output are required.
- If  $y[-k] = 0, \forall k = 1, \dots, N$ , then the system is in **initially relaxed**.

**Linear Difference Equation with Constant Coefficients (LCCDE):** for LSI system, the D.E. is linear, with constant coefficients, with zero initial conditions and the system input is zero for  $n < 0$ .

The difference equation provides the ability to describe a system with an infinite impulse response using a finite number of coefficients. This simplifies how to describe a system.

# Non-Recursive Systems

- Impulse Response with a finite number of samples: the causal LSI system is called **Finite Impulse Response (FIR)** or **non-recursive** and its output is:

$$y[n] = \sum_{m=0}^M b_m x[n - m] = \sum_{k=0}^M x[k] h[n - k]$$

- Therefore, in FIR systems the output can be calculated from the convolutional sum instead of the difference equation.
- Its impulse response is:

$$h[n] = \sum_{m=0}^M b_m \delta[n - m]$$

- The impulse response of an FIR system is a finite-duration signal. But when the input to the system is of infinite duration, then the output of the filter is also of infinite duration.

# Recursive Systems

- Impulse response with an infinite number of samples: the causal LSI system is called Infinite Impulse Response (IIR) or recursive and its output is given by:

$$y[n] = \sum_{k=0}^{+\infty} x[k] h[n - k]$$

- In recursive systems the impulse response  $h[n]$  has an infinite duration and is calculated by:

$$h[n] = \sum_{k=0}^{\infty} b_k \delta[n - k]$$



# **Study of Systems using the Method of (Linear) Convolution**

# Properties of Linear Convolution

- Homogeneity
- Commutative
- Cooperative
- Distributive
- Identity

# Homogeneity property

Convolution is a **linear** operator with the following properties

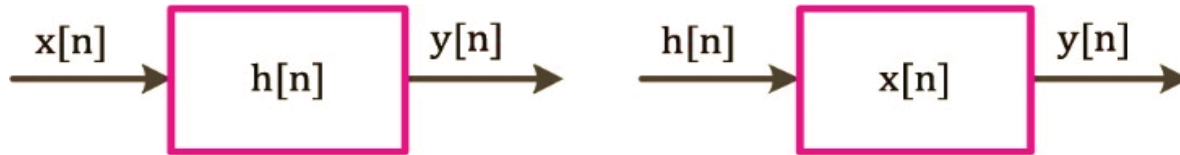
1. **Homogeneity:** Relationships apply:

$$(\alpha x[n]) * y[n] = x[n] * (\alpha y[n])$$

$$x[n] * (\alpha y[n]) = \alpha(x[n] * y[n])$$

# Commutative Property

2. Commutative:  $x[n] * h[n] = h[n] * x[n]$



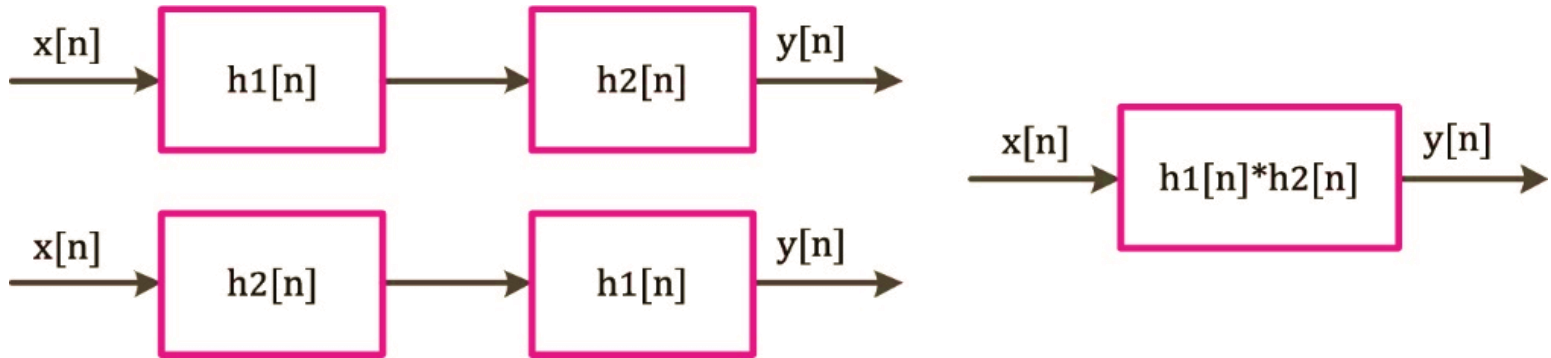
The order in which the convolution is performed does not matter.

A system with impulse response  $h[n]$  and input  $x[n]$  behaves the same as a system with impulse response  $x[n]$  and input  $h[n]$ .

# Cooperative Property

3. Cooperative:

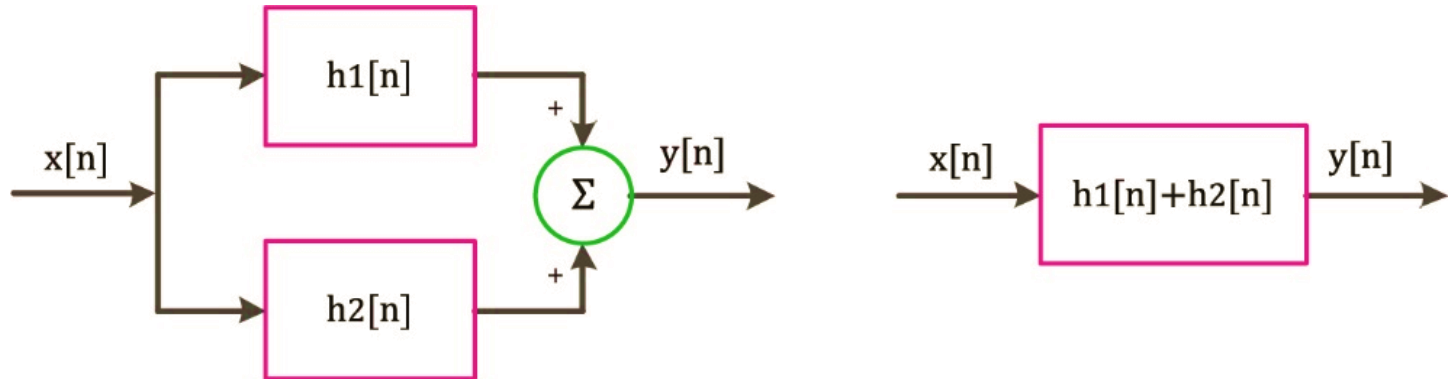
$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$



If two systems with impulse responses  $h_1[n]$  and  $h_2[n]$  are connected cascade, the resulting equivalent system has a impulse response equal to the **convolution** of  $h_1[n]$  and  $h_2[n]$ .

# Distributive and Identity Properties

4. **Distributive:**  $x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$



If two systems with impulse responses  $h_1[n]$  and  $h_2[n]$  are connected in parallel, the resulting equivalent system has a impulse response equal to the **sum** of  $h_1[n]$  and  $h_2[n]$ .

5. **Identity:**  $x[n] * \delta[n] = x[n]$

The neutral element of the convolution operation is the unit impulse function  $\delta[n]$ .

# Convolution Properties

It is easily proved that convolution is a linear and shift invariant operation, i.e. it satisfies the relations:

$$h[n] * \{a_1 x_1[n] + a_2 x_2[n]\} = a_1 y_1[n] + a_2 y_2[n]$$

$$h[n] * x[n - n_0] = y[n - n_0]$$

# Methods of Computing Linear Convolution

- Direct (analytical) calculation
- Graphical calculation
- Table method
- Calculation using Toeplitz matrix



# Convolution Calculation Methods

To calculate the linear convolution (in the time domain):

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n - k]$$

the following ways can be used:

- 1. Direct (analytical) calculation:** when the signals to be convolved are described by simple closed-form mathematical expressions.
- 2. Graphical calculation:** for signals that have an easy graphical representation and are preferably of finite length.
- 3. Table Method:** when the signals to be convoluted are of finite length and short in duration (same process as before but without graphical representation).
- 4. Calculation using Toeplitz Table.**

# Direct Convolution Calculation

It usually results in computing sums that converge or do not converge, and which contain terms of the form  $\alpha^n$  or  $n\alpha^n$ .

Below are some closed type expressions for the most commonly used series:

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1 - \alpha^N}{1 - \alpha}$$

$$\sum_{n=0}^{+\infty} \alpha^n = \frac{1}{1 - \alpha}, |a| < 1$$

$$\sum_{n=0}^{N-1} n \alpha^n = \frac{(N - 1)\alpha^{N+1} - N\alpha^N + \alpha}{(1 - \alpha)^2}$$

$$\sum_{n=0}^{+\infty} n \alpha^n = \frac{\alpha}{(1 - \alpha)^2}, |a| < 1$$

$$\sum_{n=0}^{N-1} n = \frac{1}{2}N(N - 1)$$

$$\sum_{n=N_1}^{N_2} \alpha^n = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1 - \alpha}$$

# Example 10

Calculate the output of the system with impulse response  $h[n] = u[n]$  and input signal:

$$x[n] = a^n u[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

**Answer:** We substitute the given sequences into the definition of the convolutional sum and find:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} a^k u[k] u[n-k]$$

to  $n \geq 0$ :

$$y[n] = \sum_{k=0}^n a^k = \frac{1 - a^{n+1}}{1 - a}$$

So the output is:

$$y[n] = \frac{1 - a^{n+1}}{1 - a} u[n]$$

# Example 11

Calculate the convolution between  $x[n] = (0.9)^n u[n]$  and  $h[n] = n u[n]$ .

**Answer:** Convolution is:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k] = \sum_{k=-\infty}^{+\infty} \{(0.9)^k u[k]\} \{[n-k] u[n-k]\}$$

Since  $u[k] = 0 \forall k < 0$  and  $u[n-k] = 0 \forall k > n$ , we have:

$$y[n] = \sum_{k=0}^n [n-k] (0.9)^k = n \sum_{k=0}^n (0.9)^k - \sum_{k=0}^n k(0.9)^k \quad \forall n \geq 0$$

Using the series formulas, we have:

$$\begin{aligned} y[n] &= n \frac{1 - (0.9)^{n+1}}{1 - 0.9} - \frac{n(0.9)^{n+2} - [n+1](0.9)^{n+1} + 0.9}{[1 - 0.9]^2} \\ &= 10n\{1 - (0.9)^{n+1}\} - 100\{n(0.9)^{n+2} - [n+1](0.9)^{n+1} + 0.9\} \quad n \geq 0 \\ &= \{10n - 90 + 90(0.9)^n\} u[n] \end{aligned}$$

# Graphical Convolution Calculation

To calculate the convolution **graphically**, we follow the following steps:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n - k]$$

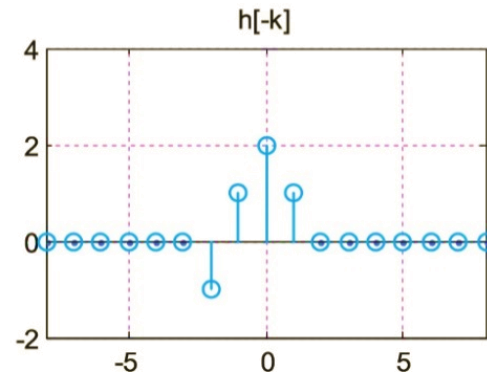
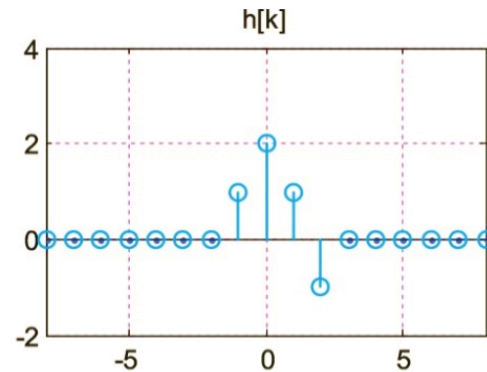
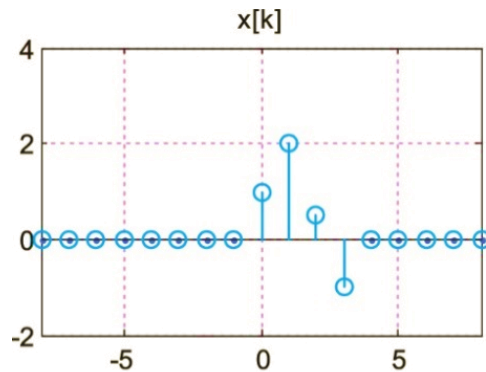
1. We draw the sequence  $x[k]$ .
2. We draw the reflection  $h[-k]$  in terms of  $k = 0$ , of  $h[k]$ .
3. We shift it  $h[-k]$  by  $n_0$  points and get the time shift  $h[n_0 - k]$  which we are drawing.
4. We multiply the graphs of  $x[k]$  and  $h[n_0 - k]$  and calculate the partial product  $u_{n_0}[k] = x[k] h[n_0 - k]$ .
5. For each value of  $k$  we calculate the total sum of the individual partial products and find the value of  $y[n_0] = \sum_{k=-\infty}^{+\infty} x[k] h[n_0 - k]$ .

We repeat steps 3, 4 and 5 for the next value of  $n_0$ . We stop when the sum of the products in step 4 gives zero for all the remaining values of  $k$ .

# Example 12

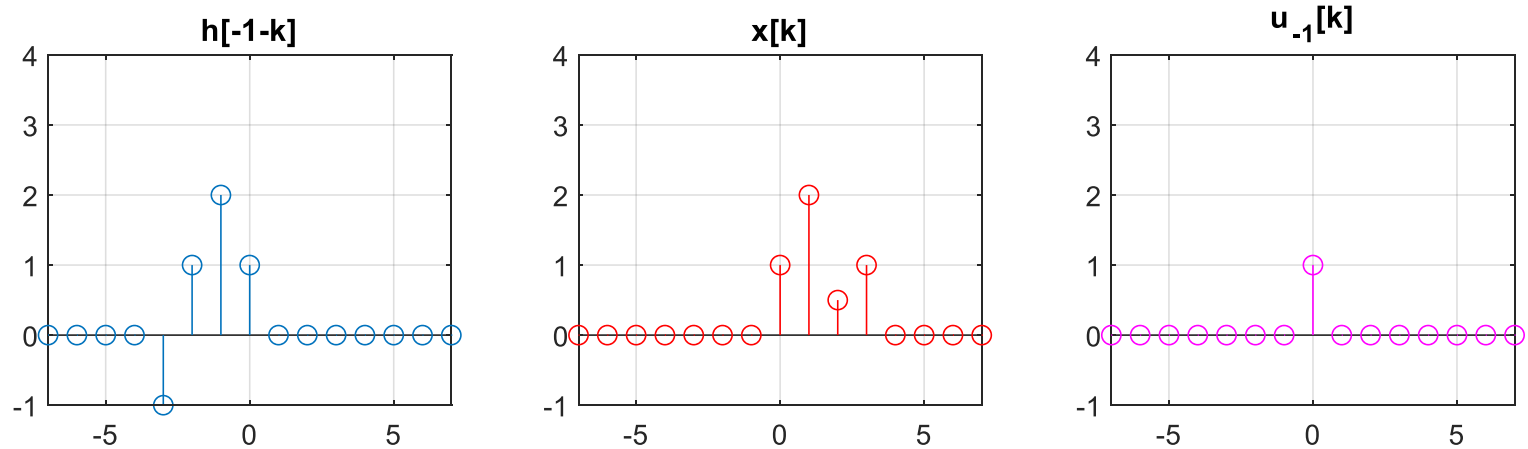
Calculate the convolution between the signals  $h[n] = \{\dots 0, 1, \hat{2}, 1, 1, 0 \dots\}$  and  $x[n] = \{\dots 0, \hat{1}, 2, 0.5, -1, 0 \dots\}$  with the graphical method.

**Answer:** We draw the signals  $x[k]$  και  $h[k]$  in their original form, as well as the reflection  $h[-k]$  with respect to  $k = 0$ .



# Example 12 (continued)

We plot the signals for time-shift  $n = -1$ , i.e.  $h[-1 - k]$  and the product  $u_{-1}[k] = x[k] h[-1 - k]$ .

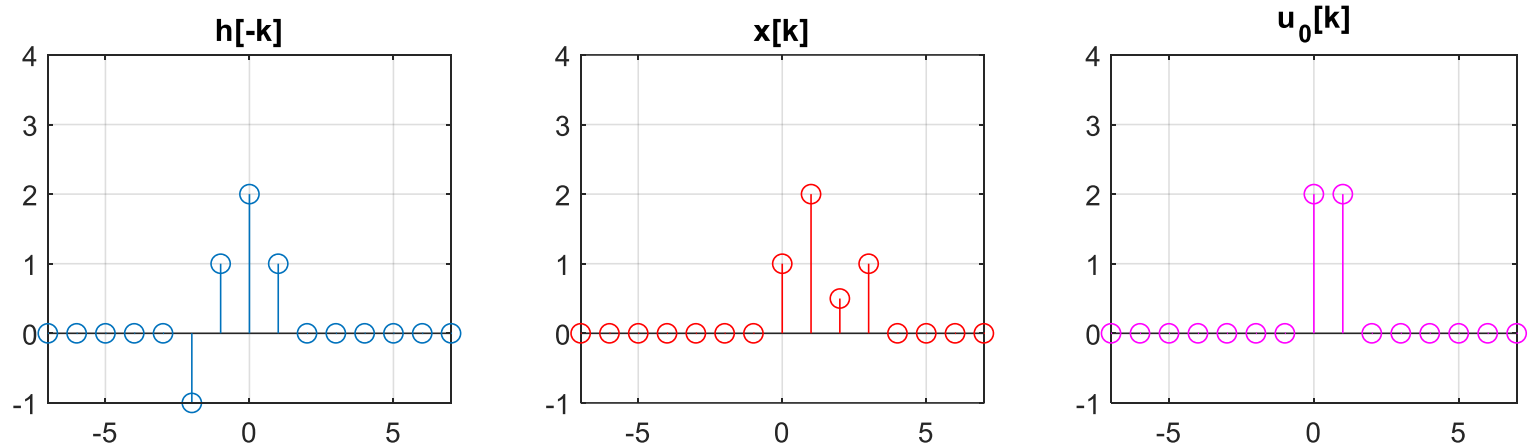


The value of the output  $y[n]$  is calculated by summing the samples of the signal  $u_{-1}[k]$  i.e.:

$$y[-1] = \sum_{k=-\infty}^{+\infty} u_{-1}[k] = 1$$

# Example 12 (continued)

We draw the signals for time-shift  $n = 0$ , i.e.  $h[-k]$  and the product  $u_0[k] = x[k] h[-k]$ .



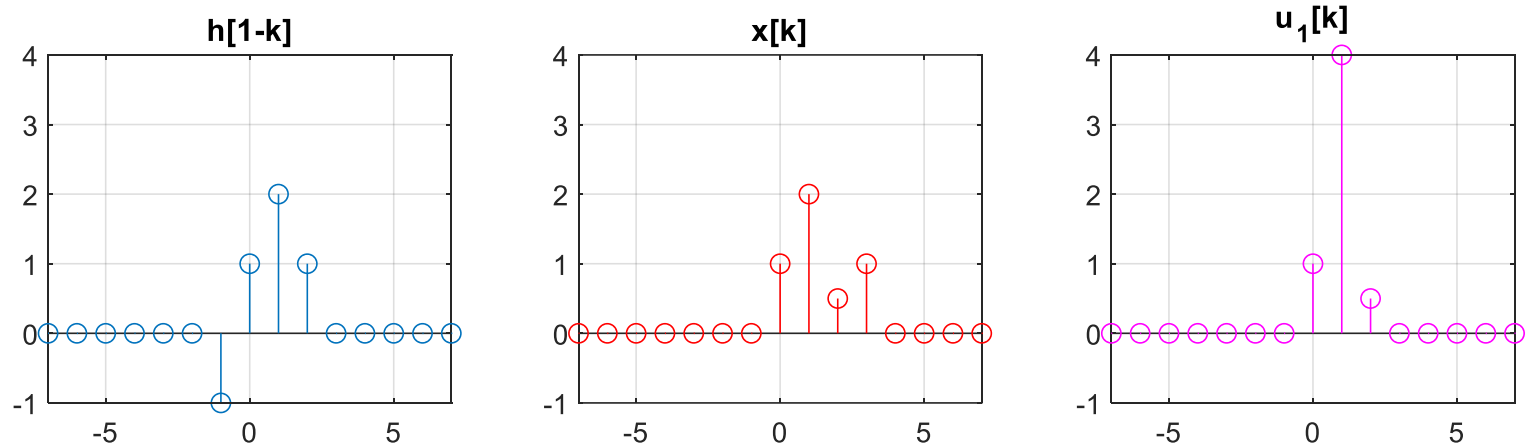
The value of the output  $y[n]$  is calculated by summing the samples of the signal  $u_0[k]$  i.e.:

$$y[0] = \sum_{k=-\infty}^{+\infty} u_0[k] = 4$$



# Example 12 (continued)

We plot the signals for time-shift  $n = 1$ , i.e.  $h[1 - k]$  and the product  $u_1[k] = x[k] h[1 - k]$ .

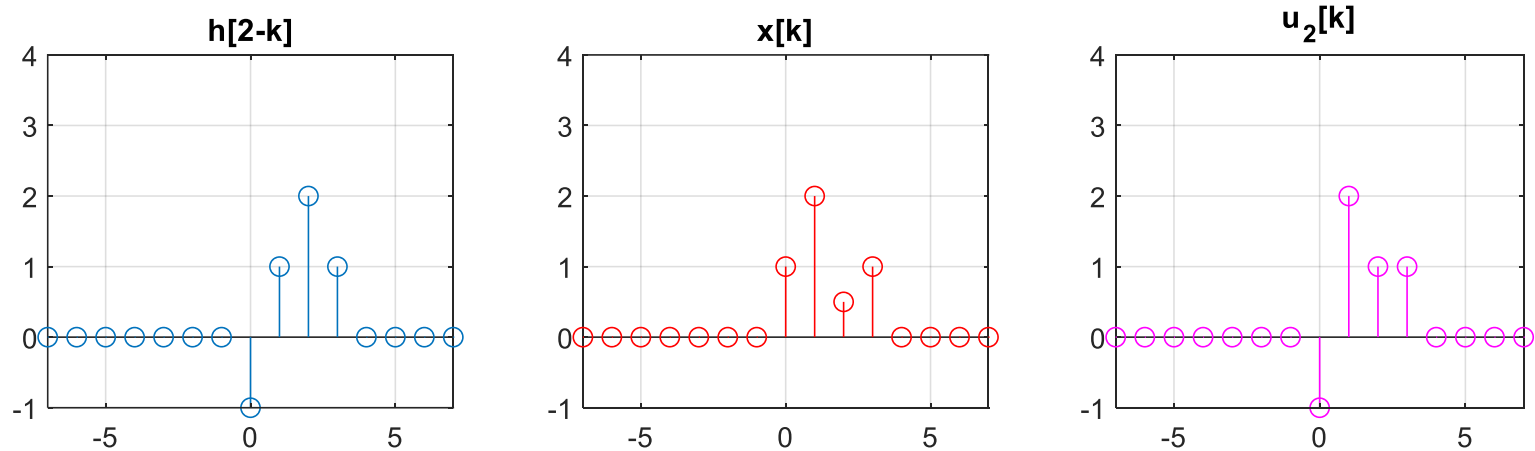


The value of the output  $y[n]$  is calculated by summing the samples of the signal  $u_1[k]$  i.e.:

$$y[1] = \sum_{k=-\infty}^{+\infty} u_1[k] = 5.5$$

# Example 12 (continued)

We plot the signals for time-shift  $n = 2$ , i.e.  $h[2 - k]$  and the product  $u_2[k] = x[k] h[2 - k]$ .



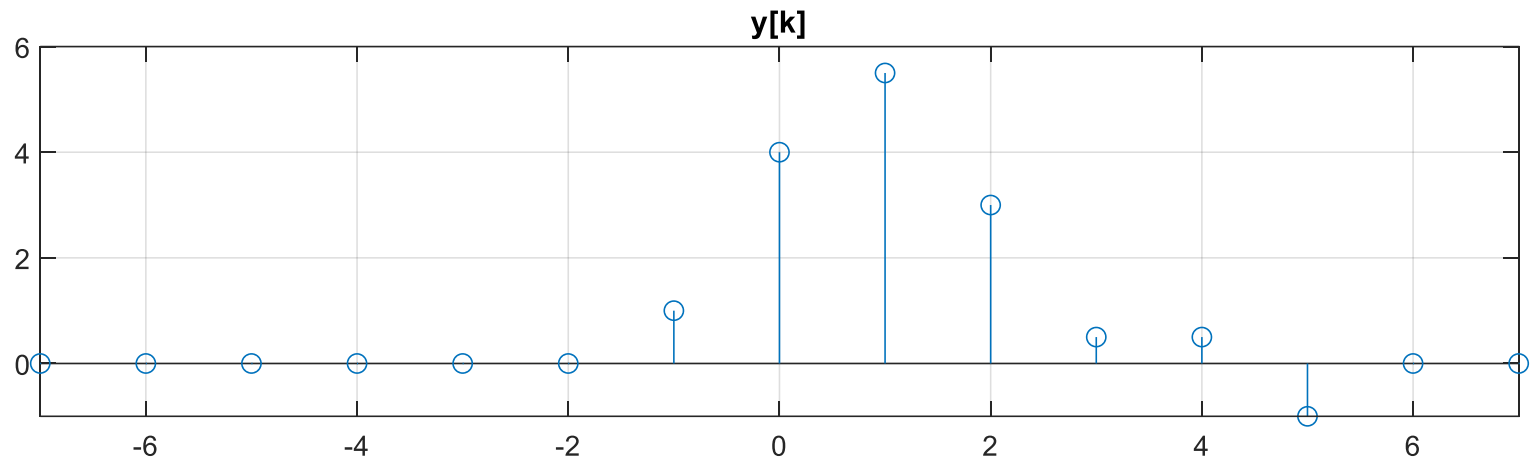
The value of the output  $y[n]$  is calculated by summing the samples of the signal  $u_2[k]$  i.e.:

$$y[2] = \sum_{k=-\infty}^{+\infty} u_2[k] = 3$$

# Example 12 (continued)

Similarly for different time-shifts the result is obtained:

$$y[n] = x[n] * h[n] = \{ \dots 0, 1, \hat{4}, 5.5, 3, 0.5, 0.5, -1, 0, \dots \}$$



# Example 13

To find the convolution between the signals  $h[n] = n\{u[n] - u[n - 5]\} + 1$  and  $x[n] = u[n + 2] - u[n - 2]$  using the table method.

**Answer:** We write the given signals in the form of sequences. Is:  
 $h[n] = \{\dots 0, \hat{1}, 2, 3, 4, 5, 0 \dots\}$ ,  $x[n] = \{\dots 0, 1, 1, \hat{1}, 1, 1, 0 \dots\}$ .

We create the table of the next slide:

- In the first line we depict the time scale, let  $-7 \leq k \leq 7$ .
- In the next two lines we place the given sequences  $x[k]$  and  $h[k]$  in the next one the reflection  $h[-k]$  with respect to  $k = 0$ .
- In the following lines we put time-shifts by  $n_0$  the sequence  $h[-k]$ , that is, the sequences  $h[n_0 - k]$ , where  $n_0 = \pm 1, \pm 2, \pm 3, \dots$ .
- We repeat the process for as many values of  $n_0$  the sequences  $x[k]$  and  $h[n_0 - k]$  overlap.

# Example 13 (continued)

|         |           |    |    |    |    |    |    |   |   |   |   |   |   |   |   |     |        |           |
|---------|-----------|----|----|----|----|----|----|---|---|---|---|---|---|---|---|-----|--------|-----------|
| $k$     | -7        | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |     |        |           |
| $x[k]$  |           |    |    |    |    | 1  | 1  | 1 | 1 | 1 |   |   |   |   |   |     |        |           |
| $h[k]$  |           |    |    |    |    |    |    | 1 | 2 | 3 | 4 | 5 |   |   |   |     |        |           |
| $h[-k]$ |           |    |    | 5  | 4  | 3  | 2  | 1 |   |   |   |   |   |   |   |     |        |           |
| $n_0$   |           |    |    |    |    |    |    |   |   |   |   |   |   |   |   | $n$ | $y[n]$ |           |
| -3      | $h[-3-k]$ | 5  | 4  | 3  | 2  | 1  |    |   |   |   |   |   |   |   |   |     | -3     | <b>0</b>  |
| -2      | $h[-2-k]$ |    | 5  | 4  | 3  | 2  | 1  |   |   |   |   |   |   |   |   |     | -2     | <b>1</b>  |
| -1      | $h[-1-k]$ |    |    | 5  | 4  | 3  | 2  | 1 |   |   |   |   |   |   |   |     | -1     | <b>3</b>  |
| 0       | $h[-k]$   |    |    |    | 5  | 4  | 3  | 2 | 1 |   |   |   |   |   |   |     | 0      | <b>6</b>  |
| 1       | $h[1-k]$  |    |    |    |    | 5  | 4  | 3 | 2 | 1 |   |   |   |   |   |     | 1      | <b>10</b> |
| 2       | $h[2-k]$  |    |    |    |    |    | 5  | 4 | 3 | 2 | 1 |   |   |   |   |     | 2      | <b>15</b> |
| 3       | $h[3-k]$  |    |    |    |    |    |    | 5 | 4 | 3 | 2 | 1 |   |   |   |     | 3      | <b>14</b> |
| 4       | $h[4-k]$  |    |    |    |    |    |    |   | 5 | 4 | 3 | 2 | 1 |   |   |     | 4      | <b>12</b> |
| 5       | $h[5-k]$  |    |    |    |    |    |    |   |   | 5 | 4 | 3 | 2 | 1 |   |     | 5      | <b>9</b>  |
| 6       | $h[6-k]$  |    |    |    |    |    |    |   |   |   | 5 | 4 | 3 | 2 | 1 |     | 6      | <b>5</b>  |
| 7       | $h[7-k]$  |    |    |    |    |    |    |   |   |   |   | 5 | 4 | 3 | 2 | 1   | 7      | <b>0</b>  |

## Example 13 (continued)

From the above table we observe that for values  $n_0 \leq -3$  and  $n_0 \geq 7$  the sequences  $x[k]$  and  $h[n_0 - k]$  do not overlap. Thus  $x[k] h[n_0 - k] = 0$ , because at least one of the two sequences is zero. Therefore, for the given example the values of  $n_0$  which produce a non-zero result are those which satisfy the inequality  $-2 \leq n_0 \leq 6$ .

Then, for a value  $n_0$  of satisfying the inequality, we multiply point by point the sequences  $h[n_0 - k]$  and  $x[k]$  and obtain the product  $x[k] h[n_0 - k]$ . We add the products for  $-7 \leq k \leq 7$  and find the point  $y[n_0] = \sum_{k=-7}^7 x[k] h[n_0 - k]$ .

We repeat this process for a new value of  $n_0$  until we finish the values of  $n_0$  which produce a non-zero result  $y[n_0]$ .

We depict them  $y[n_0]$  in the last column of the table and we have:

- $y[-3] = 0, \quad y[-2] = 1, \quad y[-1] = 3, \quad y[0] = 6, \quad y[1] = 10, \quad y[2] = 15$
- $y[3] = 14, \quad y[4] = 12, \quad y[5] = 9, \quad y[6] = 5, \quad y[7] = 0$

Therefore:  $y[n] = \{ \dots 0, 1, 3, \hat{6}, 10, 15, 14, 12, 9, 5, 0, \dots \}$

# Calculation using Toeplitz Table

The linear convolution of two sequences  $x[n]$  of length  $L_x$  and  $h[n]$  of length  $L_h$  is a sequence  $y[n]$  of length  $L_y = L_x + L_h - 1$ , which can be expressed in matrix form:

$$\mathbf{y}^T = \mathbf{H} \mathbf{x}^T$$

- the vector  $\mathbf{x}^T$  is the inverse matrix of  $x[n]$ , that is, it has the signal values  $x[n]$  and is dimensional  $[L_x, 1]$ ,
- the matrix  $\mathbf{H}$  has the signal values  $h[n]$  folded and shifted and is of dimensions  $[L_y, L_x]$ , and
- the vector  $\mathbf{y}^T$  has the values of the convolution  $y[n]$  and is dimensional.

# Example 14

Using a Toeplitz table calculate the convolution between the signals  $x[n] = \{\hat{1}, -2, 0, 3, -1\}$  and  $h[n] = \{2, \hat{3}, 0, 1\}$ .

**Answer:** The signal  $x[n]$  is of finite duration in space  $[-1, 2]$  with length  $L_x = 5$ .

Therefore, the convolution is of finite duration in the time interval  $[0 + (-1), 4 + 2] = [-1, 6]$  and has a length equal to:

$$L_y = L_x + L_h - 1 = 5 + 4 - 1 = 8 \text{ samples}$$

The vector  $x$  has dimensions  $[L_x, 1] = [5, 1]$  and are:  $x = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ -1 \end{bmatrix}$



# Example 14 (continued)

The table  $\mathbf{H}$  has dimensions  $[L_y, L_x] = [8, 5]$  and are:  $H =$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 1 & 0 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We calculate the vector  $\mathbf{y}^T$ :

$$\mathbf{y}^T = \mathbf{H} \mathbf{x}^T = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ 2 & 1 & 3 & 2 & 0 \\ 0 & 2 & 1 & 3 & 2 \\ 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ -1 \end{bmatrix} = \dots = \begin{bmatrix} 2 \\ -1 \\ -6 \\ 7 \\ 5 \\ -3 \\ 3 \\ -1 \end{bmatrix}$$

Therefore the convolution is:

$$y[n] = \{2, -\hat{1}, -6, 7, 5, -3, 3, -1\}$$