University of the Peloponnese
Electrical and Computer Engineering Department

Digital Signal Processing

Unit 02: Discrete Time Signals

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- Analog vs Digital Processing Differences
- Generation of Discrete Time Signals
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Signals and Systems

Signal:

Any physical quantity that varies with time, space and any other independent variable, e.g.

$$x_1(t) = 5t, \qquad x_2(t) = \sum_{i=1}^{N} A_i(t)\sin[2\pi F_i(t) + \theta_i(t)]$$

System:

An entity that performs an operation (transformation) on an input signal and produces one (or more) output signals.

Analog vs Digital Processing Differences

Digital signal processing:

- Flexibility in design
- Predicted accuracy
- Homogeneity in the performance of products of the same type
- Reduced implementation costs, reliability
- Implementation of complex algorithms
- Ability to store on magnetic media

Analog signal processing:

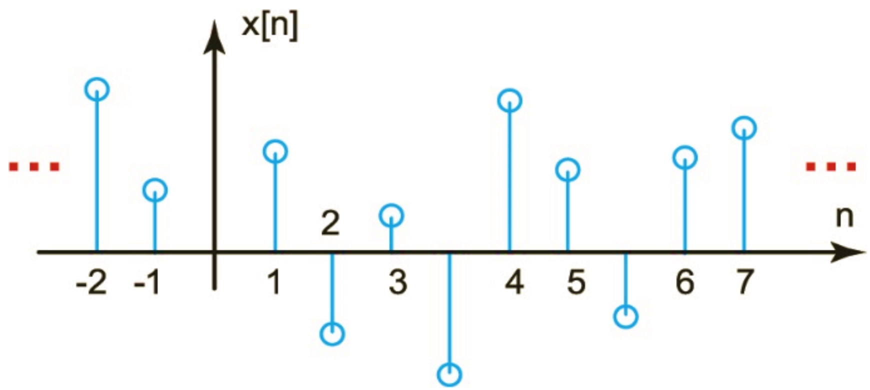
- Communication of digital systems with the analog environment
- High frequency applications

Discrete Time Signals

A **discrete time signal** (discrete-time systems)$x[n]$ is expressed mathematically by the equation:

$$x[.]\colon\ I \to R(C)$$
$$n \qquad x[n]$$

The discrete time signal$x[n]$ is a sequence of real or complex numbers. The independent variable n represents (usually) time and takes only integer values. For non-integer values of $n$the sign is undefined.



Graphical representation of a discrete-time signal

discrete-time systems notation in **vector form** $\boldsymbol{x} = [x(0), x(1), \ldots., x(N-1)]^T$

Digital Signal Production

(a) Analog Signal to Digital Conversion

The discrete-time systems $x[n]$ usually produced with the help of a Digital-to-Analog-Converter. In this process, a continuous time signal (CTS) $x(t)$is **sampled at** a rate of $f_s = 1/T_s$ samples/ sec and the discrete-time systems is produced $x[n]$. The process is described by the equation:

$$x[n] \triangleq x(nT_s) = x(t)\Big|_{t=nT_s}$$

(b) Signal production in primary digital form:

In some cases, the discrete-time systems are primarily created in a discrete form, e.g. the alphanumeric symbols produced while typing, the price of a stock on consecutive days, population statistics of a city, etc.

Discrete Time Signal Classification

- Periodic and Non-Periodic Discrete-Time Signals
- Even and Odd Discrete Time Signals
- Causal and Anticausal Signals
- Energy Signals and Power Signals

Periodic and Non-Periodic Discrete-Time Signals

discrete-time systems $x[n]$ is called **periodic** when

- It is defined for all values of n, where $-\infty < n < \infty$, that is, has **infinity duration**,
- There exists a positive integer N, such that for every integer k satisfy the equation:

$$x[n + kN] = x[n]$$

Otherwise, the discrete-time systems is called **non-periodic** or **aperiodic.**

The smallest positive integer N is called **the fundamental period.**

Periodic sequence $x[n]$ with period Nrepeats itself each time Nsamples will appear.

A periodic discrete-time systems with period N, is periodic with period $2N$, $3N$ and in general with every integer multiple of$N$.

From any sequence $x[n]$ we can always create a **periodic signal** $y[n]$ with period N, repeating it $x[n]$ as follows:

$$y[n] \ = \sum_{k=-\infty}^{+\infty} x[n - kN]$$

Periodic and Non-Periodic Discrete-Time Signals

Periodic discrete-time sinusoids with fundamental period N, defined by the equation:

$$x[n] = A\,cos\left(\frac{2\pi m}{N}n + \theta\right) = Acos(2\pi f_0 n + \theta), \qquad -\infty < n < \infty$$

where:

- $\omega_0 = 2\pi m/N$(rad) is the discrete circular frequency,
- θ is the phase and positive integers m and N are not divisible by each other.

The quotient m/N can be set as a variable $f_0 = m/N$, which is called **the discrete frequency.**

Observations on Periodic Discrete-Time Signals

- The definition of periodic discrete-time signals is the same as the definition of periodic continuous-time signals except for the fundamental period, which in discrete-time systems is <u>an integer.</u>
- Shifting a sinusoidal sequence by an integer multiple of the fundamental period does not change the sequence, since:

$$x[n+kN] = A\,cos\left(\frac{2\pi m}{N}(n+kN)+\theta\right) = Acos\left(\frac{2\pi m}{N}n + 2\pi\, mk + \theta\right) = x[n]$$

- If the discrete frequency is not in the form $2\pi m/N$ then the sine is not periodic. Continuous-time sinusoidal signals, are <u>always periodic.</u>
- Therefore, even if a discrete-time sine has resulted from sampling a continuous-time sine <u>it is not guaranteed to be periodic.</u>

Observations on Periodic Discrete-Time Signals

- The discrete frequencies repeat every 2π, that is, it applies $\omega = \omega + 2k\pi$ to every integer k. So, we only need to define the digital frequencies in the range $-\pi < \omega < \pi$, unlike the analog ones we define them for $-\infty < \Omega < \infty$.
- Unit of measurement of discrete frequency ω is radians (rad), while unit of measurement of analog frequency Ω is rad / sec.
- When we sample a continuous-time sinusoidal signal $x(t) = A\,cos(\Omega_0 t + \theta),\ -\infty < n < \infty$ we get a discrete-time **periodic sinusoidal signal:**

$$x[n] = A\,cos(\Omega_0 n T_s + \theta) = A cos\left(\frac{2\pi T_s}{T_0} n + \theta\right)$$

only if:

$$\frac{T_s}{T_0} = \frac{m}{N}$$

where m and $N$are positive integers that are not divisible by each other.

- In order for the phenomenon of frequency folding not to appear, the sampling period must also satisfy the Nyquist criterion:

$$T_s \leq \frac{\pi}{\Omega_0} = \frac{T_0}{2}$$

Sum and Product of Periodic Signals DT

If $x_1[n]$ και $x_2[n]$ are periodical discrete-time systems with periods N_1 and N_2, respectively, then the discrete-time systems $y[n] = x_1[n] + x_2[n]$ is periodic if for the reason of the periods:

$$\frac{N_2}{N_1} = \frac{p}{q}$$

where p and q are integers that have no common divisor.

If this is true, then its fundamental period $y[n]$ is given by the equation:

$$N = \frac{N_1 N_2}{\text{highest common factor}(\mathrm{N}_1, \mathrm{N}_2)}$$

Also, the product $z[n] = x_1[n]\, x_2[n]$ is periodic with period N.

Example 1

Investigate whether the following sequences are periodic. If positive, calculate their fundamental period and draw the sequences:

$$\text{(a) } x_1[n] = cos(0.25\pi n) \quad \text{(b) } x_2[n] = sin(\pi + 0.5n)$$

$$\text{(c)} x_3[n] = e^{j\pi n/8} cos(\pi n/11)$$

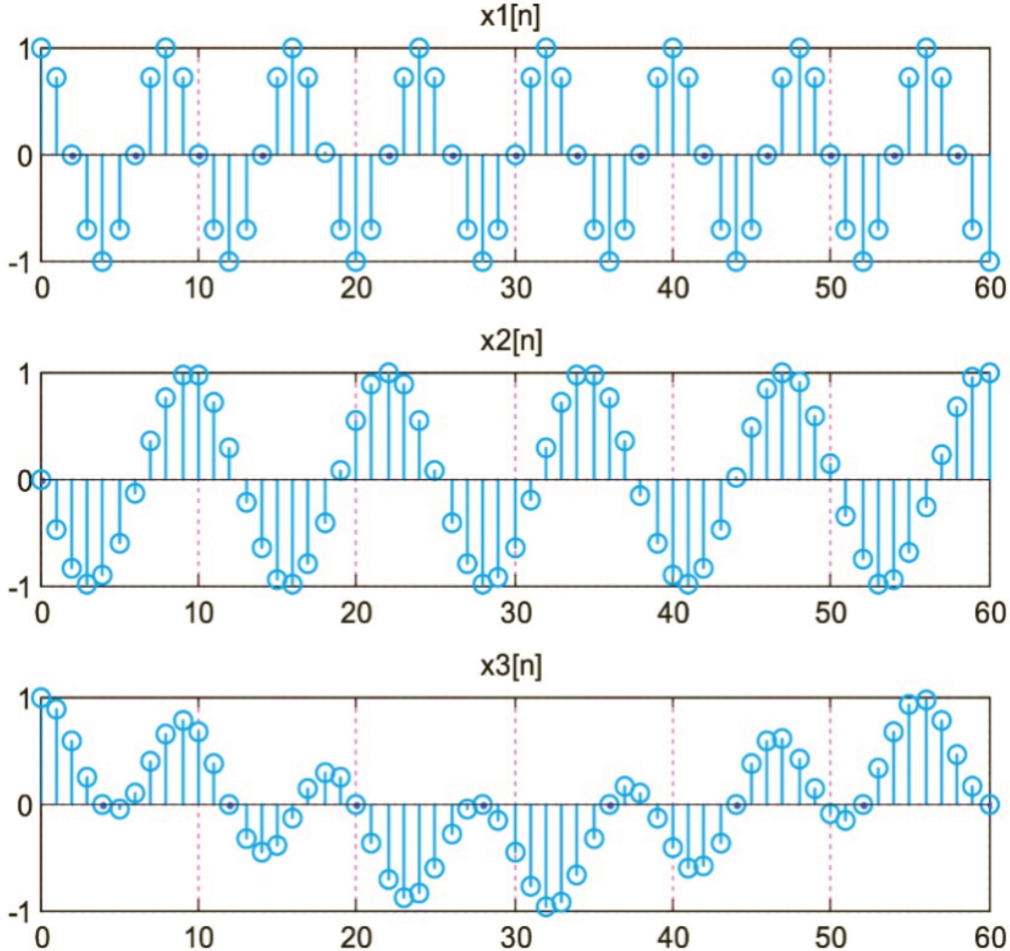
Answer: (a) Since $0.25\,\pi = \pi/4$ and holds $cos(\pi n/4) = cos(\pi(n+8)/4)$, it follows that $x_1[n]$is a period with fundamental period $N_1 = 8$.

(b) For be periodic $x_2[n]$, a value N must be found such that the equation is satisfied $sin(\pi + 0.5n) = sin(\pi + 0.5(n + N))$. The function sin() is periodic with period 2π. The quantity $0.5N$ must be an integer multiple of 2π. Since π is an irrational number, there is no integer value N for it to verify the equality. So, it $x_2[n]$ is non-periodic.

(c) The signal $x_3[n]$ is the product of the sequences $e^{j\pi n/8}$ και $cos(\pi n/11)$. Both sequences are periodic, with periods $N_1 = 16$ and $N_2 = 22$, respectively. So the product $x_3[n]$ is also periodic with period N_3:

$$N_3 = \frac{16.22}{\text{highest common factor}(16, 22)} = \frac{352}{2} = 176$$

Example 1 (continued)



(a)

(b)

(c)

Signals: (a) $x_1[n] = cos(0.25\pi n)$,
(b) $x_2[n] = sin(\pi + 0.5n)$, (c)$x_3[n] = e^{j\pi n/8}\, cos(\pi n/11)$

Example 2

Investigate whether the following sequence is periodic and, if so, determine its fundamental period.

$$x[n] = Re\left\{e^{\frac{jn\pi}{12}}\right\} + Im\left\{e^{\frac{jn\pi}{18}}\right\}$$

Answer: $x[n]$ is the sum of two periodic signals:

$$x[n] = \cos\left(\frac{n\pi}{12}\right) + \sin\left(\frac{n\pi}{18}\right)$$

Periods are $N_1 = 24$ and $N_2 = 36$.

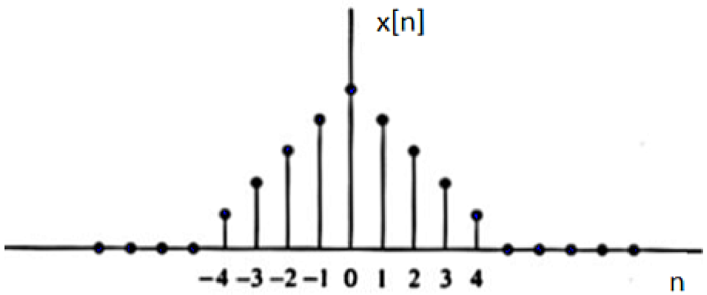
Therefore, the period of the sum, i.e. of xn is:

$$N = \frac{N_1 N_2}{\text{ΜΚΔ}(N_1, N_2)} = \frac{24.36}{\text{ΜΚΔ}(24{,}36)} = \frac{864}{12} = 72$$

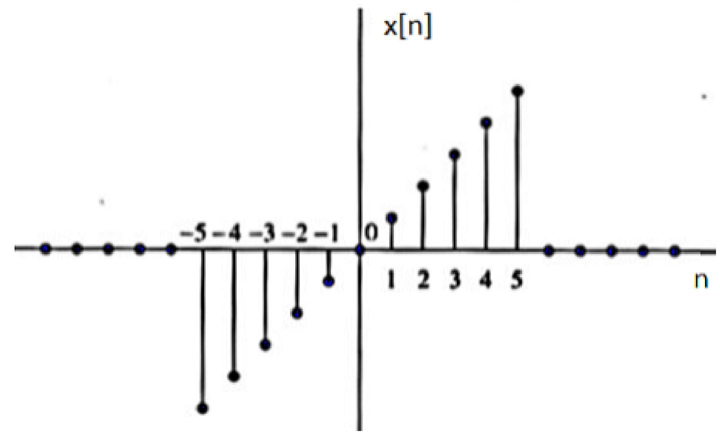
Even and Odd Discrete Time Signals

Perfect symmetry: $x[n] = x[-n], \ \forall n$

Unnecessary symmetry: $x[n] = -x[-n], \ \forall n$



(a)



- Even signals have a plot symmetrical about the vertical axis.
- Odd signals have a diagram symmetrical about the point of intersection (center) of the axes.

Even and Odd Discrete Time Signals

Each signal $x[n]$ can be written as the sum of an even $x_e[n]$ and an odd component $x_o[n]$, according to the equation:

$$x[n] = x_e[n] + x_o[n]$$

The even $x_e[n]$ and odd $x_o[n]$components of the signal $x[n]$are given by the relations:

$$x_e[n] = \frac{1}{2}\big[x[n] + x[-n]\big] \quad \text{and} \quad x_o[n] = \frac{1}{2}\big[x[n] - x[-n]\big]$$

If the signal $x[n]$ is **complex** the symmetries are defined similarly. In particular, a complex sequence exhibits **even symmetry** if it satisfies the equation:

$$x[n] = x^*[-n], \qquad \forall n$$

A complex sequence displays **redundant symmetry** if it satisfies the equation:

$$x[n] = -x^*[-n], \qquad \forall n$$

Example 3

Find the even and the odd part of the discrete-time systems: $x[n] = u[n]$.

Answer: The even part is given by the equation:

$$x_e[n] = \frac{1}{2}\big[x[n] + x[-n]\big] = \frac{1}{2}[u[n] + u[-n]] =$$

$$= \begin{cases} 1, & n = 0 \\ 1/2, & n \neq 0 \end{cases} = \frac{1}{2} + \delta[n]$$

The odd part is given by the equation:

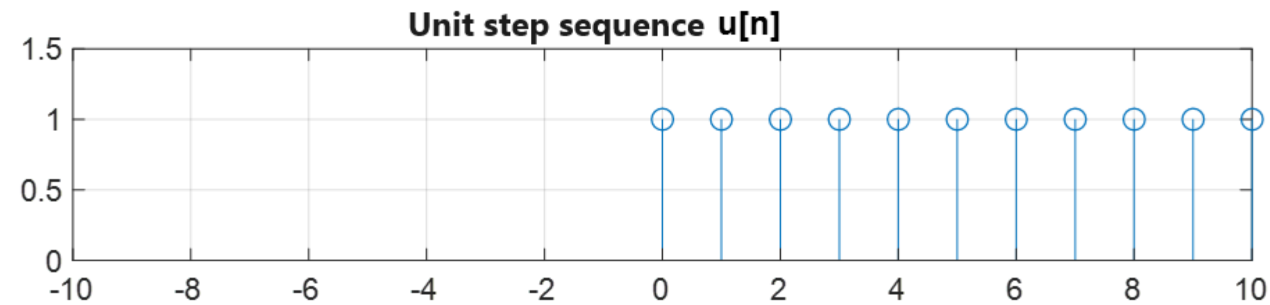
$$x_o[n] = \frac{1}{2}\big[x[n] - x[-n]\big] = \frac{1}{2}\big[u[n] - u[-n]\big] =$$

$$= \begin{cases} 1/2, & n > 0 \\ 0, & n = 0 \\ -1/2, & n < 0 \end{cases} = \frac{1}{2} sgn(n)$$

where sgn(n) is the sign function, which returns: $+1$ when $n > 0$, 0 when $n = 0$, and -1 when $n < 0$.

Example 3 (continued)

(a)



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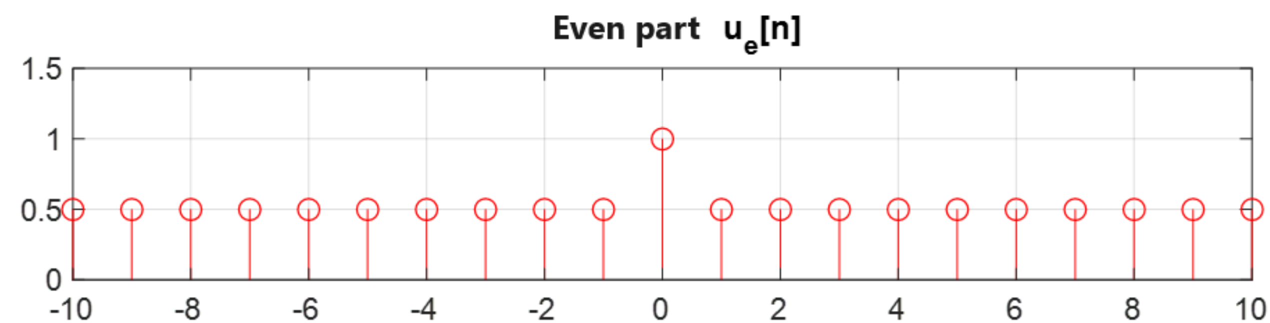
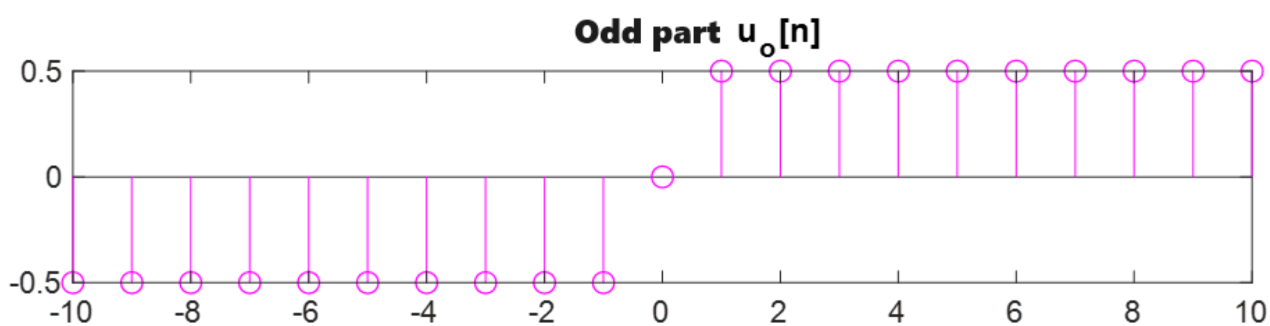


(c)



(a) Unit step sequence $u[n]$, (b) Even part $u_e[n]$, (c) Odd part $u_o[n]$

Example 4

Find the even and the odd part of the discrete-time systems:$x[n] = \alpha^n u[n]$.

Answer: The even part is given by the equation:

$$x_e[n] = \frac{1}{2}\big[x[n] + x[-n]\big] = \frac{1}{2}[a^n u[n] + a^{-n} u[-n]] =$$

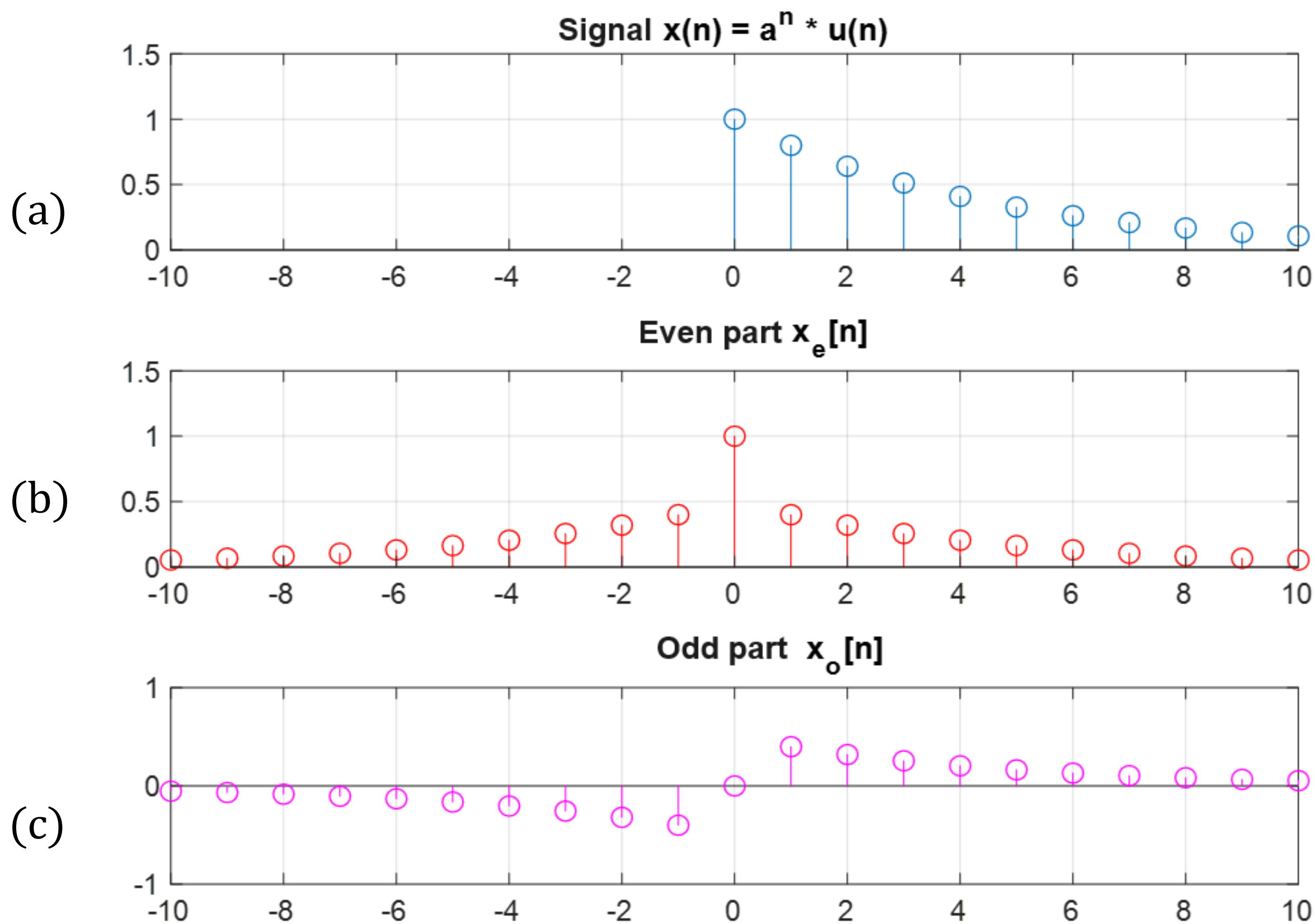
$$= \begin{cases} \frac{1}{2}a^n, & n > 0 \\ 1, & n = 0 \\ \frac{1}{2}a^{-n}, & n < 0 \end{cases} = \frac{1}{2}a^{|n|} + \delta(n)$$

The odd part is given by the equation:

$$x_o[n] = \frac{1}{2}\big[x[n] - x[-n]\big] = \frac{1}{2}\big[a^n u[n] - a^{-n} u[-n]\big] =$$

$$= \frac{1}{2}a^{|n|} sgn(n)$$

Example 4 (continued)



(a) Unit step sequence $u[n]$, (b) Even part $u_e[n]$, (c) Odd part $u_o[n]$

Example 5

If $x_1[n]$ even sign and $x_2[n]$ odd, what follows for $y[n] = x_1[n]\, x_2[n]$?

Answer:

Since $x_1[n]$ is an even sign, the following applies: $x_1[n] = x_1[-n]$

Since the $x_2[n]$ signal is odd, it holds: $x_2[n] = -x_2[-n]$

Therefore:

$$y[n] = x_1[n]\, x_2[n] = -x_1[-n]\, x_2[-n] = -y[-n]$$

So the signal $y[n]$is odd.

Characteristics Sizes of Discrete Time Signals

- Average value
- Active value
- Energy
- Instant Power
- Average Power

Characteristic Values of Discrete-Time Signals

Average Value of a discrete-time signal $x[n]$ in the range $[0, N]$:

$$\bar{x}[n] = \frac{1}{N+1}\sum_{n=0}^{N} x[n]$$

- $N + 1$ is the number of points (samples) of the signal.
- For a sinusoidal signal the mean value is zero ($\bar{x}[n] = 0$), when as the calculation period is considered to be one period.
- For a stationary signal $x[n] = A$, its average value is $\bar{x}[n] = A$.

Active value of a discrete-time signal $x[n]$ in the range of values $[0, N]$:

$$\bar{\bar{x}}[n] = \frac{1}{N+1}\left[\sum_{n=0}^{N} x^2[n]\right]^{1/2}$$

Characteristic Values of Discrete-Time Signals

Energy of a discrete-time signal $x[n]$ on the value interval $[0, N]$:

$$E_x = \sum_{n=0}^{N} x^2[n]$$

For discrete-time signals of infinite duration derived from sampling with period T_s, the following applies:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \lim_{N \to \infty} \left[T_s \sum_{n=-N}^{N} |x(n\, T_s)|^2 \right]$$

- T_s is the sampling period.
- Energy can be infinite or finite.
- A signal $x[n]$is called **an energy signal** if it contains finite energy, i.e. when applicable $0 < E_x < \infty$.

Characteristic Values of Discrete-Time Signals

Instant Power: $P[n] = x^2[n]$

Average Power of an discrete-time systems $x[n]$ in the range of values $[0, N]$:

$$P_x = \overline{P[n]} = \frac{1}{N+1} \sum_{n=0}^{N} x^2[n]$$

If the signal spans the time interval $(-\infty, +\infty)$, the following applies:

$$P_x = \overline{P[n]} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N \to \infty} \frac{1}{2N+1} E_x$$

- It $x[n]$is called **a power signal** if it is valid $0 < P < \infty$.
- The average power **of a periodic signal** is finite and equal to the average power over one period N:

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

Energy Signals and Power Signals

The signal $x[n]$ is called **a power signal** if it is valid $0 < P_x < \infty$.

The signal $x[n]$ is called **a power signal** if it is valid $0 < E_x < \infty$.

Remarks:

- A signal can be an energy signal or a power signal or neither.
  It cannot be an energy signal and a power signal at the same time.
- A signal with finite magnitude and duration is an energy signal. If the magnitude is finite but the duration is infinite then a necessary (but not sufficient) condition is that the signal magnitude decays to zero for $n \rightarrow \pm \infty$.
- An energy signal has zero power because finite energy is divided by infinite time.
- A power signal has infinite energy because its finite power is multiplied by infinite time.
- A signal that exhibits no periodicity but has infinite duration and its magnitude is absolutely blocked is a power signal.
- Signals with infinite energy can have finite average power.

Energy Signals and Power Signals

Notes on Periodic Signals:

- Periodic signals are power signals.
- If a periodic signal takes finite values, then its energy in one period is finite.
- The energy of the signal over time is infinite because it is the sum of the energy of infinite periods. Therefore a periodic signal is not energy.

Example 6

Determine the energy and average power of the unit step $u[n]$.

Answer: The signal energy $u[n]$ is calculated from the equation:

$$E_u = \sum_{n=-\infty}^{\infty} |u[n]^2| = \sum_{n=0}^{\infty} 1 = \infty$$

Because the energy is infinite, the signal is not an energy signal.

The average signal strength $u[n]$ is calculated from the relationship:

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{\infty} |u[n]^2| = \lim_{N\to\infty} \frac{N+1}{2N+1} = \lim_{N\to\infty} \frac{1+\frac{1}{N}}{2+\frac{1}{N}} = \frac{1}{2}$$

Because the average power is finite, the unit step $u[n]$ is a power signal.

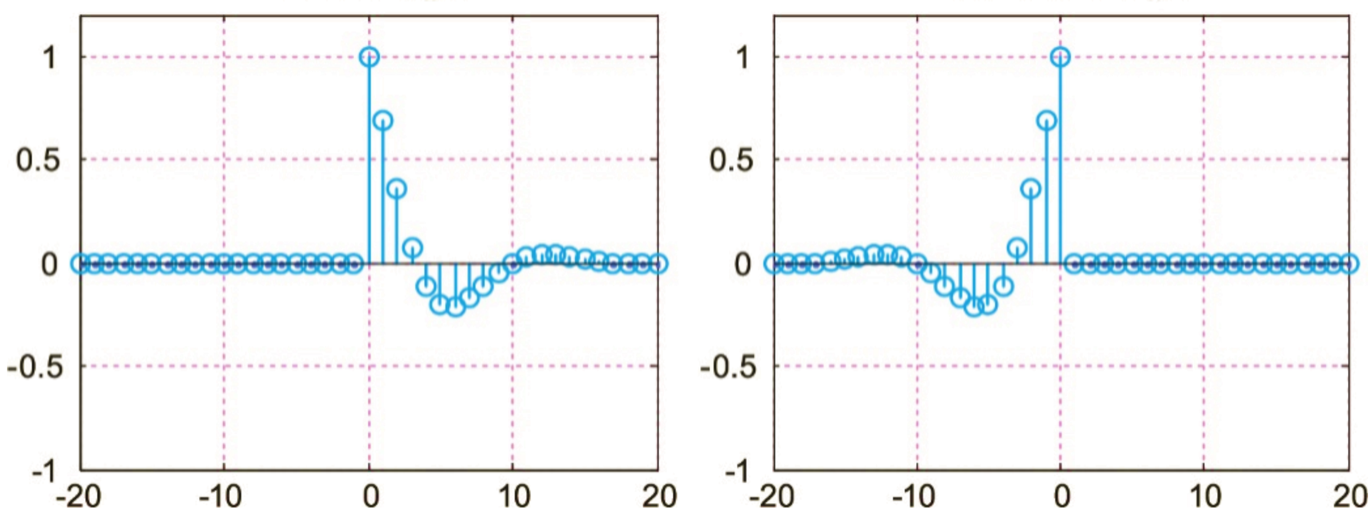
Causal and Anti -causal Signals

A discrete-time signal is called **causal** or **right-sided** if it satisfies the equation:

$$x[n] = 0, \qquad \gamma\iota\alpha\ n < 0$$

and **anti -causal** or **left-sided** if it satisfies the equation:

$$x[n] = 0, \qquad \gamma\iota\alpha\ n > 0$$



Causal and anti-causal discrete-time signal

Operations on Discrete Time Signals

- Summation
- Multiplication
- magnitude Scaling

Operations on Discrete Time Signals

1. **Summation:** $y[n] = x_1[n] + x_2[n] \qquad -\infty < n < \infty$

2. **Multiplication:** $y[n] = x_1[n]\, x_2[n] \qquad -\infty < n < \infty$

3. **magnitude Scaling:** $y[n] = a\, x[n] \qquad -\infty < n < \infty, \quad a \in \mathrm{R}$

All of the above operations affect the width of the samples.

Transformations of the Independent Variable

- Time Shift
- Reflection
- Scaling in Time

Time Shift – Time Reversal

1. Time shift or sliding of an discrete-time systems $x[n]$ occurs when we replace the independent variable $n$with the quantity $n - n_0$, that is:

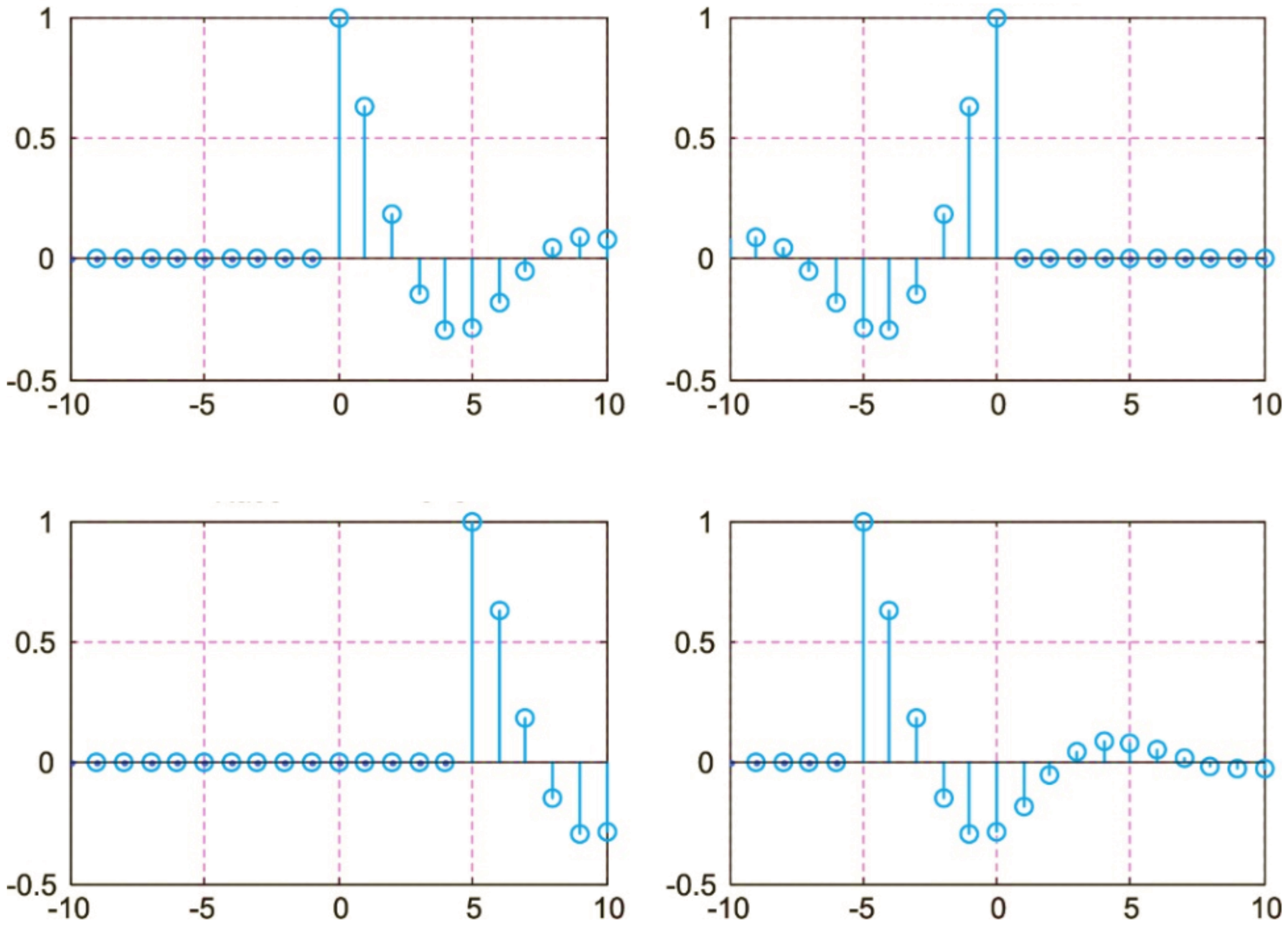
$$y[n] = x[n - n_0]$$

- If $n_o > 0$ then it is $x[n]$ shifted to the right and the sequence $y[n]$ is **delayed** relative to the sequence $x[n]$.
- If $n_o < 0$ then it $x[n]$is shifted to the left and the $y[n]$ shows **progress** (preview) relative to the sequence $x[n]$.

2. An inversion or reflection of a discrete-time systems $x[n]$ occurs when we replace the independent variable n with the quantity $-n$, i.e.:

$$y[n] = x[-n]$$

Example of Time Shift and Reversal



(a) Original signal $x[n]$, (b) Reflection $y[n] = x[-n]$
(c) Delay ($n_0 = 5$), $y[n] = x[n-5]$, (d) Advance ($n_0 = -5$), $y[n] = x[n+5]$,

Time Scaling

3. Time Scaling: Changing the time scale in discrete-time systems is more complicated than continuous-time signals, because in discrete-time systems the time contraction and expansion is related to the change in the sampling period T_s.

- If we change the sampling period from T_s to MT_s, where M integer with $M > 1$, then the number of samples will decrease. The process is called **frequency division** or **down-sampling** and is defined by the equation:

$$y[n] = x[Mn], \qquad \text{όπου } M \in N, \qquad M > 1$$

The $y[n]$ is formed by taking each time the M-order sample of $x[n]$.

- If we change the sampling period from $T_s$to T_s/N,where Ninteger with $N > 1$, then the number of samples will increase. The process is called **frequency multiplication** or **up-sampling** and is defined by the equation:

$$y[n] = x\left[\frac{n}{N}\right], \text{where } N \in N, \qquad N > 1$$

The$y[n]$ is formed by interposing N-1 zeros between each sample of $x[n]$.

Time Scaling

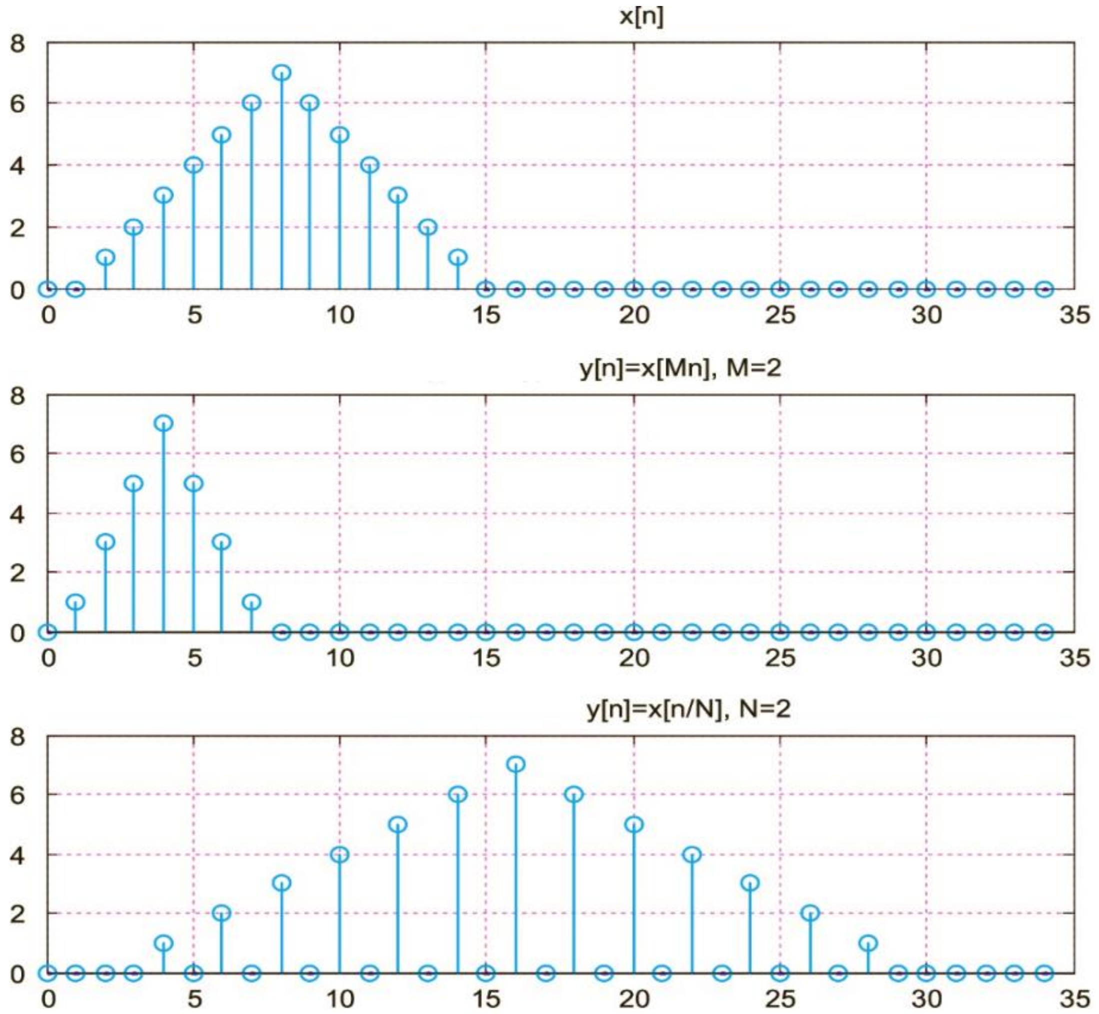
If the quotient n/N is a non-integer, then the discrete-time systems is not defined. When n/N is an integer, i.e. is n a multiple of N() $n = 0, \pm N, \pm 2N, \ldots$), we should determine its values $y[n]$ for the intermediate values of n. In this case it is $y[n]$ formed by interposing a number $N - 1$ of zero values between two consecutive samples of $x[n]$:

$$y[n] = \begin{cases} x[n/N], & n = 0, \pm L, \pm 2N, \ldots \\ 0, & \text{elsewhere} \end{cases}$$

<u>**Remarks:**</u>

- Time shifting and time scaling are not commutative, so they depend on the **order** in which they are performed.
- If the signal $x[n]$ is requested to generate $y[n] = x(an - b)$, one must first time-shift, generate the signal $z[n] = x(n - b)$, and then scale in time to $n = an$ generate the requested signal $y[n] = x(an - b)$.
- Between time scaling and inversion (reflection) the commutative property holds, so the order in which they are performed does not matter.

Example of Down-sampling and Up-sampling



(a)

(b)

(c)

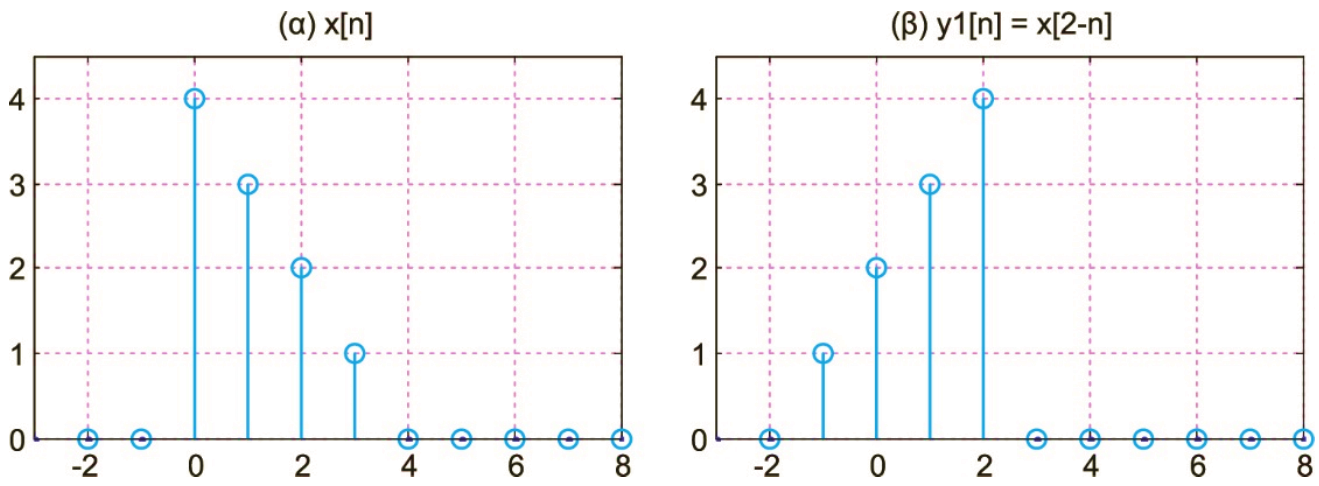
Example of (b) Down-sampling and (c) Up-sampling of a discrete-time signal

Example 7

The signal is given $x[n] = [4-n]\big[u[n] - u[n-4]\big]$. Draw the signals:

(a) $y_1[n] = x[2-n]$ (b) $y_2[n] = x[2n-1]$ (c) $y_3[n] = x[6-2n]$

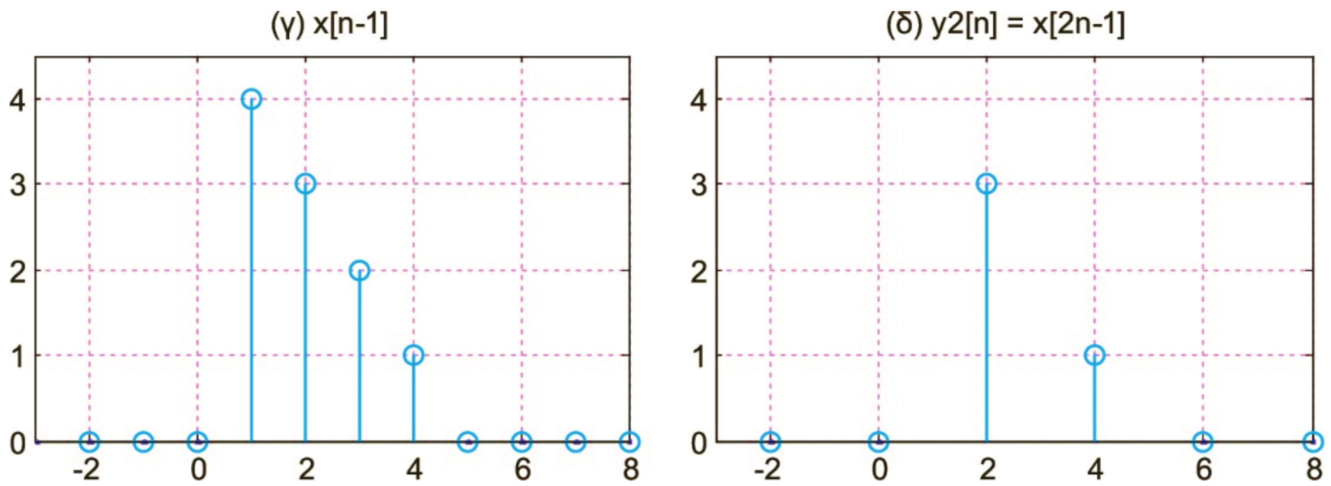
Answer: (a) The signal $x[n]$ [figure (a)], is a linearly decreasing sequence, starting at $n = 0$ and ending at $n = 4$. The signal $y_1[n] = x[2-n]$ [figure (b)] is obtained by shifting it $x[n]$ by two points and reversing it in time. For $n = 2$n $y_1[n]$ is equal to $x[0]$. Therefore, a $y_1[n]$ takes the value 4 for $n = 2$ and decreases linearly to the left up to the point $n = -1$, beyond which a $y_1[n]$ equals zero.



Signals: (a) $x[n]$, (b) $x[2-n]$

Example 7 (continued)

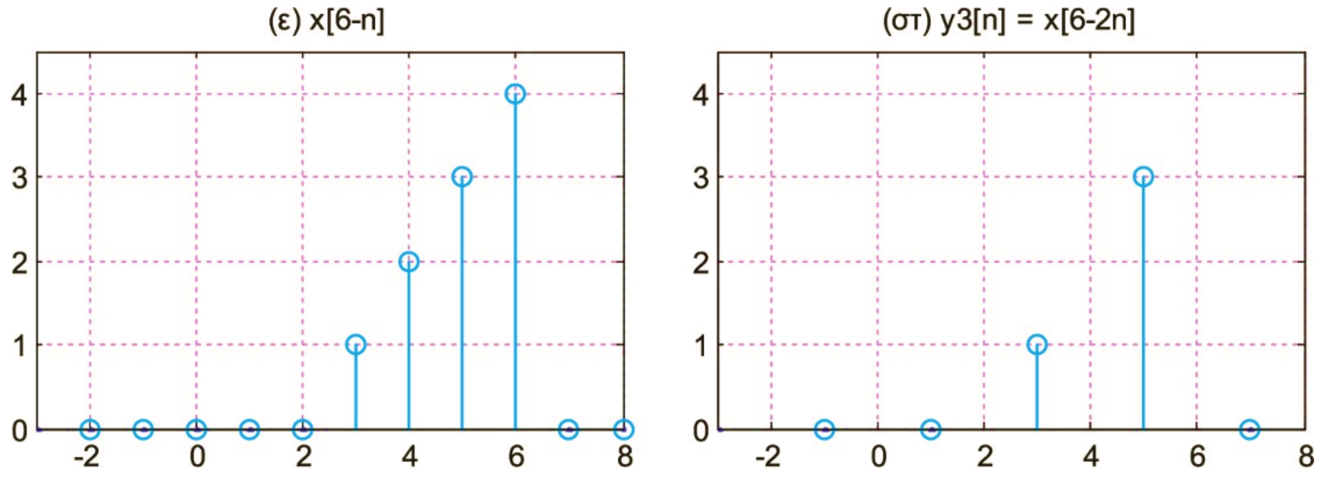
(b) The signal $y_2[n] = [2n - 1]$ [figure (c)] results from the combination of time shift and frequency division. Therefore, it $y_2[n]$ is drawn by first shifting it $x[n]$ to the right by one point (delay). Then, $y_2[n]$ frequency division by a factor of 2 is applied to the signal. Keeping only the even terms in figure (c) results in the graph of $y_2[n]$ [figure (d)].



Signals: (c) $x[n-1]$, (d) $x[2n-1]$

Example 7 (continued)

(c) The signal $y_3(n) = x[6-2n]$ [figure (e)] results from a combination of time-shifting, frequency division, and time-reversal. To represent it graphically $y_3[n]$ we start by drawing $x[6-2n]$, which is formed by shifting it $x[n]$to the left by six points (advance) and with reversal in time. Finally $y_3[n]$ [figure (f)] is found by computing every second sample of it $x[6-2n]$.



Signals: (e) $x[6-n]$ (f) $x[6-2n]$

Fundamental Discrete Time Signals

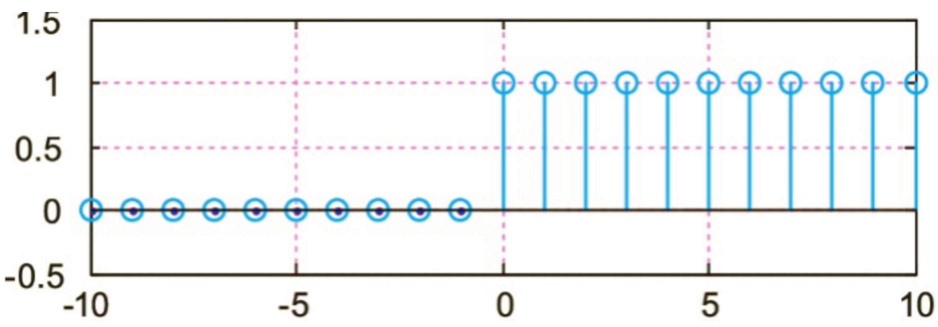
- Unit step sequence
- Unit impulse sequence
- Unit ramp sequence
- Real exponential sequence
- Complex exponential sequence
- Sine sequence

Unit Step Sequence

Unit step sequence:

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Relative to the corresponding continuous-time step function $u(t)$, the discrete-time unit step $u[n]$ does not exhibit any mathematical discontinuity.



Unit step sequence$u[n]$

Unit Impulse Sequence

Unit impulse sequence:

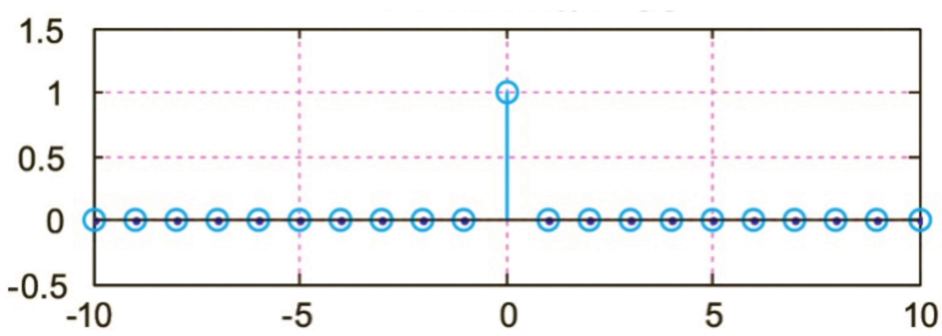
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

With respect to $\delta(t)$ η $\delta[n]$ is defined in a very simple way and does not show a discontinuity at $t = 0$, nor does it take an infinite value of magnitude.

Signals $\delta[n]$ και $u[n]$ are connected through relationships:

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m], \qquad \text{όπου } m = n - k$$



Unit impulse sequence $\delta[n]$

Discrete Time vs Continuous Time Signals

Signals $\delta[n]$ και $u[n]$ are connected through relationships:

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m], \qquad \text{όπου } m = n - k$$

The corresponding relations for the continuous-time signals $\delta(t)$and $u(t)$, are:

$$\delta(t) = \frac{du(t)}{dt}$$

$$u(t) = \int_{0}^{\infty} \delta(t-\tau)\, d\tau = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

The above relations are similar to each other but the first ones are simpler mathematically as instead of the derivative and the integral we use differences and sums.

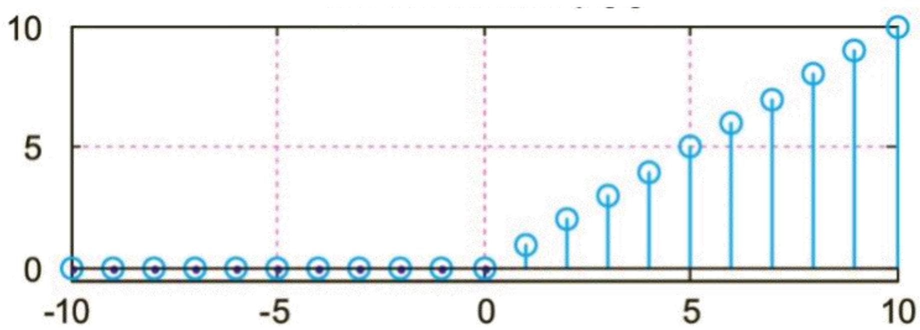
Moreover, there are no discontinuities in the definitions of $\delta[n]$ και $u[n]$.

The $\delta[n]$ and $u[n]$ are not sampled versions of $\delta(t)$ και $u(t)$but are defined independently.

Unit Ramp Sequence

Unit ramp sequence:

$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Sequence of unit ramp $r[n]$

Analysis of discrete-time systems in Unit Impulses

Each discrete-time systems $x[n]$ can be represented as a sum of suitably shifted unit impulses $\delta[n]$ multiplied by a weight factor corresponding to the respective value of the signal $x[n]$.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$$

- Each term $x[k]\, \delta[n-k]$is assumed to be a signal with magnitude $x[k]$ at time $n = k,$ and is set to zero for any other value of $n.$
- The discrete-time signal notation above follows from the **shift property** of the function $\delta[n]$, namely:

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

and is used to calculate the output of a linear and time invariant (LTI) system.

Example 8

Express the signal $x[n] = \{.., 0, \hat{1}, 2, 3, 0, \ldots\}$: (a) as a sum of shifted unit impulses and (b) as a sum of unit step sequences.

Answer: (a) As the sum of unit impulses it is:

$$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$$

(b) In the above solution we set $\delta[n] = u[n] - u[n-1]$ and we get:

$$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] =$$

$$= u[n] - u[n-1] + 2u[n-1] - 2u[n-2] + 3u[n-2] - 3u[n-3]$$

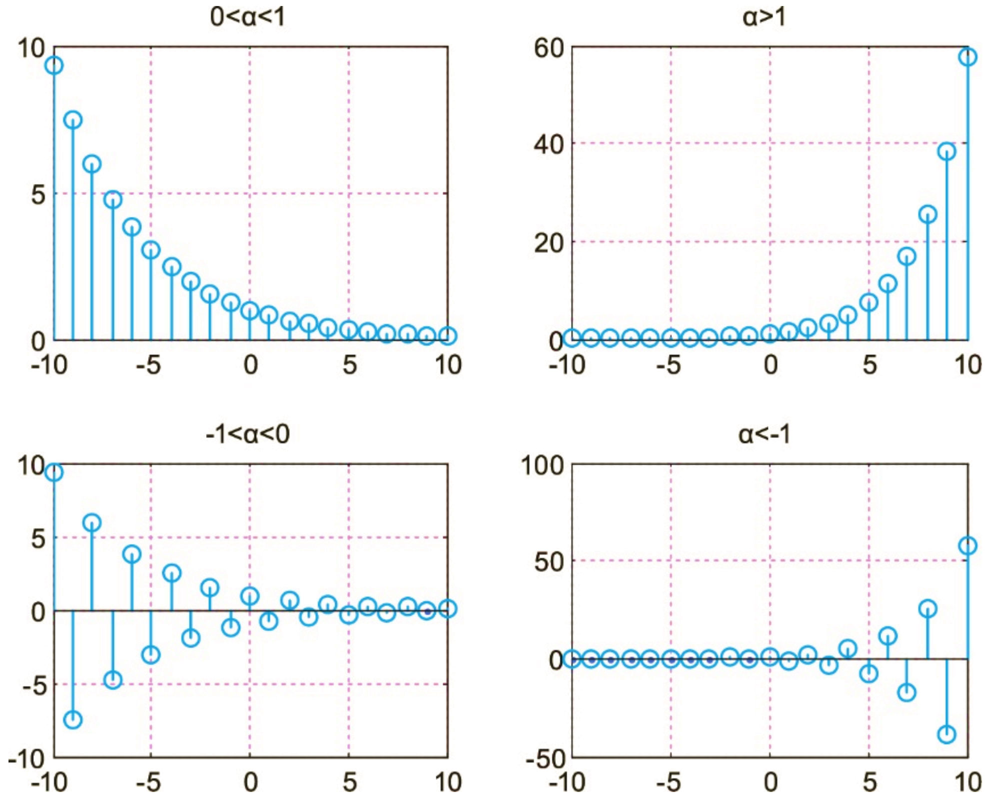
$$= u[n] + u[n-1] + u[n-2] - 3u[n-3]$$

Real Exponential Sequence

Real exponential sequence: $x[n] = A\,\alpha^n$, where $A, \alpha \in R$

A:expresses the width of the sequence

α: expresses the slope rate of the plot of the sequence



Forms of the sequence $x[n] = \alpha^n$ for various values of α.

Summary of Complex Analysis

A complex sequence $x[n]$ can be expressed either in **cartesian form,** i.e. in real and imaginary part:

$$a[n] + jb[n] = Re\{x[n]\} + j\ Im\{x[n]\}$$

either in **polar form,** i.e. in magnitude and phase:

$$x[n] = |x[n]|\ e^{j\ \varphi_x[n]}$$

where the width$|x[n]|$ and the phase arg$\{x[n]\}$ are given by the relations:

$$|x[n]| = \sqrt{Re^2\{x[n]\} + Im^2\{x[n]\}}\text{abs()}$$

$$\varphi_x[n] = \tan^{-1}\frac{Im\{x[n]\}}{Re\{x[n]\}}\text{angle()}$$

The polar form is preferred because it simplifies the calculations and produces the magnitude and phase spectral representations.

The conjugate sequence $x^*[n]$ is given by the equation:

$$x^*[n] = Re\{x[n]\} - j\ Im\{x[n]\} = |x[n]|\ e^{-j\varphi_x[n]}$$

The magnitude remains the same but there is a phase reversal.

Complex Exponential Sequence

Complex Exponential Sequence: $x[n] = a^n$, όπου $\alpha \in$ C.

Setting $\alpha = |\alpha|e^{j\omega_0}$ we have:

$$x[n] = \alpha^n = |\alpha|^n e^{j(n\omega_0)} = |\alpha|^n\{\cos(n\omega_0) + j\sin(n\omega_0)\}$$

<u>Cartesian Form:</u>

- Real part: $x_R[n] = |\alpha|^n \, cos(n\omega_0)$
- Fantastic part: $x_I[n] = |\alpha|^n \, sin(n\omega_0)$

<u>Polar Form:</u>

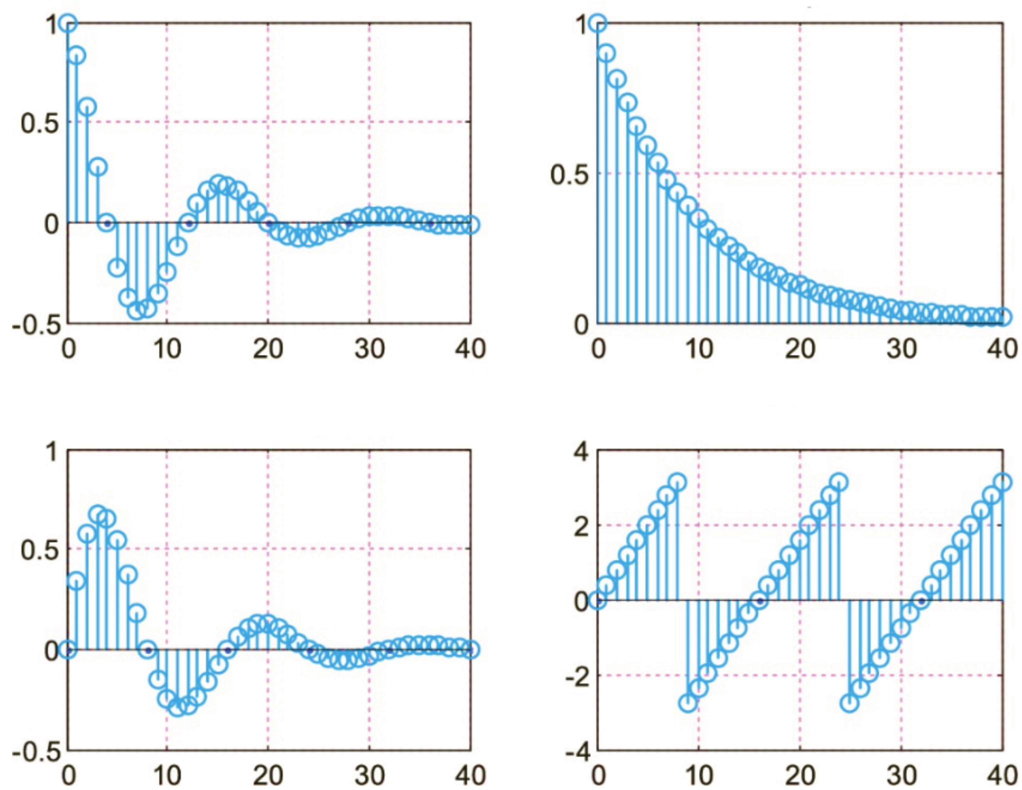
- Magnitude: $|x[n]| = \alpha^n$
- Phase: $\varphi(n) = n\omega_0$

Comments on the Complex Exponential Sequence

- The discrete-time complex exponential sequence is **different** from the continuous-time complex function:
$$x(t) = e^{-st} = e^{(-\sigma + j\Omega_0)t}$$
where Ω_0 is the continuous frequency.
- By sampling the complex function $x(t)$ with sampling period T_s, yields the sampled complex sequence:
$$x[n] = x(nT_s) = e^{(-\sigma n T_s + j\Omega_0 n T_s)} = (e^{-\sigma T_s})^n e^{j(\Omega_0 T_s)n} = \alpha^n e^{j\omega_0 n}$$
where $\alpha = e^{-\sigma T_s}$ and $\omega_0 = \Omega_0\, T_s$.
- The discrete-time complex exponential sequence, unlike the continuous-time complex function, is **always periodic** with period 2π, since it $e^{j2\pi n} = 1, \forall n \in Z$, satisfies the equation:
$$e^{j(\omega_0 + 2\pi)n} = e^{j\omega_0 n} e^{j2\pi n} = e^{j\omega_0 n}$$
- The negative frequency indicates the clockwise direction of rotation of the vector $e^{-jn\omega_0}$ with velocity $n\omega_0$.
- The term $e^{jn\omega_0}$ describes a vector that rotates counterclockwise with speed $n\omega_0$ and expresses the positive frequencies.

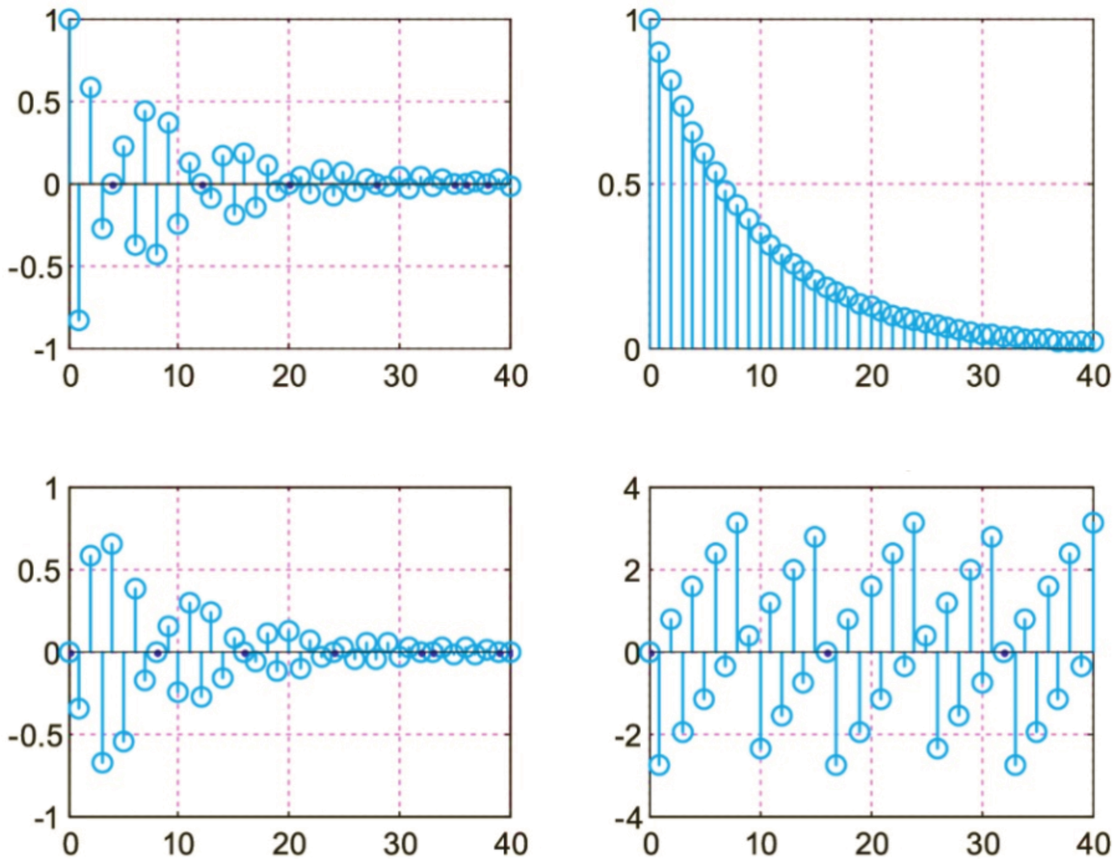
Example 9

To calculate the complex signal $x[n] = a^n e^{j\omega_0 n}$, with $\omega_0 = \pi/8$ in the time interval $0 \leq n \leq 40$ and to draw the diagrams of real-imaginary part and magnitude-phase for values of $\alpha = 0.9, -0.9, 1.2, -1.2$.



Complex exponential sequence graph for $\alpha = 0.9$,
(Up: Real-part, magnitude, Down: Imaginary-part, Phase)

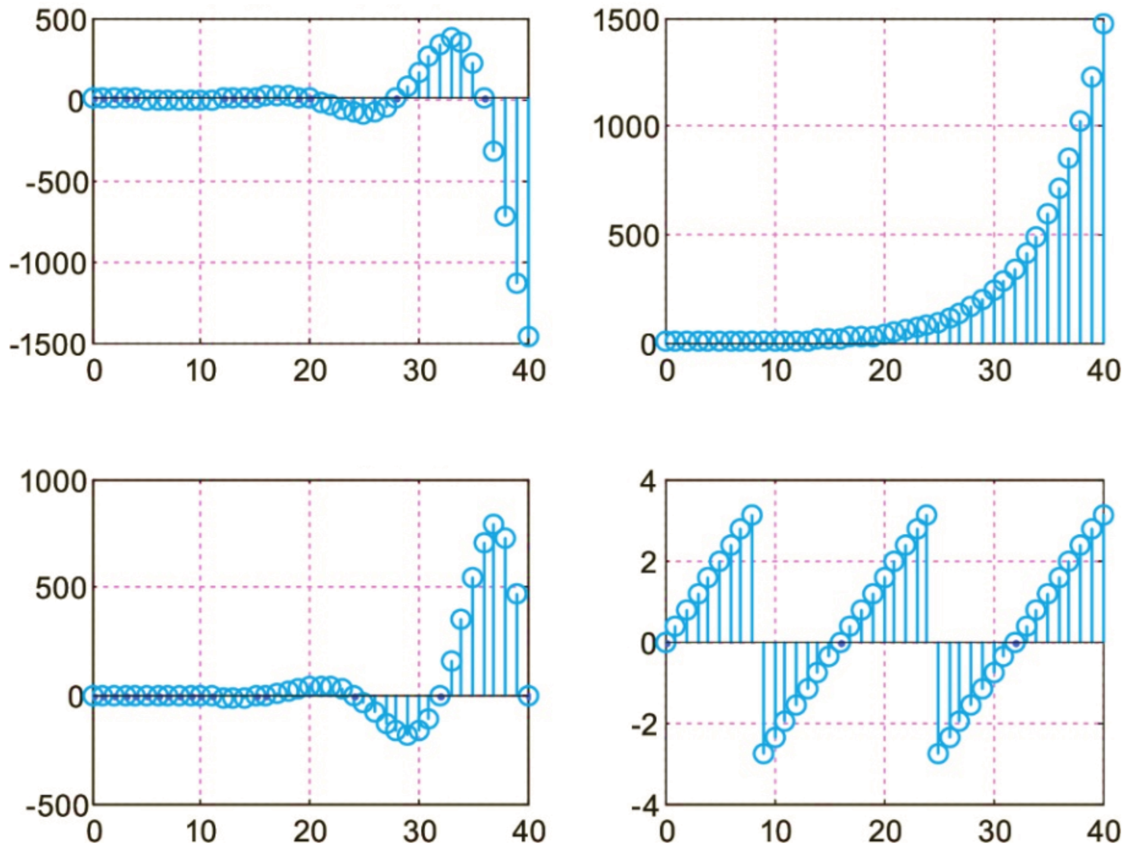
Example 9 (continued)



Complex exponential sequence graph for $\alpha = -0.9$

(Up: Real−part, magnitude, Down: Imaginary−part, Phase)

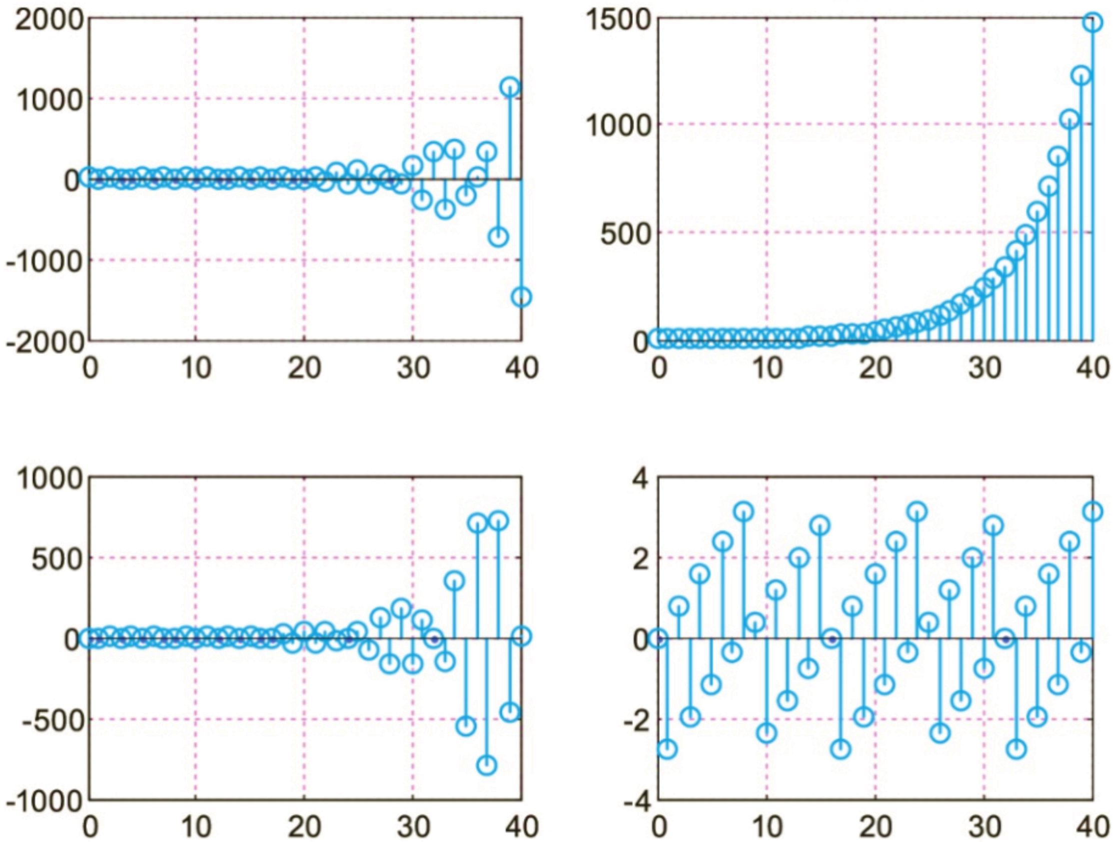
Example 9 (continued)



Complex exponential sequence graph for $\alpha = 1.2$

(Up: Real-part, magnitude, Down: Imaginary-part, Phase)

Example 9 (continued)



Complex exponential sequence graph for $\alpha = -1.2$

(Up: Real−part, magnitude, Down: Imaginary−part, Phase)

Sine Sequence

Sine and cosine sequences can be considered as special cases of complex exponential sequences.

The cosine sequence is the real part of the complex exponential sequence, while the sine is the imaginary part.

$$x[n] = A\alpha^n = |A|e^{j(n\omega_0+\theta)} = |A|\{\cos(n\omega_0+\theta) + j\sin(n\omega_0+\theta)\}$$

There is a phase difference of π/2 between the two sequences, according to:

$$\mathrm{A}\cos(n\omega_0+\theta) = A\sin(n\omega_0+\theta+\pi/2)\,, \qquad -\infty < n < \infty$$

where $\omega_0 = 2\pi(m/N)$(rad) is the discrete frequency.

The variable $f_0 = m/N$ is called **frequency** (Hz) and is valid $\omega_0 = 2\pi f_0$.

<u>**Important:**</u> If the numbers m and N have no common divisor, then the sinusoidal signal is not periodic. Therefore, **discrete-time sinusoidal sequences are not always periodic functions,** as are continuous-time sinusoidal functions.

Sine Sequence

Because the variable ω is expressed in rad, it follows that the discrete frequency repeats with period 2π, i.e. satisfies the equation $\omega_0 + 2\pi k = \omega_0$, where k is a positive or negative integer.

To avoid the ambiguity of reference to a specific period we assume that the discrete frequency ω satisfies the equation $-\pi < \omega \leq \pi$and transform every other frequency $\omega_1$outside this region, with the equation:

$$\omega_1 = \omega + 2\pi k \quad \text{ή} \quad \omega_1 \equiv \omega\ (mod\ 2\pi)$$

The frequency ω is called **the equivalent or apparent frequency.** For example:

- If the frequency takes value $\omega_0 = 2\pi$, then it can be written as $\omega_0 = 0 + 2\pi$, so the equivalent frequency is 0 (rad).
- If the frequency is $\omega_0 = 7\pi/2$, then it can be written as $\omega_0 = (8-1)\pi/2 = 2x2\pi - \pi/2$, so the equivalent frequency is $-\pi/2$.

Example 10

Consider the sinusoidal sequences: $x_1[n] = sin(0.1\pi n)$, $x_2[n] = sin(0.2\pi n)$, $x_3[n] = sin(0.6\pi n)$ and $x_4[n] = sin(0.7\pi n)$ for $-\infty < n < \infty$.

(a) To find whether they are periodic or not.

(b) Plot the sequences in the time domain $n = 0, \dots 40$.

(c) Can these sequences be sampled versions of the corresponding continuous - time functions?

Answer: (a) The given sequences are written:

$$x_1[n] = sin(0.1\pi n) = \sin\left(\frac{2\pi}{20} n\right)$$

$$x_2[n] = sin(0.2\pi n) = \sin\left(\frac{2\pi}{20} 2n\right)$$

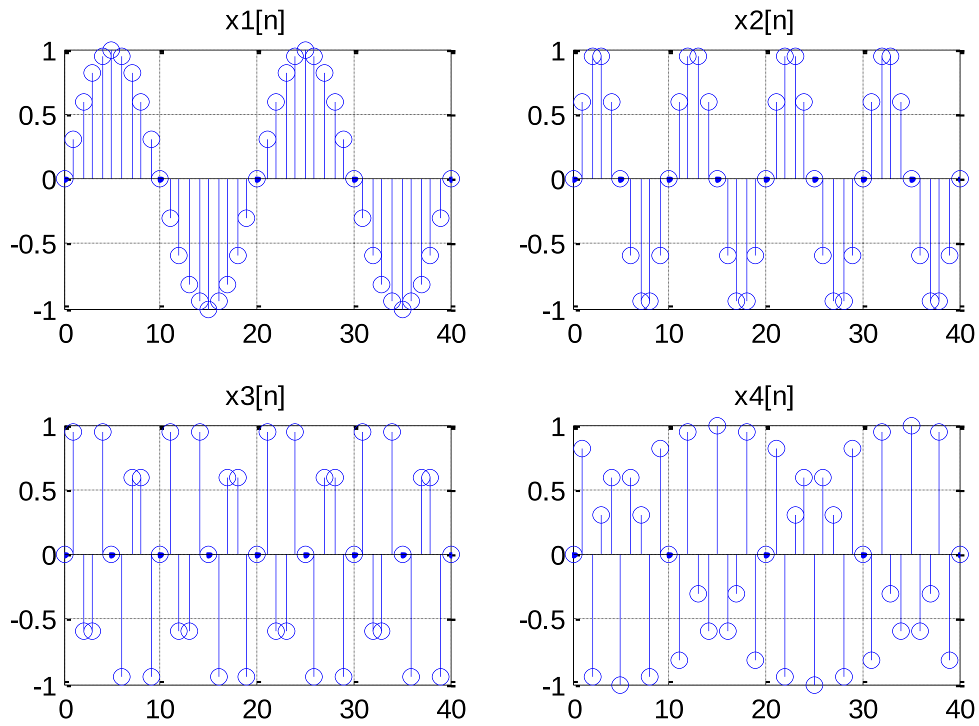
$$x_3[n] = sin(0.6\pi n) = \sin\left(\frac{2\pi}{20} 6n\right)$$

$$x_4[n] = sin(0.7\pi n) = \sin\left(\frac{2\pi}{20} 7n\right)$$

Therefore, the sequences are periodic and harmonically connected to each other.

Example 10 (continued)

Sine sequence for different frequency values f_0:



It follows from the figure that the sequences and $x_1[n]$ are $x_2[n]$ the sampled versions of the corresponding continuous-time functions.

However, the same is not true for the sequences $x_3[n]$ and $x_4[n]$.

Example 10 (continued)

It would be wrong to assume that this occurs due to a violation of the Nyquist rule, i.e., because of incorrect sampling frequency.

Let's explain why this happens: To obtain the discrete sequence $\sin(\omega_0 n)$, we need to sample the continuous-time function $\sin(\Omega_0 t)$ with a sampling period T_s=1 according to the Nyquist condition.:

$$T_s = 1 \leq \frac{\pi}{\Omega_0}$$

where π/Ω_0 is the maximum allowed value of the sampling period for which the aliasing phenomenon does not occur.

For the sequence $x_3[n] = sin(0.6\pi n) = sin(0.6\pi t)|_{t=nT_s=n}$ when $T_s = 1$, it holds:

$$T_s = 1 \leq \frac{\pi}{0.6\,\pi} \approx 1{,}66$$

Conversely, in the case of the sequence $x_2[n]$ we have: $x_2[n] = sin(0.2\pi n) = sin(0.2\pi t)|_{t=nT_s=n}$ when $T_s = 1$, then we have:

$$T_s = 1 \leq \frac{\pi}{0.2\,\pi} = 5$$

Example 10 (continued)

Therefore, generating the sequence $x_2[n]$ is done by taking a larger number of samples from the function $sin(0.2\pi t)$, than generating the sequence $x_3[n]$ from the function $sin(0.6\pi t)$, using in both cases the same sampling period.

This results in the sequence $x_2[n]$ being more like an analog sinusoid than $x_3[n]$, however in both cases no distortion effect occurs.

Important note: The analog frequency Ω of analog sinusoid waves varies in the range $[0, \infty)$, while the discrete (digital) frequencies ω are radial and vary in the range $[0, \pi]$.

Negative frequencies are needed in the analysis of real-valued signals and thus we end up with frequency ranges:

(a) for the continuous-time signals: $-\infty < \Omega < \infty$, and

(b) for discrete-time signals: $-\pi < \omega \leq \pi$.