University of the Peloponnese
Electrical and Computer
Engineering Department

Digital Signal Processing

Unit 01: Analog to Digital Conversion

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Lecture Contents

- Conversion of Analog Signal to Digital
- Sampling Types
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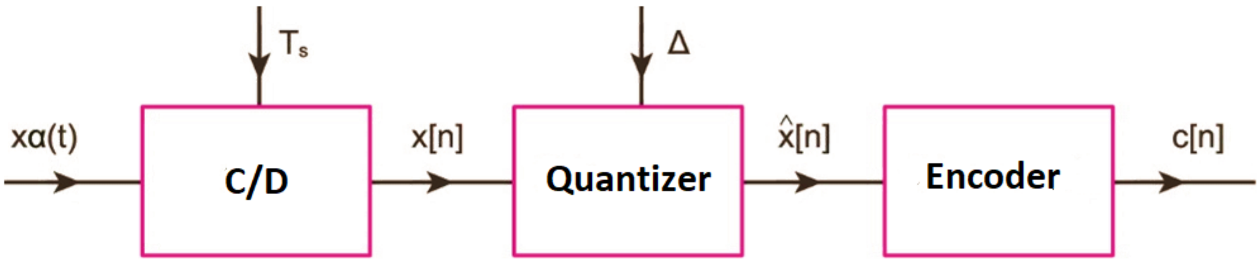
    - Practical Sampling
    - Flat-top Sampling
- Quantization
    - Uniform and Non-uniform Quantization
    - Quantization Parameters
- Coding
- Reconstruction of Analog Signal from Digital form

Analog to Digital Conversion

Most discrete-time signals (DTS) are generated from continuous-time signals (CTS) through the processing of the following three stages:



Analog-to-Digital Converter (ADC)

- **Sampling**: Continuous to Discrete Conversion. Generates $x[n] = x_a(nT_s)$, where T_s is the sampling period.
- **Quantization**: Maps the continuous amplitude $x_a(nT_s)$ to a discrete set of values $\hat{x}[n]$. Characteristics include: Δ, the quantization step, and word length (bits).
- **Coding**: Produces a sequence $c[n]$ of binary code words transmitted over the communication channel.

Sampling

Sampling

Sampling is the process of converting a continuous-time and continuous-amplitude signal (analog signal) into a discrete-time signal. An analog signal $x_a(t)$ with a finite bandwidth $X_a(\Omega)$ and a maximum frequency Ω_{max} , undergoes sampling at a rate $f_s = 1/T_s$ (samples per second). As a result, the **discrete-time signal** $x_a[n]$: is produced:

$$x_a[n] \triangleq x_\alpha(nT_s) = x_\alpha(t)|_{t=nT_s}$$

The value of the sampling period T_s is determined by the **Nyquist criterion or Sampling Theorem.** According to this, if the analog signal $x_\alpha(t)$ has a strictly limited bandwidth [i.e., $X_a(\Omega) = 0$ for | $|\Omega| > \Omega_{max}$], then $x_a(t)$ can be fully recovered from the samples $x_a(nT_s)$, if the sampling frequency f_s satisfies the relationship:

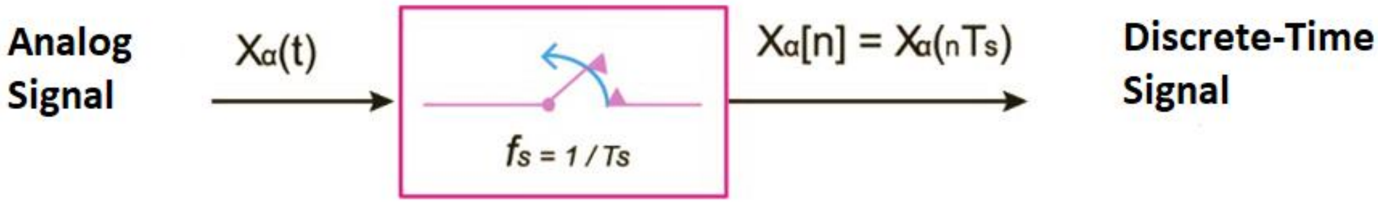
$$f_s \geq 2f_{max} \text{ or equivalent } \Omega_s \geq 2\Omega_{max}$$

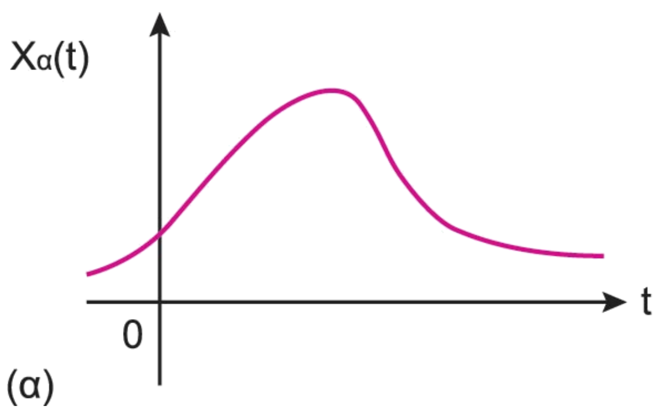
where f_{max} is the maximum frequency of the continuous-time signal $x_\alpha(t)$.

The Nyquist rate is defined by the equation: $f_n = 2\, f_{max}$

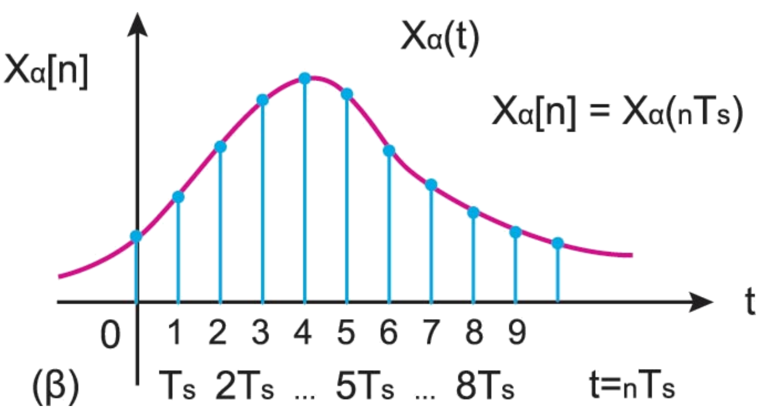
Sampling

In practice, sampling can be implemented by successively opening and closing an ideal switch at regular intervals of time T_s. The samples of the signal are taken when the switch is closed, while no samples are obtained when the switch is open.





(a) Analog Signal $x_a(t)$



(b) Discrete-Time Signal $x_a[n]$

Relationship between Analog and Digital frequency

The correspondence between **the continuous $\boldsymbol{\Omega}$** (rad/sec) of the continuous-time signal $x_a(t)$ and the **discrete frequency $\boldsymbol{\omega}$** (rad) of the discrete-time signal $x_a[n] = x_a(nT_s)$, is given by the relationship:

$$\omega = \Omega T_s, \quad (rad/sec)\, x\, (sec) \; = \; (rad)$$

Note that the values of the discrete frequency $\boldsymbol{\omega}$ result as **samples** of the continuous frequency $\boldsymbol{\Omega}$, taken at intervals equal to the sampling period Ts.

Ideal Sampling

Ideal Sampling

Ideal sampling is the process of generating samples $x_s(nT_s)$ of continuous-time signals $x_\alpha(t)$, instantaneously and uniformly, meaning one sample every T_s, by multiplying $x_\alpha(t)$ with a sampling function $\delta_{T_s}(t)$:

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

The ideally sampled signal $x_s(t)$ is given by the equation:

$$x_s(t) = x_a(t)\, \delta_{T_s}(t) = \sum_{n=-\infty}^{+\infty} x_a(nT_s)\, \delta(t - nT_s)$$

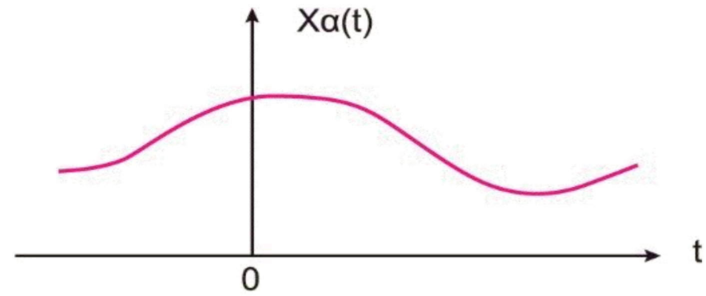
The process is termed **ideal** because it relies on the Dirac delta function $\delta(t)$, which has significant theoretical value but cannot be practically implemented. The Fourier transform $X_s(\Omega)$ of the ideally sampled signal is given by:

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\,(\Omega - k\Omega_s)$$

where $X_a(\Omega)$ is the Fourier transform of the continuous-time signal $x_a(t)$.

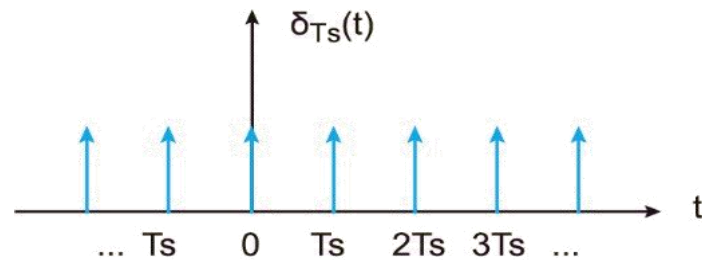
Ideal Sampling (Time Domain)

(a) Continuous-time signal x_α(t)



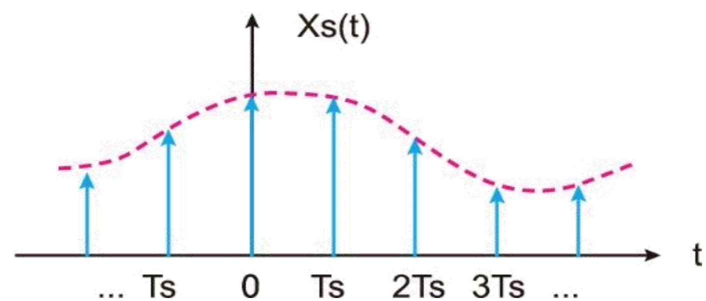
(b) Sampling function:

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$



(c) Ideally sampled signal:

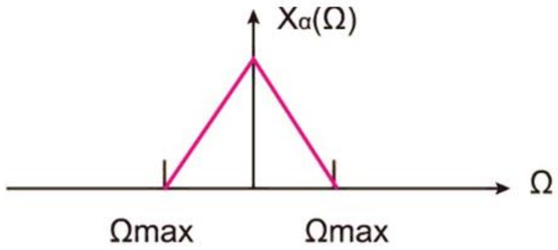
$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_a(nT_s)\,\delta(t - nT_s)$$



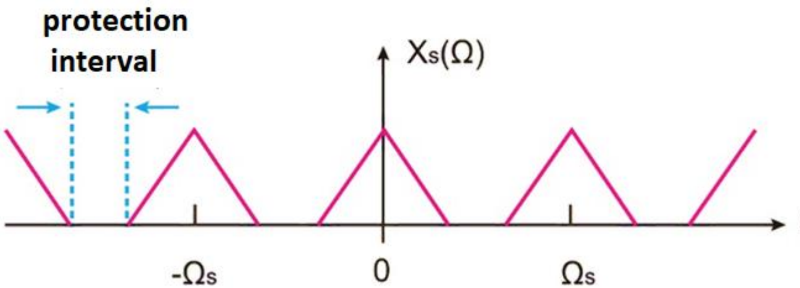
The sampling frequency is given by: $\Omega_s = 2\pi/T_s$

Ideal Sampling (Frequency Domain)

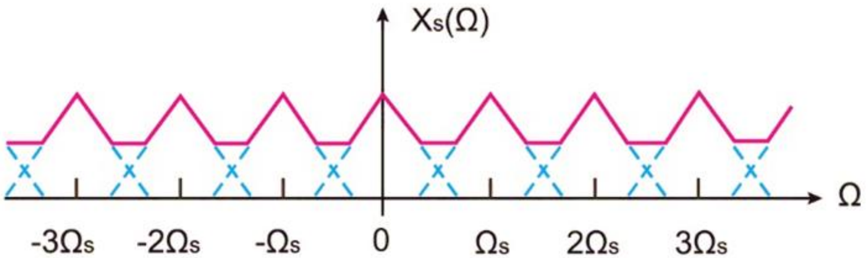
(a) Spectrum of the analog signal $x_\alpha(t)$ with $X_a(f) = 0$ for $|f| > f_x$



(b) Spectrum $X(f)$ of the sampled signal when $f_s \geq 2f_x$



(c) Spectrum $X(f)$ of the sampled signal when $f_s < 2f_x$ ⇒(aliasing effect)



We observe that the spectrum $X_s(\Omega)$ of the ideally sampled signal $x_s(t)$, arises as the sum of repetitions of the spectrum $X_a(\Omega)$ of the original continuous-time signal $x_a(t)$, at positions that are integer multiples of the sampling frequency Ω_s.

Ideal Sampling (Frequency Domain)

<u>Case (a)</u>: $\boldsymbol{\Omega_s} \geq \mathbf{2}\ \boldsymbol{\Omega_{max}}$ or $f_s \geq 2f_{max}$

The spectrum $X_s(\Omega)$[Figure (b)] is formed by successive repetitions of $X_a(\Omega)$, located at integer multiples of the sampling frequency Ω_s. There is **no overlap** between the spectral repetitions.

- **Protection Interval:** The distance $\Omega_s - \Omega_{max}$ between two successive spectral repetitions of $X_a(\Omega)$.
- The absence of overlap between the repetitions of the spectrum $X_a(\Omega)$ ensures the **recovery** of $X_a(\Omega)$ from $X_s(\Omega)$ and, consequently, the recovery of the original signal $x_a(t)$ from the ideally sampled signal $x_s(t)$.
- Recovery is achieved using a **low-pass filter** with **cutoff frequency** Ω_c , where:

$$\Omega_{max} < \Omega_c < \Omega_s \quad \text{ή} \quad f_{max} < f_c < f_s$$

- **Nyquist Frequency:** $\Omega_N = 2\Omega_{max}$ **Nyquist Criterion:** $\Omega_s \geq \Omega_N$ or $\Omega_s \geq 2\ \Omega_{max}$

$$\text{or } T_s \leq \frac{\pi}{\Omega_{max}} = \frac{1}{2f_{max}}$$

- The Nyquist criterion ensures that the signal $x_s(t)$ contains all the information of the original signal $x_a(t)$, and the original signal can **be fully recovered** from the ideally sampled signal.

Ideal Sampling (Frequency Domain)

<u>Case (b)</u>: $\boldsymbol{\Omega_s} < \mathbf{2}\ \boldsymbol{\Omega_{max}}$ or $f_s < 2f_{max}$

- In this case, there is overlap between successive spectral repetitions of the spectrum $X_\alpha(\Omega)$ [Figure (c)].

- Recovery of the original signal $x_\alpha(t)$ from the signal $x_s(t)$ is **impossible**.

- **Aliasing Effect**: The overlap of successive spectral repetitions. It causes <u>permanent</u> and <u>irreversible</u> distortion of the signal.

- If an attempt is made to filter with a low-pass filter to recover $x_\alpha(t)$, frequencies that did not exist in the original signal, aliasing frequencies, will be introduced.

Ideal Sampling (Frequency Domain)

- The above study was conducted under the assumption that the signal $x_\alpha(t)$ is a **low-frequency signal**, meaning it satisfies the relationship $X_\alpha(\Omega) = 0$, for $|\Omega| > \Omega_{max}$.
- If the signal $x_\alpha(t)$ is spectrally **unlimited**, i.e., it contains frequencies of any high value, then the phenomenon of frequency aliasing will occur for any (sufficiently large) sampling frequency.
- In practical terms, for such signals, we first filter the signal with a low-pass filter and then proceed with sampling.
- This, however, results in the permanent loss of high frequencies in the signal.
- Another issue that makes ideal sampling impractical in practice is that the reconstruction low-pass filter was considered ideal, which is naturally not the case, as an ideal low-pass filter has a non-causal and infinite impulse response. In reality, the ideal low-pass filter is approximated by a practical low-pass filter.

Example 1

If the Nyquist rate for the signal $x(t)$ is Ω_s, find the Nyquist rates for the signals:

(a) $y(t) = dx(t)/dt$ (b) $y(t) = x(t)\cos(\Omega_0 t)$

Answer: (a) To calculate the DTFT of $y(t)$ we use the differentiation property of the DTFT, which yields:

$$Y(\Omega) = j\Omega\, X(\Omega)$$

It can be observed that there is no change in the frequency domain; therefore, the Nyquist frequency remains unchanged.

(b) The given operation implies amplitude modulation, specifically amplitude modulation by a cosine term. It is known that during the modulation of a signal $x(t)$ by a term $\cos(\Omega_0 t)$, there is a frequency shift in the spectrum of $x(t)$ by $\pm\Omega_0$. Therefore, the Nyquist frequency of $y(t) = x(t)\cos(\Omega_0 t)$ will be $\Omega_s + 2\Omega_0$.

Example 2

Find the Nyquist rate of the signal $x_a(t) = 5\, cos\, 1000\pi t \;\; cos\, 4000\pi t$

Answer: From the trigonometric property:

$$cos\, A\, .\, cosB = \frac{1}{2}[cos(A+B) + cos(A-B)]\text{, we have:}$$

$$x_a(t) = \frac{5}{2}(cos(1000\pi t + 4\,000\pi t) + cos(1000\pi t \; - 4\,000\pi t))$$
$$= 2{,}5(cos\, 5000\pi t + cos\, 3000\pi t)$$

Thus, $x_a(t)$ is a signal with maximum frequency $f_{max} = 2{,}500\; Hz$.

Consequently, the Nyquist rate is $2\; x\; 2{,}500 = 5{,}000\; Hz$

The Nyquist interval (period) is $1/5{,}000$ sec $=\; 0.2\; ms$

Example 3

Find the Nyquist rate for the analog signal:

$$x_a(t) = \frac{\sin 200\pi t}{\pi t}$$

Answer: From Fourier analysis we know that:

$$\frac{\sin at}{\pi t} \overset{\mathrm{F}}{\longleftrightarrow} P_a(\omega) = \begin{cases} 1 & |\omega| < \alpha \\ 0 & |\omega| > \alpha \end{cases}$$

The $x_a(t)$ is a signal with maximum frequency $f_{max} = 100\ Hz$.

So the Nyquist rate is $200\ Hz$, and the Nyquist interval is $1/200$ sec.

Example 4

Find the Nyquist rate for the analog signal:

$$x_a(t) = \left(\frac{\sin 200\pi t}{\pi t}\right)^2$$

Answer: From the convolution theorem of the Fourier transform, we have:

$$x_1(t)\, x_2(t) \overset{F}{\longleftrightarrow} \frac{1}{2\pi}\, X_1(\Omega)\, X_2(\Omega)$$

and combined with the previous paradigm, we find that the signal $x_a(t)$ is also band-limited and that its bandwidth is twice that of the signal from the previous paradigm, i.e. it is 200 Hz .

So, the Nyquist rate is 400 Hz and the Nyquist interval is 1/400 sec.

Example 5

The analog signal $x_a(t) = 2cos(20\pi t)\cos(30\pi t) + \sin(40\pi t)$is sampled at a rate of 20 samples per second. Determine the resulting discrete time signal.

Answer: We will write the given signal as a sum of sinusoidal functions. The product $cos(20\pi t)$cos(30πt) is written:

$$2cos(20\pi t)\ \cos(30\pi t) = \cos(50\pi t) + \cos(10\pi t)$$

So the analog signal is $x_a(t) = \cos(50\pi t) + \cos(10\pi t) + \sin(40\pi t)$ and contains the frequencies $f_1 = 25\ Hz, f_2 = 5\ Hz$ και $f_3 = 20\ Hz$.

The Nyquist frequency is $f_N = 2x25\ Hz = 50\ Hz$.

The discrete-time signal resulting from sampling with frequency $f_s = 20\ Hz\ (T_s = 1/20\ sec)$, is:

$$x(n) = x_a(t)\Big|_{t=nTs} = \cos\left(\frac{50\pi}{20}n\right) + \cos\left(\frac{10\pi}{20}n\right) + \sin\left(\frac{40\pi}{20}n\right)$$

$$= \cos\left(\frac{5\pi}{2}n\right) + \cos\left(\frac{\pi}{2}n\right) + \sin(2\pi n) = \cdots = 0$$

The sampling frequency chosen does not satisfy the Nyquist criterion and the frequencies produced resulted in a zero signal value.

We used the well-known relation:$cosA\ cosB = (1/2)[\cos(A + B) + \cos(A - B)]$.

Example 6

Repeat the previous paradigm for a sampling frequency (rate) of 50 samples per second.

Answer: The discrete time signal resulting from sampling with frequency $f_s = 50\ Hz$ ($T_s = 1/50\ sec$), is:

$$x(n) = x_a(t)\Big|_{t=nTs} = \cos\left(\frac{50\pi}{50}n\right) + \cos\left(\frac{10\pi}{50}n\right) + \sin\left(\frac{40\pi}{50}n\right)$$

$$= \cos(\pi n) + \cos\left(\frac{\pi}{5}n\right) + \sin\left(\frac{4\pi}{5}n\right)$$

The frequency of the component $\cos(\pi n)$ is:

$$\omega_1 = \pi \Rightarrow \Omega_1 T_s = \pi \Rightarrow 2\pi f_1 \frac{1}{f_s} = \pi \Rightarrow f_1 = \frac{f_s}{2} \Rightarrow f_1 = \frac{50}{2} = 25\ Hz$$

The frequency of the component $\cos(\pi n/5)$ is:

$$\omega_2 = \frac{\pi}{5} \Rightarrow \Omega_2 T_s = \frac{\pi}{5} \Rightarrow 2\pi f_2 \frac{1}{f_s} = \frac{\pi}{5} \Rightarrow f_2 = \frac{f_s}{10} \Rightarrow f_2 = \frac{50}{10} = 5\ Hz$$

The frequency of the component $\cos(4\pi n/5)$ is:

$$\omega_3 = \frac{4\pi}{5} \Rightarrow \Omega_3 T_s = \frac{4\pi}{5} \Rightarrow 2\pi f_3 \frac{1}{f_s} = \frac{4\pi}{5} \Rightarrow f_3 = \frac{2f_s}{5} \Rightarrow f_3 = \frac{100}{5} = 20\ Hz$$

We notice that the frequencies of the discrete-time signal are the same as the frequencies of the analog signal, which is because the sampling frequency chosen satisfies the Nyquist criterion .

Example 7

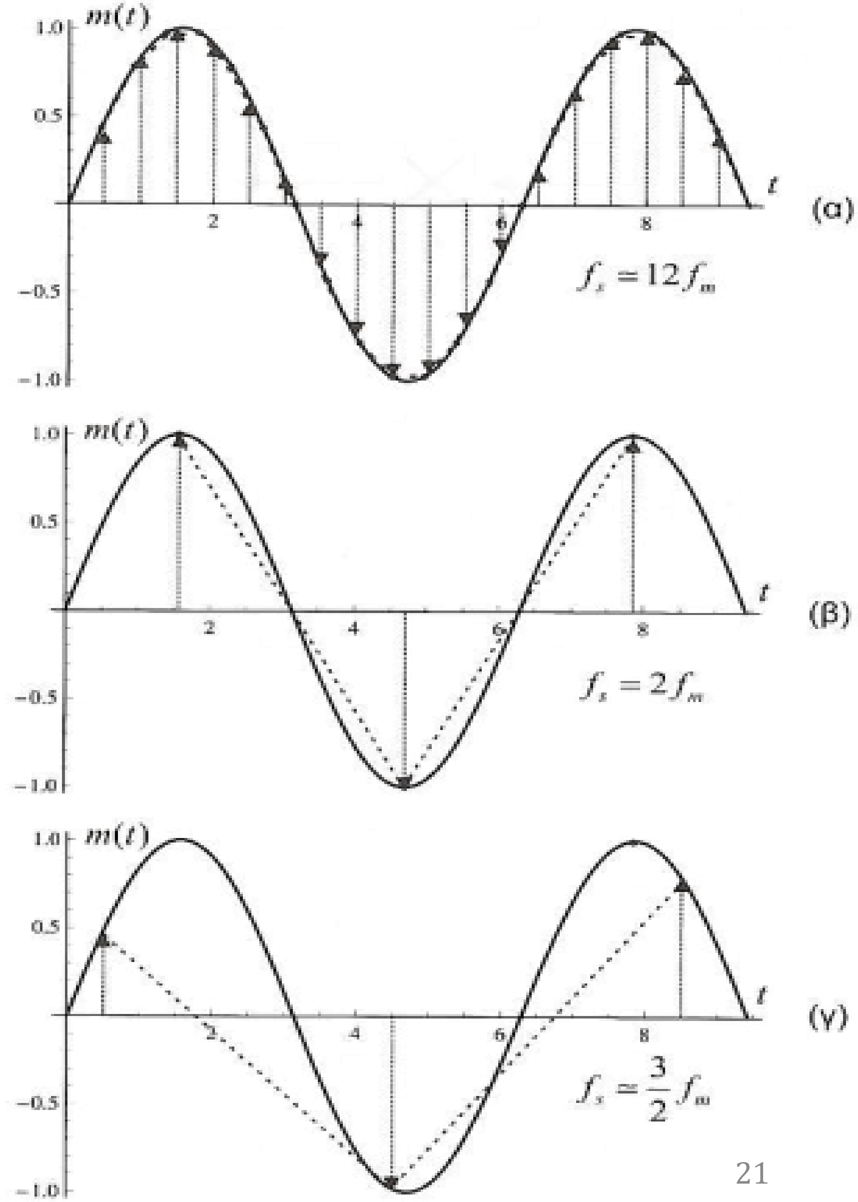
A sinusoidal signal $m(t)$with frequencyf_m sampled with frequency:

(a) $f_s = 12f_m$

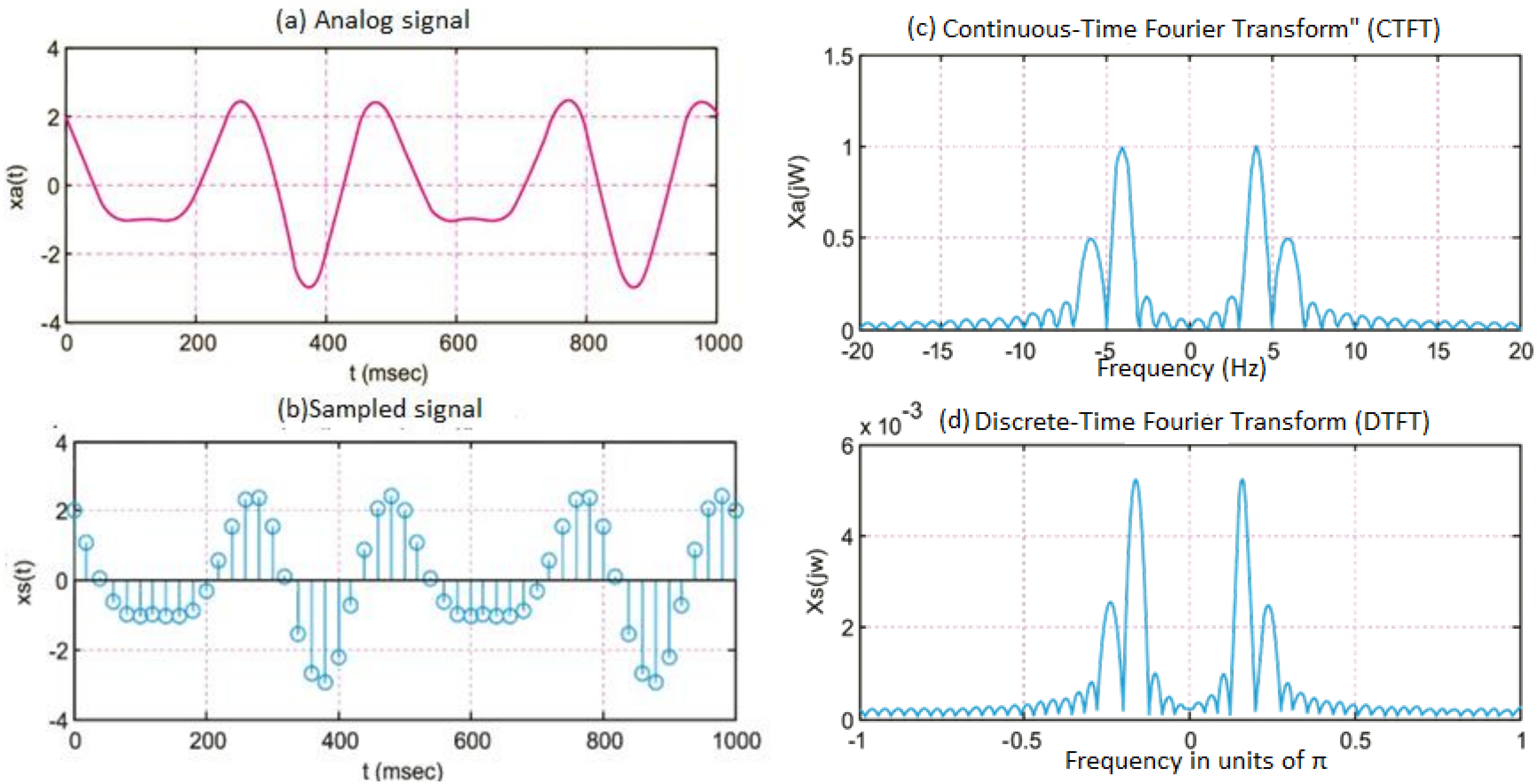
(b) $f_s = 2f_m$

(c) $f_s = \frac{3}{2}f_m$

Comment on the cases where the Nyquist condition is satisfied or not.



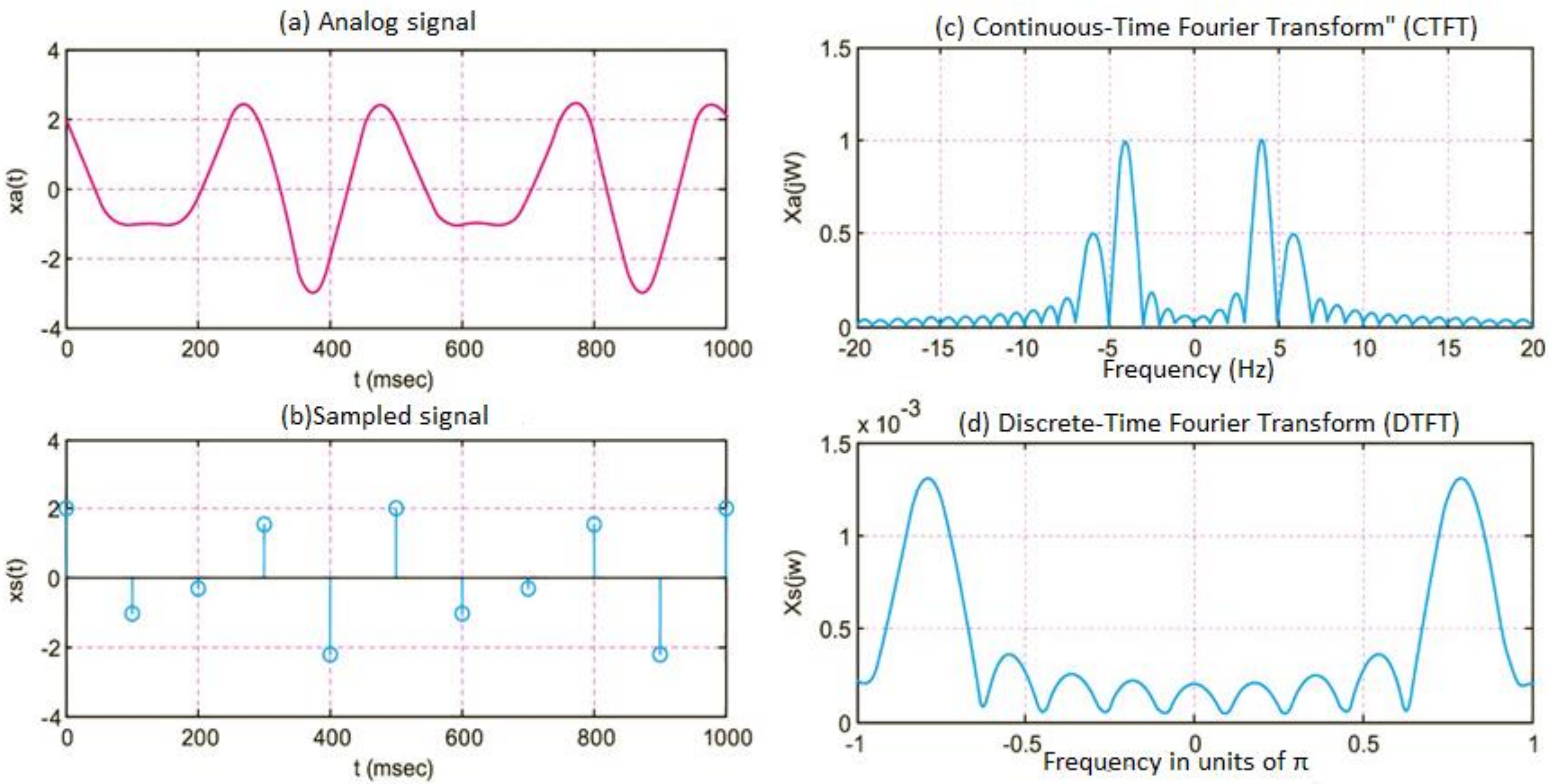
Sampling without aliasing



(a) Analog signal $x_a(t)$,
(b) Sampled signal $x_s(t)$ with $T_s = 0.02\ sec/sample$

(c) FT $X_a(\Omega)$ of the analog signal $x_a(t)$,
(d) FT $X_s(e^{j\omega})$ of the sampled signal $x_s(t)$

Sampling with aliasing



(a) Analog signal $x_a(t)$,
(b) Sampled signal $x_s(t)$ with $T_s = 0.1\ sec/sample$

(a) FT $X_a(\Omega)$ of the analog signal $x_a(t)$,
(b) FT $X_s(e^{j\omega})$ of the sampled signal $x_s(t)$

Quantization

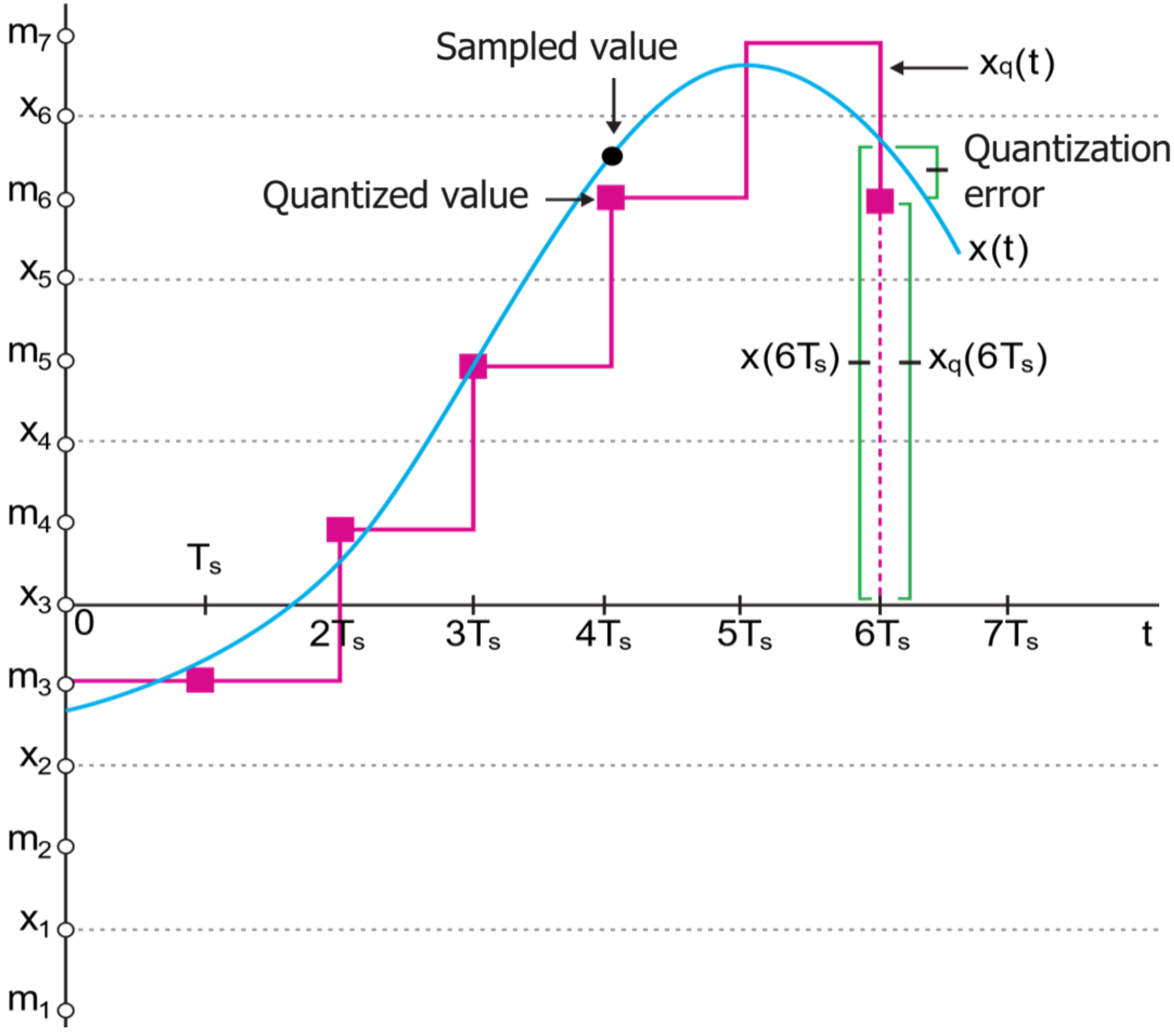
Quantization

Quantization is a non-linear and irreversible process, which transforms a $x_\alpha(n)$ continuous-amplitude input sequence for which $x(n) \in (-m_p, m_p)$, into a **discrete-amplitude sequence** $m(n) = Q[x_\alpha(n)]$.

- **L levels of decision (zones)** x_1, x_2 ...,x_L divide its range of width values $x[n]$into Lintervals $I_k = [x_k, x_{k+1}],\ \ k = 1,2, \dots, L$.
- For an input $x_a[n]$ that lies within I_k, a level is assigned $m(k) \in I_k$.
- The amplitude of the signal (dynamic range) is given by the relation:

  $|x_{max}(n)| = 2\, m_p$

Uniform Quantization



Quantization Parameters (1/2)

- **Number of levels** :$L = 2^B$ where B is the length (in bits) of each **level** $m[n]$. The relation holds:

$$B = \log_2(L)$$

- **Quantization step** :

$$\Delta = x_{k+1} - x_k$$

For equidistant levels (uniform quantization), it holds:

$$\Delta = \frac{|x_{max}[n]|}{2^B}$$

- **error (noise)** :

$$e[n] = m[n] - x_a[n] \text{ and } -\frac{\Delta}{2} < e(n) < \frac{\Delta}{2}$$

Quantization Parameters (2/2)

- **Quantization root mean square error or quantization noise power** :

$$\mathrm{E}[(x[n]\ - m[n])^2] = \sigma_e^2 = \frac{\Delta^2}{12}$$

Also applies:

$$\sigma_e^2 = m_p^2/3L^2$$

- **Signal to Noise Ratio** (in dB) Signal to Noise Ratio (SNR) :

$$SNR = 10log\frac{\sigma_x^2}{\sigma_e^2} = 6{,}02\ B + 10{,}81 - 20\ log\frac{|x_{max}[n]|}{\sigma_x}$$

- Therefore the SNR increases (improves) by **~6 dB** for **each additional bit** which is added to the description of each quantized sample $m[n]$.

Coding

Coding

- Each quantized level $m[n]$ is represented by a **code word**.
- If L is the number of quantization stations, then each sample is described by $\boldsymbol{log_2 L = B}$ bits , where B is an integer.

- **Information transmission rate** at encoder output:

  $R = f_s \, log_2 L = f_s \, B$ (bits/s) , where f_s the sampling frequency

- **Minimum bandwidth** for the signal resulting at the encoder output to be transmitted in PCM modulation:

$$W_{PCM} = \frac{1}{2} f_s B$$

Coding

Most DSP systems use **two** 's complement representation of numbers.

In this system, with a code word $c = [b_0, b_1, \ldots b_B]$ of length B+1 bits:

- The most significant digit is the sign digit.
- The remaining digits correspond to the numerical value of binary integers or fractions.
- Considering binary fractions, the code word $b_0, b_1, b_2, \ldots b_B$ has the value:
  $x = (-1)b_0 + b_1 2^{-1} + b_2 2^{-2} + \ldots + b_B 2^{-B}$

Example 8

The analog signal is given:

$$x_a(t) = -\frac{3}{2} + cos(100\pi t)\cos(200\pi t) + \frac{1}{2}\sin\left(200\pi t - \frac{\pi}{2}\right) + \cos(300\pi t)$$

(a) Determine the Nyquist frequency and the minimum acceptable value of the sampling frequency.

(b) What frequencies will result if the analog signal is sampled with a sampling frequency of 150 Hz .

(c) What is the discrete-time signal that will result from question (b)?

(d) If the signal amplitude is expressed in volts and each sample of the discrete signal is quantized to 8 bits , how many volts does the quantization step correspond to?

Answer : (a) To determine the Nyquist frequency the maximum frequency of the signal must be found. For this reason we will express the given signal as a sum of sinusoidal functions. The product $cos(100\pi t)\cos(200\pi t)$is written:

$$cos(100\pi t)\ \cos(200\pi t) = \frac{1}{2}[\cos(300\pi t) + \cos(100\pi t)]$$

So the analog signal is written:

$$x_a(t) = -\frac{3}{2} + \frac{1}{2}\cos(300\pi t) + \frac{1}{2}\cos(100\pi t) + \frac{1}{2}\sin\left(200\pi t - \frac{\pi}{2}\right) + \cos(300\pi t)$$

$$= -\frac{3}{2} + \frac{1}{2}\cos(100\pi t) + \frac{1}{2}\sin\left(200\pi t + \frac{\pi}{2}\right) + \frac{3}{2}\cos(300\pi t) \qquad (1)$$

Example 8 (continued)

Therefore the frequencies of the signal are:

$f_1 = 0\ Hz, f_2 = 50\ Hz, f_3 = 100\ Hz$ και $f_4 = 150\ Hz$. So the Nyquist frequency and minimum acceptable value of the sampling frequency is:

$$f_{s(min)} = f_N = 2f_4 = 300\ Hz.$$

(b) For sampling frequency $f_s = 150\ Hz$, only frequencies $f_1 = 0\ Hz$ και $f_2 = 50\ Hz$, that lie within the range will be correctly represented $[-f_s/2, f_s/2] = [-75Hz\ , 75\ Hz]$. The frequencies $f_3 = 100\ Hz$ και $f_4 = 150\ Hz$will be convoluted and appear to correspond to the pseudo-labeled frequencies:

$$f_3' = f_3 - kf_s = 100 - 150 = -50Hz$$
$$f_4' = f_4 - kf_s = 150 - 150 = 0Hz$$

Based on the above, it follows that the sampled signal will contain a continuous component ($0\ Hz$)and a sinusoidal frequency component 50 Hz, i.e. the frequencies 100 Hz and 150 Hz will no longer appear in the sampled signal.

(c) For a sampling frequency $f_s = 150\ Hz$ (i.e., sampling period $T_s = 1/150\ sec$), the discrete-time signal is:

$$x(n) = x_a(t)\Big|_{t=nTs} = -\frac{3}{2} + \frac{1}{2}\cos\left(\frac{100\pi}{150}n\right) + \frac{1}{2}\sin\left(\frac{200\pi}{150}n - \frac{\pi}{2}\right) + \frac{3}{2}\cos\left(\frac{300\pi}{150}n\right)$$
$$= -\frac{3}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{3}n\right) + \frac{1}{2}\cos\left(\frac{4\pi}{3}n\right) + \frac{3}{2}\cos(2\pi n) = -\frac{3}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{3}n\right) + \frac{1}{2}\cos\left(2\pi - \frac{2\pi}{3}n\right) + \frac{3}{2}$$
$$= \frac{1}{2}\cos\left(\frac{2\pi}{3}n\right) + \frac{1}{2}\cos\left(\frac{2\pi}{3}n\right) = \cos\left(\frac{2\pi}{3}n\right)$$

Example 8 (continued)

The frequency of this signal can be calculated as follows:

$$\omega = \frac{2\pi}{3} \Rightarrow \Omega T_s = \frac{2\pi}{3} \Rightarrow 2\pi f \frac{1}{f_s} = \frac{2\pi}{3} \Rightarrow f = \frac{f_s}{3} \Rightarrow f = \frac{150}{3} = 50\ Hz$$

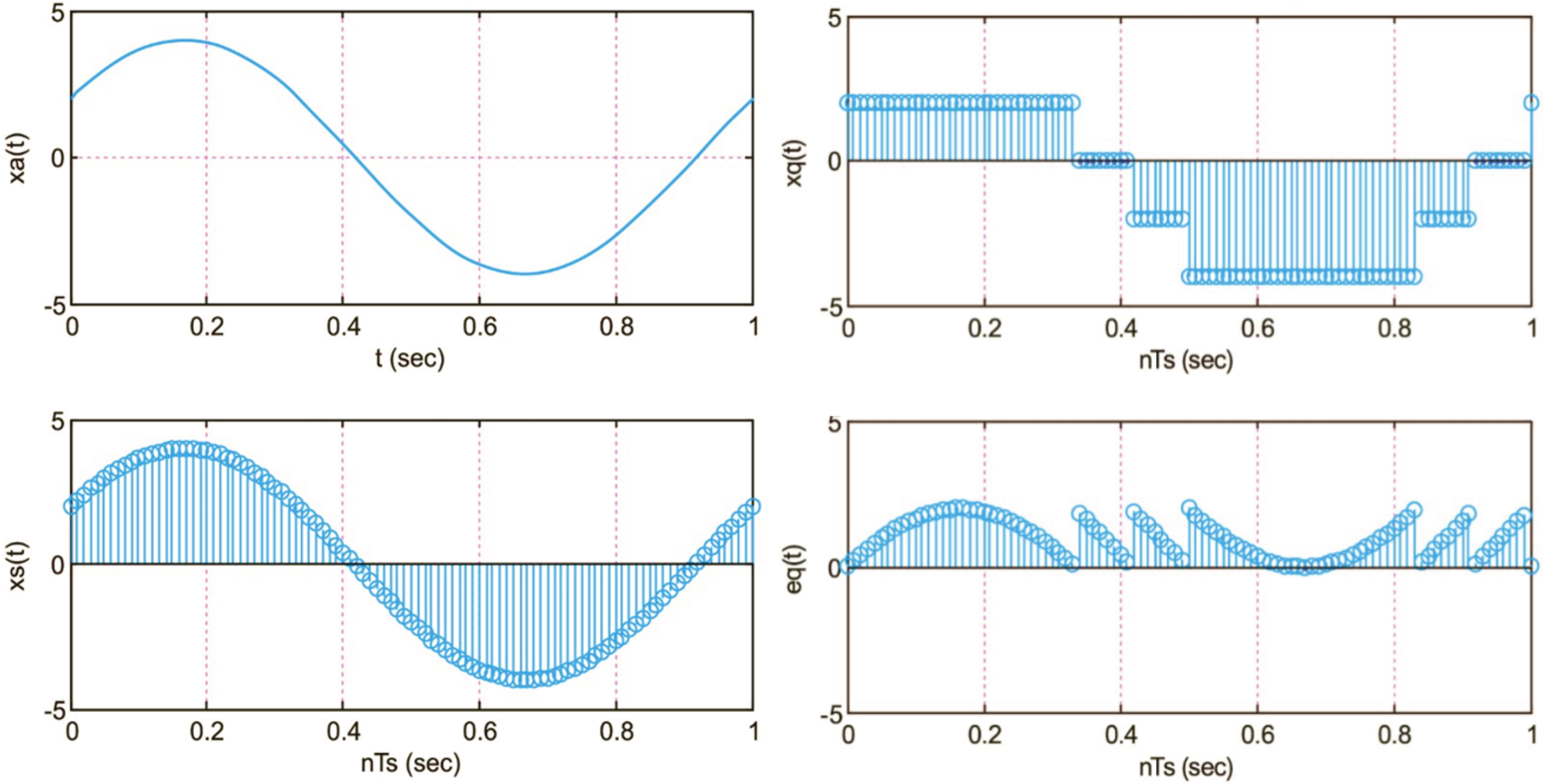
(d) From relation (1) it follows that the analog signal takes a maximum value of +1 Volt (when each trigonometric term takes a value of +1) and a minimum value of -4 Volts (when each trigonometric term takes a value of -1). Therefore, the dynamic range of the analog signal is 5 Volts and the quantization step Δ is calculated as:

$$\Delta = \frac{x_{max} - x_{min}}{2^L - 1} = \frac{1-(-4)}{2^8 - 1} = \frac{5}{255} = 19{,}61\ mV$$

Usually quantizers work assuming that the amplitude values of the signal are symmetrical, i.e. $\pm 5\ V, \pm 10V,$ κλπ.In the case of the above signal where the amplitude of the signal ranges from +1 Volt to -4 Volts we must use a quantizer. So the quantization step for 8 $\pm 5\ Volts$. Quantization step of converter is:

$$\Delta = \frac{x_{max} - x_{min}}{2^L - 1} = \frac{10}{2^8 - 1} = \frac{10}{255} = 39{,}22\ mV$$

Analog to Digital Signal Conversion



Up: Analog signal $x_a(t)$, Sampled signal $x_s(t)$,
Down: Quantized signal $x_q(t)$at 4 levels, Quantization error $e_q(t) = x_a(t) - x_q(t)$

Analog Signal Reconstruction from Digital

Analog Signal Reconstruction from Digital

An analog signal sampled according to the Nyquist criterion (i.e. $f_s \geq 2f_x$), can be **recovered** from its samples, by the steps:

(1) The samples $x[n]$are transformed into a function $x_r(t)$through the relation:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\, \delta(t - n\, T_s)$$

(2) The function $x_r(t)$passes through an ideal LPF with shock response:

$$h_{LPF}(t) = \frac{\sin(\pi t T_s)}{\pi t T_s} = \mathrm{sinc}(tT_s)$$

Fourier transform :

$$H_{LPF}(\Omega) = \begin{cases} T_s, & -\Omega_s/2 < \Omega < \Omega_s/2 \\ 0, & \end{cases}$$

The reconstructed analog signal at the filter output is given by:

$$\hat{x}_\alpha(t) = \sum_{n=-\infty}^{\infty} x_r(nT_s)\, h_{LPF}(t - nT_s) = \sum_{n=-\infty}^{\infty} x_r(nT_s)\ \mathrm{sinc}((t - nT_s)/T_s)$$

and the Fourier transform of the reconstructed signal is:

$$\hat{X}_\alpha(\Omega) = X_r(\Omega)\, H_{LPF}(\Omega)$$

Analog Signal Reconstruction from Digital

In real conditions, accurate reconstruction of the original signal is possible because:

- The original signal was not of finite bandwidth, so it was not possible to determine the Nyquist frequency and therefore the minimum value of the sampling frequency, so as not to cause the distortion effect.
- The sampling rate was not constant throughout the sampling period and some variation in the time interval between successive samples may have occurred.
- The reconstruction filter used is a real filter and not an ideal one as required by the theoretical analysis.

However, the most important reason for the non-accurate reconstruction of the original signal is due to the quantization of the signal samples.

Quantization process always introduces quantization noise, which cannot be removed.

Analog Signal Reconstruction from Digital

(a) Analog signal $x_a(t)$

(b) Sampled signal $x_s(t)$

(c) Reconstructed signal $\hat{x}_\alpha(t)$with zero-order interpolation