8 ΟΡΙΣΜΕΝΑ ΟΛΟΚΛΗΡΩΜΑΤΑ

8.1 Ορισμοί

Έστω ότι η $f(x)$ είναι ορισμένη στο διάστημα $a \le x \le b$. Αν το διάστημα αυτό χωρισθεί σε n ίσα υποδιαστήματα με μήκος $\Delta x = (b-a)/n$, το *ορισμένο ολοκλήρωμα* της $f(x)$ από το $x = a$ έως το $x = b$ ορίζεται με τη σχέση

$$\int_a^b f(x)dx = \lim_{n\to\infty}\{f(a)\Delta x + f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x + \cdots f(a+(n-1)\Delta x)\Delta x\}$$

Αν η $f(x)$ είναι τμηματικά συνεχής (συνεχής σε υποδιαστήματα του διαστήματος ολοκλήρωσης), το όριο υπάρχει.

Αν $f(x) = \dfrac{d}{dx}g(x)$, τότε σύμφωνα με το *θεμελιώδες θεώρημα του ολοκληρωτικού λογισμού* είναι

$$\int_a^b f(x)\,dx = \int_a^b \frac{d}{dx}g(x) = g(x)\Big|_a^b = g(b) - g(a)$$

Αν το διάστημα είναι άπειρο ή αν η $f(x)$ έχει ένα ανώμαλο σημείο στο διάστημα ολοκλήρωσης, το ορισμένο ολοκλήρωμα λέγεται *γενικευμένο ολοκλήρωμα* και μπορεί να ορισθεί κατάλληλα με όρια. Έτσι π.χ. είναι

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

$$\int_{-\infty}^\infty f(x)dx = \lim_{\substack{a\to-\infty\\ b\to\infty}}\int_a^b f(x)dx$$

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to0+}\int_a^{b-\varepsilon} f(x)\,dx \qquad \text{αν } b \text{ είναι ένα ανώμαλο σημείο}$$

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to0+}\int_{a+\varepsilon}^{b} f(x)\,dx \qquad \text{αν } a \text{ είναι ένα ανώμαλο σημείο}$$

Η *πρωτεύουσα τιμή του Cauchy* (που μπορεί να υπάρχει ακόμα και όταν το γενικευμένο ολοκλήρωμα δεν υπάρχει) ενός γενικευμένου ολοκληρώματος $\int_{-\infty}^{\infty} f(x)\,dx$ ορίζεται ως το όριο $\lim_{N\to\infty}\int_{-N}^{N} f(x)\,dx$. Όμοια, για ένα ανώμαλο σημείο c στο εσωτερικό του διαστήματος ολοκλήρωσης η πρωτεύουσα τιμή του Cauchy ορίζεται ως το όριο $\lim_{\varepsilon\to0+}\left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx\right]$.

8.2 Γενικοί Κανόνες και Ιδιότητες

$$\int_a^b [f(x) \pm g(x) \pm h(x) \pm \cdots]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \pm \int_a^b h(x)\,dx \pm \cdots$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx \qquad \text{[ολοκλήρωση κατά παράγοντες]}$$

$$\int_a^b f(x)\,dx = (b-a)f(c), \qquad \text{για κατάλληλο } c \text{ μεταξύ των } a \text{ και } b$$

Αυτό είναι το *θεώρημα της μέσης τιμής*για ορισμένα ολοκληρώματα και ισχύει, αν η $f(x)$ είναι συνεχής στο διάστημα $a \le x \le b$.

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx, \qquad \text{για κατάλληλο } c \text{ μεταξύ των } a \text{ και } b$$

Η σχέση αυτή αποτελεί γενίκευση του θεωρήματος της μέσης τιμής και ισχύει, αν οι $f(x)$ και $g(x)$ είναι συνεχείς στο $a \le x \le b$ και $g(x) \ge 0$.

$$\frac{d}{d\alpha}\int_{\varphi_1(\alpha)}^{\varphi_2(\alpha)} F(x,\alpha)\,dx = \int_{\varphi_1(\alpha)}^{\varphi_2(\alpha)} \frac{\partial F}{\partial \alpha}\,dx + F(\varphi_2,\alpha)\frac{d\varphi_2}{d\alpha} - F(\varphi_1,\alpha)\frac{d\varphi_1}{d\alpha} \qquad \text{[κανόνας του Leibnitz]}$$

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = \{f(0)-f(\infty)\}\ln\frac{b}{a}$$

Αυτό είναι το *ολοκλήρωμα του Frullani* και ισχύει, αν η $f'(x)$ είναι συνεχής και το $\int_1^\infty \frac{f(x)-f(\infty)}{x}\,dx$ συγκλίνει.

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx \le M(b-a), \qquad \text{για } |f(x)| \le M \text{ στο } a < x < b$$

8.3 Διάφορα Ολοκληρώματα

Με αλγεβρικές συναρτήσεις

$$\int_0^1 x^m(1-x)^n\,dx = \int_0^1 \frac{x^m + x^n}{(1+x)^{m+n+2}}\,dx = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}, \qquad m>-1,\ n>\cdot$$

$$\int_0^1 \frac{x^n\,dx}{1+x} = (-1)^n\left[\ln 2 - 1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^n}{n}\right], \qquad n = 1, 2, \ldots$$

$$\int_0^1 \frac{x^\lambda\,dx}{(1-x)^{\lambda+1}} = \frac{\pi}{\sin(\lambda+1)\pi}, \qquad -1<\lambda<0$$

$$\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \frac{\pi}{2}$$

$$\int_0^a \sqrt{a^2-x^2}\,dx = \frac{\pi a^2}{4}$$

$$\int_0^a x^m(a^n-x^n)^p\,dx = \frac{a^{m+1+np}\Gamma[(m+1)/n]\Gamma(p+1)}{n\Gamma[(m+1)/n+p+1]}$$

$$\int_{-a}^a (a+x)^{m-1}(a-x)^{n-1}\,dx = (2a)^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$$

$$\int_0^\infty \frac{x^\lambda\,dx}{1+x} = \frac{\pi}{\sin(\lambda+1)\pi}, \qquad -1<\lambda<0$$

$$\int_0^1 x^m(1-x^n)^p\,dx = \frac{\Gamma(p+1)\Gamma\left(\frac{m+1}{n}\right)}{n\Gamma\left(\frac{m+1}{n}+p+1\right)}, \qquad m>-1,\ n>0,\ p>-1$$

$$\int_0^\infty \frac{x^m\,dx}{x^n+a^n} = \frac{\pi a^{m+1-n}}{n\sin[(m+1)\pi/n]}, \qquad 0<m+1<n$$

$$\int_0^\infty \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}}\cot^{-1}\frac{b}{\sqrt{4ac-b^2}}, \qquad a>0,\ 4ac-b^2>0$$

$$\int_0^\infty \frac{x^n\,dx}{x^2+2x\cos\beta+1} = \frac{\pi}{\sin n\pi}\,\frac{\sin n\beta}{\sin\beta}$$

$$\int_0^\infty \frac{x^m\,dx}{(x^n+a^n)^p} = \frac{(-1)^{p-1}\pi a^{m-np+1}\Gamma[(m+1)/n]}{n\sin[(m+1)\pi/n](p-1)!\Gamma[(m+1)/n-p+1]}, \qquad 0<m+1<np$$

Με τριγωνομετρικές συναρτήσεις

Όλα τα γράμματα-σύμβολα θεωρούνται θετικά εκτός αν δηλωθεί το αντίθετο.

$$\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^\pi \sin^2 x\,dx = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \cos^2 x\,dx = \frac{1}{2}\int_0^\pi \cos^2 x\,dx = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \sin^2 nx\,dx = \frac{1}{2}\int_0^\pi \sin^2 nx\,dx = \frac{1}{4}\int_0^{2\pi} \sin^2 nx\,dx = \frac{\pi}{4}, \qquad n=1,2,\ldots$$

$$\int_0^{\pi/2} \cos^2 nx\,dx = \frac{1}{2}\int_0^\pi \cos^2 nx\,dx = \frac{1}{4}\int_0^{2\pi} \cos^2 nx\,dx = \frac{\pi}{4}, \qquad n=1,2,\ldots$$

$$\int_0^\pi \sin mx\sin nx\,dx = \begin{cases} 0, & m\neq n \\ \pi/2, & m=n \end{cases}, \qquad m,\ n=1,2,\ldots$$

$$\int_0^\pi \cos mx\cos nx\,dx = \begin{cases} 0, & m\neq n \\ \pi/2, & m=n \end{cases}, \qquad m,\ n=1,2,\ldots$$

$$\int_0^\pi \sin mx\cos nx\,dx = \begin{cases} 0, & m+n=2p \\ 2m/(m^2-n^2), & m+n=2p+1 \end{cases}, \qquad m,\ n,\ p=1,2,\ldots$$

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \dfrac{2\cdot4\cdot6\cdots(n-1)}{3\cdot5\cdot7\cdots n}, & n=2p+1,\ p=0,1,2\ldots \\ \dfrac{1\cdot3\cdot5\cdots(n-1)}{2\cdot4\cdot6\cdots n}\,\dfrac{\pi}{2}, & n=2p,\ p=1,2\ldots \\ \dfrac{\sqrt{\pi}}{2}\,\dfrac{\Gamma((n+1)/2)}{\Gamma(n/2+1)}, & n>-1 \end{cases}$$

$$\int_0^{\pi} x \sin^n x\, dx = \pi \int_0^{\pi/2} \sin^n x\, dx$$

$$\int_0^{\pi/2} \tan^p x\, dx = \int_0^{\pi/2} \cot^p x\, dx = \frac{\pi}{2\cos(p\pi/2)} \qquad |p|<1$$

$$\int_0^{\pi/2} \frac{dx}{1+\tan^n x} = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \frac{x\, dx}{\sin x} = 2\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right) = 2G \approx 1.8319311884$$

$$\int_0^{\pi/2} \frac{x^2\, dx}{\sin^2 x} = \pi \ln 2$$

$$\int_0^{\pi/2} \frac{x\, dx}{\tan x} = \frac{\pi}{2} \ln 2$$

$$\int_0^{\pi} \sin^p x \sin mx\, dx = \frac{\pi}{2^p} \frac{\sin\frac{m\pi}{2}\Gamma(p+1)}{\Gamma\left(\frac{p+m}{2}+1\right)\Gamma\left(\frac{p-m}{2}+1\right)}, \qquad \begin{cases} p+1>0 \\ p+m+2>0 \\ p-m+2>0 \end{cases}$$

$$\int_0^{\pi/2} \cos^p x \cos mx\, dx = \frac{\pi}{2^{p+1}} \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+m}{2}+1\right)\Gamma\left(\frac{p-m}{2}+1\right)}, \qquad \begin{cases} p+1>0 \\ p+m+2>0 \\ p-m+2>0 \end{cases}$$

$$\int_0^{\pi/2} \sin^p x \cos mx\, dx = \frac{\pi}{2^p} \frac{\cos\frac{m\pi}{2}\Gamma(p+1)}{\Gamma\left(\frac{p+m}{2}+1\right)\Gamma\left(\frac{p-m}{2}+1\right)}, \qquad \begin{cases} p+1>0 \\ p+m+2>0 \\ p-m+2>0 \end{cases}$$

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)}, \qquad \begin{cases} p+1>0 \\ q+1>0 \end{cases}$$

$$\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{\cos^{-1}(b/a)}{\sqrt{a^2-b^2}}$$

$$\int_0^{2\pi} \frac{dx}{a+b\sin x} = \int_0^{2\pi} \frac{dx}{a+b\cos x} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\int_0^{\pi} \frac{dx}{a^2 \pm 2ab\cos x + b^2} = \frac{\pi}{|a^2-b^2|}, \qquad a \neq \pm b$$

$$\int_0^{\pi} \frac{x\sin x\,dx}{1-2a\cos x+a^2} = \begin{cases} (\pi/a)\ln(1+a) & |a|<1 \\ \pi\ln(1+1/a) & |a|>1 \end{cases}$$

$$\int_0^{\pi} \frac{\cos mx\,dx}{1-2a\cos x+a^2} = \frac{\pi a^m}{1-a^2}, \qquad a^2<1,\ m=0,1,2,\dots$$

$$\int_0^{2\pi} \frac{dx}{(a+b\sin x)^2} = \int_0^{2\pi} \frac{dx}{(a+b\cos x)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

$$\int_0^{\pi/2} \frac{dx}{(a\sin x+b\cos x)^2} = \frac{1}{ab}, \qquad a>0,\ b>0$$

$$\int_0^{\pi/2} \frac{dx}{1\pm a^2\sin^2 x} = \int_0^{\pi/2} \frac{dx}{1\pm a^2\cos^2 x} = \frac{\pi}{2\sqrt{1\pm a^2}}, \qquad 1\pm a^2\sin^2 x>0$$

$$\int_0^{\pi/2} \frac{dx}{(1\pm a^2\sin^2 x)^2} = \int_0^{\pi/2} \frac{dx}{(1\pm a^2\cos^2 x)^2} = \frac{\pi(2\pm a^2)}{4(1\pm a^2)^{3/2}}, \qquad 1\pm a^2\sin^2 x>0$$

$$\int_0^{\pi/2} \frac{dx}{b^2+a^2\tan^2 x} = \int_0^{\pi/2} \frac{\cos^2 x\,dx}{a^2\sin^2 x+b^2\cos^2 x} = \frac{\pi}{2b(a+b)}, \qquad a>0,\ b>0$$

$$\int_0^{\pi/2} \frac{dx}{a^2+b^2\cot^2 x} = \int_0^{\pi/2} \frac{\sin^2 x\,dx}{a^2\sin^2 x+b^2\cos^2 x} = \frac{\pi}{2a(a+b)}, \qquad a>0,\ b>0$$

$$\int_0^1 \frac{\sin^{-1} x}{x}\,dx = \frac{\pi}{2}\ln x$$

$$\int_0^1 \frac{\tan^{-1} x}{x}\,dx = \frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}\cdots = G = 0.9159655942\dots$$

$$\int_0^{\infty} \frac{\sin px}{x}\,dx = \begin{cases} \pi/2, & p>0 \\ 0, & p=0 \\ -\pi/2, & p<0 \end{cases}$$

$$\int_0^\infty \frac{\sin px \sin qx}{x}\,dx = \frac{1}{2}\ln\frac{p+q}{p-q}, \qquad p > q > 0$$

$$\int_0^\infty \frac{\sin px \cos qx}{x}\,dx = \begin{cases} \pi/2, & p > q > 0 \\ \pi/4, & p = q > 0 \\ 0, & q > p > 0 \end{cases}$$

$$\int_0^\infty \frac{\sin px \sin qx}{x^2}\,dx = \begin{cases} \pi p/2 & q \ge p > q \\ \pi q/2 & p \ge q > 0 \end{cases}$$

$$\int_0^\infty \frac{\sin^2 px}{x^2}\,dx = \frac{\pi|p|}{2}$$

$$\int_0^\infty \frac{1-\cos px}{x^2}\,dx = \frac{\pi|p|}{2}$$

$$\int_0^\infty \frac{\cos px - \cos qx}{x}\,dx = \ln\left|\frac{q}{p}\right|, \qquad p > 0,\ q > 0$$

$$\int_0^\infty \frac{\cos px - \cos qx}{x^2}\,dx = \frac{\pi(q-p)}{2}, \qquad q > p > 0$$

$$\int_0^\infty \frac{\sin^{2n+1} mx}{x}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\frac{\pi}{2}, \qquad m > 0,\ n = 1, 2, 3, \cdots$$

$$\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2a}e^{-ma}, \qquad a > 0,\ m \ge 0$$

$$\int_0^\infty \frac{x\sin mx}{x^2+a^2}\,dx = \frac{\pi}{2}e^{-ma}, \qquad a \ge 0,\ m > 0$$

$$\int_0^\infty \frac{\sin mx}{x(x^2+a^2)}\,dx = \frac{\pi}{2a^2}(1-e^{-ma}), \qquad a > 0,\ m \ge 0$$

$$\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty \frac{\sin x}{x^{3/2}}\,dx = \sqrt{2\pi}$$

$$\int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$$

$$\int_0^\infty \frac{\sin^4 x}{x^4} = \frac{\pi}{3}$$

$$\int_0^\infty \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \qquad 0 < p < 2$$

$$\int_0^\infty \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}, \qquad 0 < p < 1$$

$$\int_0^\infty \frac{\tan mx}{x}\,dx = \begin{cases} \pi/2, & m > 0 \\ 0, & m = 0 \\ -\pi/2, & m < 0 \end{cases}$$

$$\int_0^\infty \frac{\tan^{-1} px - \tan^{-1} qx}{x}\,dx = \frac{\pi}{2}\ln\frac{p}{q}$$

$$\int_0^\infty \left(\frac{1}{1+x^2} - \cos x\right)\frac{dx}{x} = \gamma$$

$$\int_0^\infty \sin(ax^2)\,dx = \int_0^\infty \cos(ax^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \qquad a > 0$$

$$\int_0^\infty \sin(ax^p)\,dx = \frac{1}{pa^{1/p}}\Gamma(1/p)\sin\frac{\pi}{2p}, \qquad p > 1$$

$$\int_0^\infty \cos(ax^p)\,dx = \frac{1}{pa^{1/p}}\Gamma(1/p)\cos\frac{\pi}{2p}, \qquad p > 1$$

$$\int_0^\infty \sin(ax^2)\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} - \sin\frac{b^2}{a}\right), \qquad a > 0$$

$$\int_0^\infty \cos(ax^2)\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} + \sin\frac{b^2}{a}\right), \qquad a > 0$$

Με εκθετικές συναρτήσεις

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a}, \qquad a>0$$

$$\int_0^\infty xe^{-ax}\,dx = \frac{1}{a^2}, \qquad a>0$$

$$\int_0^\infty \sqrt{x}e^{-ax}\,dx = \frac{\sqrt{\pi}}{2a^{3/2}}, \qquad a>0$$

$$\int_0^\infty \frac{e^{-ax}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{a}}, \qquad a>0$$

$$\int_0^\infty x^p e^{-ax}\,dx = \begin{cases} \dfrac{p!}{a^{p+1}}, & p=1,\,2,\,3,\ldots,\; a>0 \\ \dfrac{\Gamma(p+1)}{a^{p+1}}, & p>-1,\; a>0 \end{cases}$$

[$\Gamma(x)$ η συνάρτηση γάμα]

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\frac{b}{a}, \qquad a>0,\, b>0$$

$$\int_0^\infty \frac{x\,dx}{e^{ax}-1} = \frac{\pi^2}{6a^2}, \qquad a>0$$

$$\int_0^\infty \frac{dx}{e^{ax}+1} = \frac{\ln 2}{a}, \qquad a>0$$

$$\int_0^\infty \frac{x\,dx}{e^{ax}+1} = \frac{\pi^2}{12a^2}, \qquad a>0$$

$$\int_0^\infty \left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x} = \int_0^\infty \left(\frac{1}{e^x-1}-\frac{e^{-x}}{x}\right)dx = \gamma$$

$$\int_0^\infty \frac{x^{p-1}\,dx}{e^{ax}-1} = \begin{cases} \dfrac{(p-1)!}{a^p}\zeta(p) = \dfrac{2^{2n-2}\pi^{2n}}{na^{2n}}B_n, & p=2n,\; n=1,\,2,\ldots,\; a>0 \\ \dfrac{\Gamma(p)}{a^p}\zeta(p), & p>0,\; a>0 \end{cases}$$

[$\Gamma(x)$ η συνάρτηση γάμα, $\zeta(x)$ η συνάρτηση ζήτα του Riemann]

$$\int_0^\infty \frac{x^{p-1}\,dx}{e^{ax}+1} = \begin{cases} \dfrac{(2n-1)!}{a^{2n}}(1-2^{1-2n})\zeta(p) = \dfrac{(2^{2n-1}-1)\pi^{2n}}{2na^{2n}}B_n, & \begin{cases} p=2n,\ a>0 \\ n=1,\,2,\ldots \end{cases} \\ \dfrac{\Gamma(p)}{a^p}(1-2^{1-p})\zeta(p), & p>0,\ a>0 \end{cases}$$

[$\Gamma(x)$ η συνάρτηση γάμα, $\zeta(x)$ η συνάρτηση ζήτα του Riemann]

$$\int_0^1 \frac{dx}{x^x} = \frac{1}{1^1}+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\cdots = 1.2912859970627\ldots$$

$$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}, \qquad a>0$$

$$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}, \qquad a>0$$

$$\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}, \qquad a>0$$

$$\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2-b^2}{(a^2+b^2)^2}, \qquad a>0$$

$$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \tan^{-1}\frac{b}{a}, \qquad a>0$$

$$\int_0^\infty x^{p-1}e^{-ax}\sin bx\,dx = \frac{\Gamma(p)\sin p\theta}{(a^2+b^2)^{p/2}}, \qquad \begin{cases} a,\,b,\ p>0 \\ \theta=\sin^{-1}\dfrac{b}{\sqrt{a^2+b^2}}=\cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}} \end{cases}$$

$$\int_0^\infty x^{p-1}e^{-ax}\cos bx\,dx = \frac{\Gamma(p)\cos p\theta}{(a^2+b^2)^{p/2}}, \qquad \begin{cases} a,\ p>0 \\ \theta=\sin^{-1}\dfrac{b}{\sqrt{a^2+b^2}}=\cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}} \end{cases}$$

$$\int_0^\infty e^{-ax}\sin(bx+c)\,dx = \frac{a\sin c+b\cos c}{a^2+b^2}, \qquad a>0$$

$$\int_0^\infty e^{-ax}\cos(bx+c)\,dx = \frac{a\sin c-b\cos c}{a^2+b^2}, \qquad a>0$$

$$\int_0^\infty e^{-ax}\sin bx\sin cx\,dx=\frac{2abc}{[a^2+(b-c)^2][a^2+(b+c)^2]},\qquad a>0$$

$$\int_0^\infty e^{-ax}\sin bx\cos cx\,dx=\frac{b(a^2+b^2-c^2)}{[a^2+(b-c)^2][a^2+(b+c)^2]},\qquad a>0$$

$$\int_0^\infty e^{-ax}\cos bx\cos cx\,dx=\frac{b(a^2+b^2+c^2)}{[a^2+(b-c)^2][a^2+(b+c)^2]},\qquad a>0$$

$$\int_0^\infty \frac{\sin px}{e^{ax}-1}dx=\frac{\pi}{2a}\coth\frac{p\pi}{a}-\frac{1}{2p},\qquad a>p>0$$

$$\int_0^\infty \frac{\sin px}{e^{ax}+1}dx=\frac{1}{2p}-\frac{\pi}{2a\sinh(p\pi/a)},\qquad a>p>0$$

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\cos px\,dx=\frac{1}{2}\ln\left(\frac{b^2+p^2}{a^2+p^2}\right)$$

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\sin px\,dx=\tan^{-1}\frac{b}{p}-\tan^{-1}\frac{a}{p}$$

$$\int_0^\infty \frac{e^{-ax}(1-\cos x)}{x^2}dx=\cot^{-1}a-\frac{a}{2}\ln(a^2+1)$$

$$\int_0^\infty \frac{e^{-ax}}{x}(1-\cos bx)\,dx=\frac{1}{2}\ln\frac{a^2+b^2}{a^2},\qquad a>0$$

$$\int_0^\infty \frac{e^{-ax}}{x}(\cos bx-\cos cx)\,dx=\frac{1}{2}\ln\frac{a^2+c^2}{a^2+b^2},\qquad a>0$$

$$\int_0^\infty e^{-ax^2}dx=\frac{1}{2}\sqrt{\frac{\pi}{a}},\qquad a>0$$

$$\int_0^\infty xe^{-ax^2}dx=\frac{1}{2a},\qquad a>0$$

$$\int_0^\infty e^{-ax^2}\cos bx\,dx=\frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-b^2/4a},\qquad a>0$$

$$\int_0^\infty \frac{1-e^{-ax^2}}{x^2}dx=\sqrt{a\pi},\qquad a>0$$

$$\int_0^\infty e^{-(ax^2+bx+c)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{(b^2-4ac)/4a}\operatorname{erfc}\left(\frac{b}{2\sqrt{a}}\right), \qquad \operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-x^2}\,dx\ ,\ a>0$$

$$\int_{-\infty}^\infty e^{-(ax^2+bx+c)}\,dx = \sqrt{\frac{\pi}{a}}\,e^{(b^2-4ac)/4a}$$

$$\int_0^\infty e^{-(ax^2+b/x^2)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}, \qquad a>0,\ b>0$$

$$\int_0^\infty \frac{e^{-x^2}-e^{-x}}{x}\,dx = \frac{1}{2}\gamma \qquad [\gamma \text{ ο αριθμός του Euler}]$$

$$\int_0^\infty x^p e^{-ax^2}\,dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\dfrac{\pi}{a}}, & p=2n,\ n=1,\,2,\ldots,\ a>0 \\ \dfrac{n!}{2a^{n+1}}, & p=2n+1,\ n=1,\,2,\ldots,\ a>0 \\ \dfrac{1}{2}a^{-(p+1)/2}\Gamma\left(\dfrac{p+1}{2}\right), & p>-1,\ a>0 \end{cases}$$

$$\int_0^\infty x^p e^{-(ax)^q}\,dx = \frac{1}{qa^{p+1}}\Gamma\left(\frac{p+1}{q}\right), \qquad a>0,\ p>-1,\ q>0$$

Με λογαριθμικές συναρτήσεις

$$\int_0^1 \frac{\ln x}{1\pm x}\,dx = (-3\pm 1)\frac{\pi^2}{24}$$

$$\int_0^1 \frac{\ln x}{(1+x)^2}\,dx = -\ln 2$$

$$\int_0^1 \frac{\ln x}{1+x^2}\,dx = -\left[\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right] = -G = -0.9159655942\ldots$$

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}\ln 2$$

$$\int_0^1 \frac{x\ln x}{\sqrt{1-x^2}}\,dx = 1-\ln 2$$

$$\int_0^1 (\ln x)\sqrt{1-x^2}\,dx = -\frac{\pi}{8}(1+2\ln 2)$$

$$\int_0^1 (\ln x)\ln(1+x)\,dx = 2-2\ln 2-\frac{\pi^2}{12}$$

$$\int_0^1 (\ln x)\ln(1-x)\,dx = 2-\frac{\pi^2}{6}$$

$$\int_0^1 \ln\left|\ln x\right|\,dx = -\gamma$$

$$\int_0^1 x^p \ln x\,dx = \frac{-1}{(p+1)^2}, \qquad p>-1$$

$$\int_0^1 (-\ln x)^p\,dx = \Gamma(p+1), \qquad p>-1$$

$$\int_0^1 \frac{x^{p-1}\ln x}{1+x}\,dx = -p^2\frac{\cot p\pi}{\sin p\pi} \qquad 0<p<1$$

$$\int_0^1 \frac{x^n \ln x}{1+x}\,dx = (-1)^{n+1}\frac{\pi^2}{12}+(-1)^{n+1}\sum_{k=1}^{n}\frac{(-1)^k}{k^2}, \qquad n=1,2,\ldots$$

$$\int_0^1 \frac{x^n \ln x}{1-x}\,dx = -\frac{\pi^2}{6}+\sum_{k=1}^{n}\frac{1}{k^2}, \qquad n=1,2,\ldots$$

$$\int_0^1 x^n \ln(1-x)\,dx = \frac{-1}{n+1}\sum_{k=1}^{n+1}\frac{1}{k}, \qquad n=0,1,2,\ldots$$

$$\int_0^1 x^n \ln(1+x)\,dx = \begin{cases} \dfrac{1}{n+1}\left[2\ln 2+\displaystyle\sum_{k=1}^{n+1}\frac{(-1)^k}{k}\right], & n=0,2,4,\ldots \\ \dfrac{1}{n+1}\displaystyle\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k}, & n=1,3,5,\ldots \end{cases}$$

$$\int_0^1 \frac{\ln(1\pm x^p)}{x}\,dx = (-1\pm 3)\frac{\pi^2}{24p}, \qquad p>0$$

$$\int_0^1 x^p(-\ln x)^q\,dx = \begin{cases} \dfrac{n!}{(p+1)^{n+1}}, & p>-1,\ q=n=0,1,2,\ldots \\ \dfrac{\Gamma(q+1)}{(p+1)^{q+1}}, & p>-1,\ q>-1 \end{cases}$$

$$\int_0^1 \frac{x^p - x^q}{\ln x}\,dx = \ln\frac{p+1}{q+1}, \qquad p>-1,\ q>-1$$

$$\int_0^1 x^{p-1}\sin(q\ln x)\,dx = \frac{-q}{p^2+q^2}, \qquad p>0$$

$$\int_0^1 x^{p-1}\cos(q\ln x)\,dx = \frac{p}{p^2+q^2}, \qquad p>0$$

$$\int_0^{\pi/2} \ln(\sin x)\,dx = \int_0^{\pi/2}\ln(\cos x)\,dx = -\frac{\pi}{2}\ln 2$$

$$\int_0^{\pi/2} [\ln(\sin x)]^2 dx = \int_0^{\pi/2}[\ln(\cos x)]^2 dx = \frac{\pi}{2}(\ln 2)^2 + \frac{\pi^3}{24}$$

$$\int_0^{\pi} x\ln(\sin x)\,dx = -\frac{\pi^2}{2}\ln 2$$

$$\int_0^{\pi/2} (\sin x)\ln(\sin x)\,dx = \int_0^{\pi/2}(\cos x)\ln(\cos x)\,dx = \ln 2 - 1$$

$$\int_0^{2\pi} \ln(a+b\sin x)\,dx = \int_0^{2\pi}\ln(a+b\cos x)\,dx = 2\pi\ln\left(a+\sqrt{a^2-b^2}\right)$$

$$\int_0^{\pi} \ln(a\pm b\cos x)\,dx = \pi\ln\left(\frac{a+\sqrt{a^2-b^2}}{2}\right), \qquad a\ge b>0$$

$$\int_0^{\pi} \ln(a^2 \pm 2ab\cos x + b^2)\,dx = \begin{cases} 2\pi\ln a, & a\ge b>0 \\ 2\pi\ln b, & b\ge a>0 \end{cases}$$

$$\int_0^{\pi/2} \ln(1\pm\cos x)\,dx = -\frac{\pi}{2}\ln 2 \pm 2G = -\frac{\pi}{2}\ln 2 \pm 1.8319311884\ldots$$

$$\int_0^{\pi/4} \ln(1+\tan x)\,dx = \frac{\pi}{8}\ln 2$$

$$\int_0^{\pi/2} \ln(1+\tan x)\,dx = \frac{\pi}{2}\ln 2 + G = \frac{\pi}{2}\ln 2 + 0.9159655942$$

$$\int_0^{\pi/4} \ln(1-\tan x)\,dx = \frac{\pi}{8}\ln 2 - G = \frac{\pi}{8}\ln 2 - 0.457982797\ldots$$

$$\int_0^{\pi/2} \frac{\ln(1+b\cos x)-\ln(1+a\cos x)}{\cos x}\,dx = \frac{1}{2}\left\{(\cos^{-1} a)^2 - (\cos^{-1} b)^2\right\}$$

$$\int_0^{\pi/4} \ln(\sin x)\,dx = -\frac{\pi}{4}\ln 2 - \frac{G}{2} = -\frac{\pi}{4}\ln 2 - 0.457982797\ldots$$

$$\int_0^{\pi/4} \ln(\cos x)\,dx = -\frac{\pi}{4}\ln 2 + \frac{G}{2} = -\frac{\pi}{4}\ln 2 + 0.457982797\ldots$$

$$\int_0^{a} \ln\left(2\sin\frac{x}{2}\right)dx = -\left(\frac{\sin a}{1^2} + \frac{\sin 2a}{2^2} + \frac{\sin 3a}{3^2} + \cdots\right)$$

$$\int_0^{\pi/2} \ln(1+p\sin^2 x)\,dx = \int_0^{\pi/2} \ln(1+p\cos^2 x)\,dx = \pi\ln\frac{1+\sqrt{1+p}}{2}, \qquad p \ge -1$$

$$\int_0^{\pi/2} \ln(a^2\sin^2 x + b^2\cos^2 x)\,dx = \pi\ln\frac{a+b}{2}, \qquad a>0,\ b>0$$

$$\int_0^{\pi/2} \ln(a^2 + b^2\tan^2 x)\,dx = \int_0^{\pi/2} \ln(a^2 + b^2\cot^2 x)\,dx = \pi\ln(a+b), \qquad a>0,\ b>0$$

$$\int_0^{\infty} e^{-x}\ln x\,dx = -\gamma$$

$$\int_0^{\infty} xe^{-x}\ln x\,dx = 1-\gamma$$

$$\int_0^{\infty} x^2e^{-x}\ln x\,dx = 3-2\gamma$$

$$\int_0^{\infty} \frac{e^{-x}\ln x}{\sqrt{x}}\,dx = -\sqrt{\pi}(2\ln 2+\gamma)$$

$$\int_0^\infty e^{-x^2}\ln x\,dx = -\frac{\sqrt{\pi}}{4}(\gamma + 2\ln 2)$$

$$\int_0^\infty \ln(1\pm e^{-x})\,dx = (-1\pm 3)\frac{\pi^2}{24}$$

$$\int_0^\infty \frac{\ln x}{x^2+a^2}\,dx = \frac{\pi}{2a}\ln a, \qquad a>0$$

$$\int_0^\infty \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{4}\ln 2 + G = \frac{\pi}{4}\ln 2 + 0.9159655942\ldots$$

$$\int_0^\infty \frac{x^{p-1}\ln x}{1\pm x}\,dx = \frac{\pi^2}{\sin^2 p\pi}\left(-\cos^2\frac{p\pi}{2} \pm \sin^2\frac{p\pi}{2}\right), \qquad 0<p<1$$

$$\int_0^\infty \frac{\ln(1+x^p)}{x^{q+1}}\,dx = \frac{\pi}{q\sin(\pi q/p)}, \qquad 0<q<p$$

Με υπερβολικές συναρτήσεις

$$\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2}, \qquad a>0$$

$$\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2a}, \qquad a>0$$

$$\int_0^\infty \frac{x\,dx}{\cosh ax} = \frac{2G}{a^2} = \frac{1.8319311884\ldots}{a^2}, \qquad a>0$$

$$\int_0^\infty \frac{x^{p-1}\,dx}{\sinh ax} = \frac{2^p-1}{2^{p-1}a^p}\Gamma(p)\zeta(p), \qquad a>0,\ p>1$$

[$\Gamma(x)$ η συνάρτηση γάμα, $\zeta(x)$ η συνάρτηση ζήτα του Riemann]

$$\int_0^\infty \frac{x^{p-1}\,dx}{\cosh ax} = \frac{2\Gamma(p)}{a^p}\left(1-\frac{1}{3^p}+\frac{1}{5^p}-\frac{1}{7^p}+\cdots\right), \qquad a>0,\ p>0$$

$$\int_0^{\pi/2} \frac{\sinh x}{\sin x}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin ax}{\sinh bx}\,dx = \frac{\pi}{2b}\tanh\frac{a\pi}{2b}, \qquad b>0$$

$$\int_0^\infty \frac{\cos ax}{\cosh bx}\,dx = \frac{\pi}{2b}\left(\cosh\frac{a\pi}{2b}\right)^{-1}, \qquad b > 0$$

$$\int_0^\infty \frac{\sinh px}{\sinh qx}\,dx = \frac{\pi}{2q}\tan\frac{\pi p}{2q}, \qquad |p| < q$$

$$\int_0^\infty \frac{\cosh px}{\cosh qx}\,dx = \frac{\pi}{2q\cos(\pi p/2q)}, \qquad |p| < q$$

$$\int_0^\infty \sin px \tanh qx\,dx = \frac{\pi}{2q\sinh(\pi p/2q)}, \qquad q > 0$$

$$\int_0^\infty \frac{\sin px}{\tanh qx}\,dx = \frac{\pi}{2q\tanh(\pi p/2q)}, \qquad q > 0$$

$$\int_0^\infty \frac{\sinh ax}{e^{bx}+1}\,dx = \frac{\pi}{2b\sin(a\pi/b)} - \frac{1}{2a}$$

$$\int_0^\infty \frac{\sinh ax}{e^{bx}-1}\,dx = \frac{1}{2a} - \frac{\pi}{2b}\cot\frac{a\pi}{b}$$