UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

SIGNALS AND SYSTEMS
Solved Examples – Tutorial Sheet 13

Instructor: Michalis Paraskevas, Professor

LINEAR FREQUENCY-SELECTIVE FILTERS

- Ideal Linear Frequency-Selective Filters
- Practical Filters

1. Ideal Linear Frequency-Selective Filters

📖 Example 1

Compare the group delays of two systems. The first has the frequency response

$$H_1(\Omega) = e^{-j\Omega T_0}, \quad \text{with } T_0 = 4\,ms$$

while the second has:

$$H_2(\Omega) = \frac{1}{1 + j\Omega/\Omega_c}, \quad \text{with} \quad \Omega_c = 500\,rad/sec$$

Calculate the phase and group delay of both systems and comment on which system causes phase distortion.

Answer: For the first system:

$$H_1(\Omega) = e^{-j\Omega T_0}$$

the phase is:

$$\theta_1(\Omega) = -\Omega T_0$$

and the group delay is:

$$\tau_{g1}(\Omega) = -\frac{d\theta_1(\Omega)}{d\Omega} = T_0 = 4\,ms$$

All frequencies are delayed by the same amount of time, so no phase distortion occurs. Thus, the first system is an ideal time delay.

For the second system:

$$H_2(\Omega) = \frac{1}{1 + j(\Omega/\Omega_c)}$$

the phase is:

$$\theta_2(\Omega) = -tan^{-1}\left(\frac{\Omega}{\Omega_c}\right)$$

and the group delay is:

$$\tau_{g2}(\Omega) = -\frac{d\theta_2(\Omega)}{d\Omega} = \frac{1/\Omega_c}{1+(\Omega/\Omega_c)^2} = \frac{\Omega_c}{\Omega_c^2+\Omega^2}$$

The phase $\theta_2(\Omega)$ is not a linear function of Ω. Therefore, the group delay $\tau_{g2}(\Omega)$ is not constant. Some frequencies are delayed more than others. This causes phase distortion, especially in signals containing many frequencies. Indeed, for $\Omega_c = 500\ rad/s$, we obtain:

- Frequency $\Omega = 0$: $\tau_{g2}(0) = \frac{1}{500} = 2\ ms$
- Frequency $\Omega = 500\ rad/s$: $\tau_{g2}(500) = \frac{500}{500^2+500^2} = 1\ ms$
- Frequency $\Omega = 1000\ rad/s$: $\tau_{g2}(1000) = \frac{500}{500^2+1000^2} = 0.4\ ms$

Comments:

- The linear phase of the first system leads to a constant group delay and hence a pure time shift of the signal. In contrast, the second system has nonlinear phase, so its group delay increases as the input signal frequency increases, causing a signal with multiple frequencies to be distorted in time at the output.
- Constant group delay is crucial in telecommunications systems where information is carried by the relative timing of multiple frequency components. Nonlinear phase distorts pulses, symbols, transients, and waveforms even when the passband magnitude is satisfactory.

📖 Example 2

A signal $x(t) = e^{-2t}u(t)$ passes through an ideal lowpass filter with cutoff frequency $\Omega_H = 1\,rad/sec$. Find the ratio of output energy to input energy (take $A = 1$ for the filter).

Answer: The Fourier transform of the signal $x(t) = e^{-2t}u(t)$ is:

$$X(\Omega) = \frac{1}{2+j\Omega}$$

Therefore, the energy spectral density of the signal $x(t)$ is:

$$S_x(\Omega) = |X(\Omega)|^2 = \left|\frac{1}{2+j\Omega}\right|^2 = \frac{1}{\Omega^2+4}$$

Since the magnitude of the lowpass filter frequency response is given by:

$$|H(\Omega)| = \begin{cases} 1, & |\Omega| \le \Omega_H = 1 \\ 0, & \text{elsewhere} \end{cases}$$

the energy spectral density of the output signal is:

$$S_y(\Omega) = |X(\Omega)|^2\,|H(\Omega)|^2 = \begin{cases} \dfrac{1}{\Omega^2+4}, & |\Omega| \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

The total energy of the output signal $y(t)$ is given by Parseval's theorem:

$$E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\Omega)\,d\Omega = \frac{1}{\pi}\int_0^1 \frac{1}{\Omega^2+4}\,d\Omega = \frac{1}{2\pi}\tan^{-1}(0.5)$$

Similarly, the energy of the input signal $x(t)$ is:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_0^{\infty} e^{-4t}dt = \frac{1}{4}$$

Dividing the last two expressions gives the required ratio:

$$\frac{E_y}{E_x} = \frac{(1/2\pi)\tan^{-1}(0.5)}{1/4} = \frac{2}{\pi}\tan^{-1}(0.5) \approx 0.295$$

We observe that the ideal filter removes 70.5% of the input energy, and only 29.5% reaches the output.

Comments:

- This example connects spectral analysis with signal energy, showing that a filter affects both the signal waveform in time and the fraction of its energy that reaches the output. The spectrum fully explains the energy loss: frequency components outside the passband are removed, while those within the passband are retained.
- In telecommunications systems, bandwidth determines how much signal energy is actually transmitted through a channel or filter. Choosing the cutoff frequency therefore involves a crucial tradeoff: a narrow bandwidth reduces transmitted energy and may distort the signal, while a wider bandwidth preserves more information but requires more spectral resources.

📖 Example 3

A signal $x(t) = e^{-2t}u(t)$ passes through an ideal lowpass filter with cutoff frequency $\Omega_H = 1\,rad/sec$. Find the ratio of output energy to input energy (take $A = 1$ for the filter).

Answer: The Fourier transform of the signal $x(t) = e^{-2t}u(t)$ is:

$$X(\Omega) = \frac{1}{2+j\Omega}$$

Therefore, the energy spectral density of the signal $x(t)$ is:

$$S_x(\Omega) = |X(\Omega)|^2 = \left|\frac{1}{2+j\Omega}\right|^2 = \frac{1}{\Omega^2+4}$$

Since the magnitude of the lowpass filter frequency response is given by:

$$|H(\Omega)| = \begin{cases} 1, & |\Omega| < \Omega_H = 1 \\ 0, & \text{elsewhere} \end{cases}$$

the energy spectral density of the output signal is:

$$S_y(\Omega) = |X(\Omega)|^2\ |H(\Omega)|^2 = \begin{cases} \dfrac{1}{\Omega^2+4}, & |\Omega| < 1 \\ 0, & \text{elsewhere} \end{cases}$$

The total energy of the output signal $y(t)$ is given by Parseval's theorem:

$$E_y = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_y(\Omega)\, d\Omega = \frac{1}{\pi}\int_0^1 \frac{1}{\Omega^2+4} d\Omega = \frac{1}{2\pi}\tan^{-1}(0.5)$$

Similarly, the energy of the input signal $x(t)$ is:

$$E_x = \int_{-\infty}^{+\infty} x^2(t)dt = \int_0^{+\infty} e^{-4t}dt = 0.25$$

Dividing the last two expressions gives the required ratio:

$$\frac{E_y}{E_x} = \frac{(1/2\pi)\tan^{-1}(0.5)}{0.25} = \frac{2}{\pi}\tan^{-1}(0.5) = 0.29$$

We observe that the ideal filter absorbs 71% of the input energy, and only 29% reaches the output.

📖 Example 4 (*)

A continuous-time signal described by:

$$x(t) = 2cos(20t) + 10sinc\left(\frac{10}{\pi}t\right) + 5sinc^2\left(\frac{5}{\pi}t\right)$$

passes through an ideal lowpass filter with frequency response:

$$H(\Omega) = rect\left(\frac{\Omega}{32}\right)$$

(a) Calculate the spectrum $X(\Omega)$.
(b) Plot the frequency response $H(\Omega)$ and the spectrum $X(\Omega)$.
(c) Calculate and plot the amplitude spectrum $Y(\Omega)$ of the filter output.
(d) Calculate the signal $y(t)$ in the time domain.
(e) Calculate the signal $y(t)$ in the time domain if the filter has frequency response:

$$H(\Omega) = 1 - rect\left(\frac{\Omega}{32}\right)$$

Answer: (a) We will solve this part using known Fourier transform pairs. Since:

$$A\ cos(\Omega_0 t) \overset{F}{\longleftrightarrow} \pi A[\delta(\Omega-\Omega_0) + \delta(\Omega+\Omega_0)]$$

the Fourier transform of the first term is:

$$2\ cos(20t) \overset{F}{\longleftrightarrow} 2\pi\ [\delta(\Omega+20) + \delta(\Omega-20)]$$

Since:

$$\left(\frac{B}{2\pi}\right) sinc\left(\frac{Bt}{2\pi}\right) \overset{F}{\longleftrightarrow} rect\left(\frac{\Omega}{B}\right)$$

the Fourier transform of the second term is:

$$10\ sinc\left(\frac{10}{\pi}t\right) = \pi\ \frac{20}{2\pi} sinc\left(\frac{20}{2\pi}t\right) \overset{F}{\longleftrightarrow} \pi\ rect\left(\frac{\Omega}{20}\right)$$

Finally, since:

$$\left(\frac{B}{2\pi}\right) sinc^2\left(\frac{Bt}{2\pi}\right) \overset{F}{\longleftrightarrow} tri\left(\frac{\Omega}{B}\right)$$

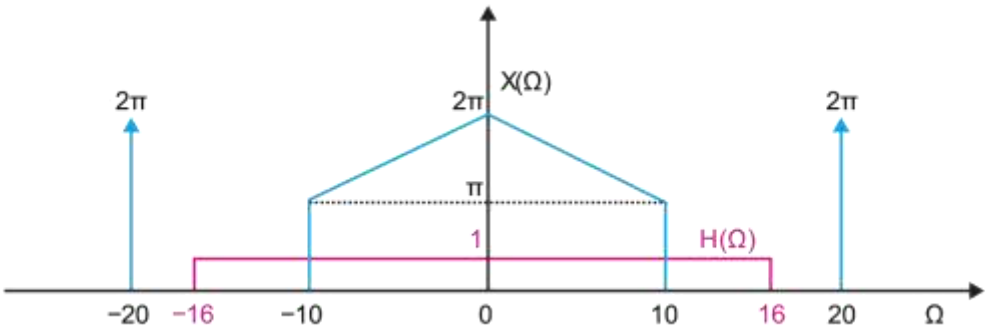
the Fourier transform of the third term is:

$$5sinc^2\left(\frac{5}{\pi}t\right) = \pi\left(\frac{10}{2\pi}\right) sinc^2\left(\frac{10t}{2\pi}\right) \overset{F}{\longleftrightarrow} \pi\ tri\left(\frac{\Omega}{10}\right)$$

By the linearity of the Fourier transform, we add the individual results to obtain the total spectrum $X(\Omega)$:

$$X(\Omega) = 2\pi\ [\delta(\Omega + 20) + \delta(\Omega - 20)] + \pi\ rect\left(\frac{\Omega}{20}\right) + \pi\ tri\left(\frac{\Omega}{10}\right)$$

(b) The lowpass filter has cutoff frequency $\Omega_H = 16\ (rad/sec)$ and unity gain. The spectra $X(\Omega)$ and $H(\Omega)$ are plotted below:



Spectra $X(\Omega)$ and $H(\Omega)$

(c) The output spectrum is given by:

$$Y(\Omega) = X(\Omega)H(\Omega)$$

From the preceding figure, we obtain:

$$Y(\Omega) = \pi\ rect\left(\frac{\Omega}{20}\right) + \pi\ tri\left(\frac{\Omega}{10}\right)$$

(d) From the preceding part, since the lowpass filter passes the input signal frequencies within the passband and rejects those within the stopband, the frequency corresponding to the term $2cos(20t)$

is removed. The time-domain signal $y(t)$ is:

$$y(t) = 10sinc\left(\frac{10}{\pi}t\right) + 5sinc^2\left(\frac{5}{\pi}t\right)$$

(e) In this case, the filter is highpass, with cutoff frequency $\Omega_L = 16\ (rad/sec)$, and passes only the frequency 20 rad/sec. Therefore, the time-domain signal $y(t)$ is:

$$y(t) = 2cos(20t)$$

Comments:

- A filter selectively processes frequencies: it retains some frequency components and rejects others rather than modifying the signal uniformly. Thus, the output is a qualitatively different signal, not merely a reduced version of the input, because specific frequency components are missing.
- Frequency-domain analysis allows the output to be predicted directly without convolution: we only need to know which frequencies pass through the filter.

2. Practical Filters

📖 Example 5

Design a Butterworth lowpass filter satisfying these specifications: maximum passband attenuation $A_p = 1\ dB$ at $\Omega_p = 1000\ rad/sec$, and minimum stopband attenuation $A_s = 40\ dB$ at $\Omega_s = 2000\ rad/sec$. Find the minimum order N and an acceptable half-power frequency Ω_c.

Answer: For a Butterworth filter:

$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$$

The specifications yield the familiar expressions:

$$N \geq \left(\frac{1}{2}\right) \cdot \log_{10}\left(\frac{10^{A_s/10} - 1}{10^{A_p/10} - 1}\right) / \log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)$$

$$\Omega_c \in \left[\frac{\Omega_p}{(10^{A_p/10} - 1)^{1/2N}}, \frac{\Omega_s}{(10^{A_s/10} - 1)^{1/2N}}\right]$$

First, we calculate:

$$10^{A_p/10} - 1 = 10^{0.1} - 1 \approx 0.2589$$

$$10^{A_s/10} - 1 = 10^4 - 1 \approx 9999$$

Thus, the order is:

$$N \geq \left(\frac{1}{2}\right) \cdot \frac{\log_{10}\left(\frac{9999}{0.2589}\right)}{\log_{10}\left(\frac{2000}{1000}\right)} = \left(\frac{1}{2}\right) \cdot \frac{\log_{10}(38618)}{\log_{10}(2)} \approx 7.62$$

so the minimum integer order is:

$$N = 8$$

For $N = 8$, the allowable interval for the half-power frequency is:

$$\Omega_c \geq \frac{1000}{(0.2589)^{1/16}} \approx 1088\, rad/s$$

$$\Omega_c \leq \frac{2000}{(9999)^{1/16}} \approx 1125\, rad/s$$

An acceptable choice is:

$$\Omega_c = 1100\, rad/s$$

Thus, an acceptable filter is an eighth-order Butterworth filter with a half-power frequency of approximately 1100 rad/s. If the full H(s) is required, the poles of the normalized eighth-order Butterworth prototype are placed in the left half-plane, followed by denormalization:

$$H(s) = H_n\left(\frac{s}{\Omega_c}\right)$$

Comments:

- The filter order N expresses how sharp the transition from passband to stopband must be. Here, the requirement of 40 dB at twice the passband frequency leads to an eighth-order filter.
- The specifications define an allowable interval for Ω_c, rather than a unique value. Any value within [1088, 1125] rad/s satisfies the requirements.
- For a Butterworth filter, Ω_c is the half-power frequency, where attenuation is approximately 3 dB. It does not necessarily coincide with the passband frequency Ω_p.

Example 6

For the specifications $A_p = 1\ dB,\ A_s = 40\ dB,\ \Omega_p = 1000\ rad/s$ and $\Omega_s = 1500\ rad/s$, compare the minimum orders of Butterworth and Chebyshev Type I filters. Calculate the respective orders and interpret the result.

Answer: For the Butterworth filter:

$$N_B \geq \left(\frac{1}{2}\right) \cdot \frac{\log_{10}\left(\frac{10^{A_s/10} - 1}{10^{A_p/10} - 1}\right)}{\log_{10}(\Omega_s/\Omega_p)}$$

With $A_p = 1\ dB,\ A_s = 40\ dB,\ \Omega_s/\Omega_p = 1.5$, the order is calculated from:

$$N_B \geq \left(\frac{1}{2}\right) \cdot \frac{\log_{10}\left(\frac{9999}{0.2589}\right)}{\log_{10}(1.5)} \approx 13.02$$

Thus, the minimum order is:

$$N_B = 14$$

For the Chebyshev Type I filter:

$$\varepsilon = \sqrt{10^{\frac{A_p}{10}} - 1} = \sqrt{0.2589} \approx 0.5088$$

and

$$N_C \geq \frac{acosh\left(\sqrt{\frac{10^{A_s/10} - 1}{10^{A_p/10} - 1}}\right)}{acosh(\Omega_s/\Omega_p)}$$

Therefore:

$$N_C \geq \frac{acosh\left(\sqrt{\frac{9999}{0.2589}}\right)}{acosh(1.5)} = \frac{acosh(196.5)}{acosh(1.5)} \approx 6.21$$

Thus, the minimum order is:

$$N_C = 7$$

For the same specifications, the Chebyshev filter achieves the transition with a much lower order than the Butterworth filter because it allows passband ripple. The large difference between orders 14 and 7 highlights the fundamental tradeoff between a maximally flat response and a sharp transition. The Butterworth filter has a maximally flat passband response, allowing no ripple. In contrast, the Chebyshev Type I filter permits controlled passband ripple, here up to $A_p = 1\ dB$.

Comments:

- The Butterworth filter has a flat passband response with no amplitude ripple, making it suitable when a smooth response is required. However, a higher order is needed to achieve a sharp transition to the stopband.
- The Chebyshev Type I filter allows passband ripple and therefore meets the same rejection specification with a lower order. It provides a sharper transition than the Butterworth filter at the expense of a non-flat passband response.
- Reducing the order from 14 to 7 is practically significant because fewer stages are needed for implementation.

Example 7

A filter has the frequency response:

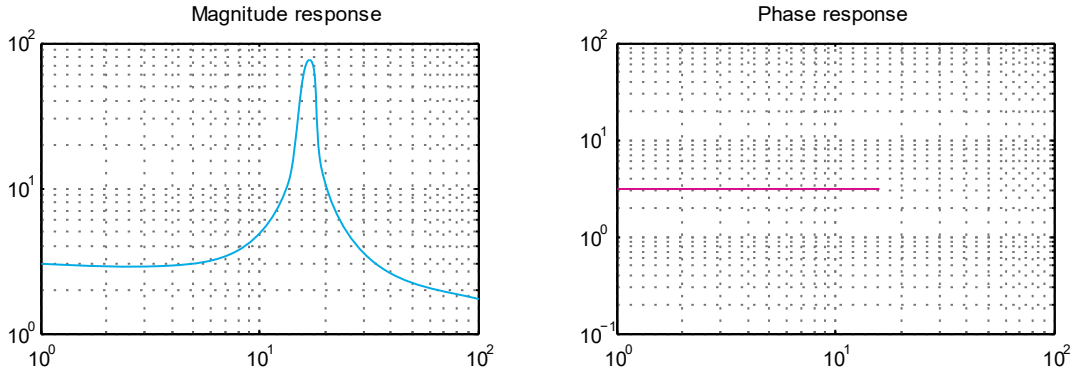
$$H(\Omega) = \frac{6\Omega^2 + 2\Omega + 15}{4\Omega^2 - 3\Omega - 5}$$

(a) Plot the frequency response $H(\Omega)$ on a logarithmic plot.

(b) Calculate and plot the output amplitude spectrum when the signal $x(t) = 3\,e^{-2t}u(t)$ is applied to the filter input.

Answer: (a) To plot the frequency response, we will use the freqs() function. First, we express the numerator and denominator terms of the given function in terms of $(j\Omega)^n$. We obtain:

$$H(\Omega) = \frac{6\Omega^2 + 2\Omega + 15}{4\Omega^2 - 3\Omega - 5} = \frac{\frac{6(j\Omega)^2}{j^2} + \frac{2j\Omega}{j} + 15}{\frac{4(j\Omega)^2}{j^2} - \frac{3j\Omega}{j} - 5}, \qquad (1)$$



Filter frequency response: (a) Magnitude, (b) Phase

From expression (1), we obtain the coefficients of the numerator and denominator polynomials and plot the amplitude and phase spectra.

```
numH = [ 6/j^2 2/j 15 ];
denH = [ 4/j^2 -3/j -5];
w = 0 : 0.1 : 10;
Hw = freqs(numH, denH, w);   % Calculate frequency response
figure(1)       % Frequency response on a logarithmic plot
subplot(121); loglog(abs(Hw));  xlim([1,100]); title('Amplitude Spectrum'); grid on
subplot(122); loglog(angle(Hw));xlim([1,100]); title('Phase Spectrum'); grid on
```

(b) Calculate and plot the output for the given input:

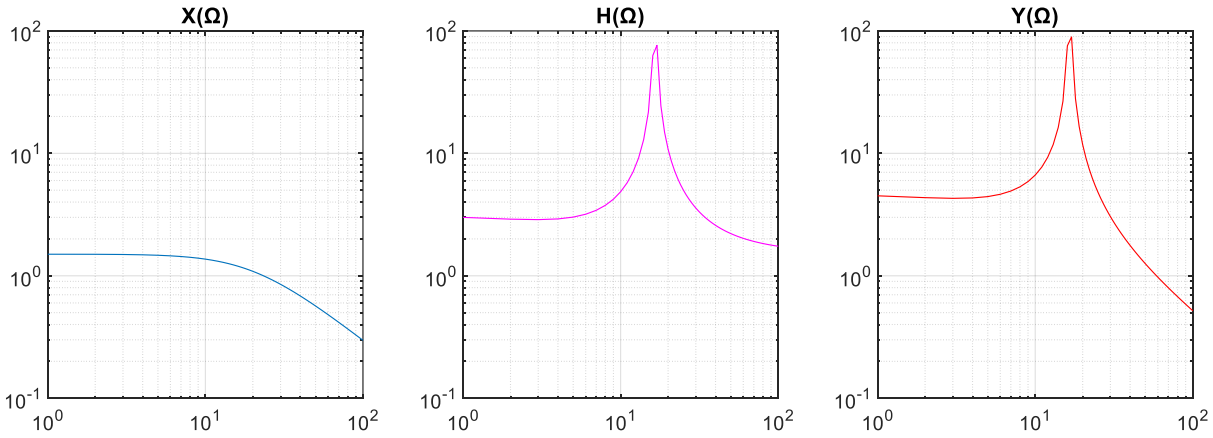
```
syms t w
x(t) = 3*exp(-2*t) * heaviside(t)
```

```
X(w) = fourier(x(t), w)     % Fourier transform of the input
```

We obtain: X(w) = 3/(w*i + 2)

We will describe the function $X(\Omega)$ using freqs():

```
numX = [3]; denX = [1 2];
w = 0:0.1:10;
Xw = freqs(numX, denX, w);
Yw = Xw .* Hw;      % Fourier transform of the output from Y(Ω) = X(Ω) H(Ω)
% Plot the amplitude spectra of the input, filter, and output
figure(3)
subplot(131); loglog(abs(Xw)); axis([1, 100, 10^-1, 10^2]); title('X(Ω)');grid on
subplot(132); loglog(abs(Hw),'m'); axis([1, 100, 10^-1, 10^2]); title('H(Ω)');grid on
subplot(133); loglog(abs(Yw),'r'); axis([1, 100, 10^-1, 10^2]); title('Y(Ω)');grid on
```



Signal amplitude responses: (a) Input, (b) Filter, (c) Output

The figure shows that the output signal amplitude spectrum is the product of the input signal amplitude spectrum and the filter frequency response magnitude.

Comment:

- This example shows that the frequency response is a tool for visually understanding system behavior across the entire spectrum, as well as a mathematical function.
- Using a logarithmic scale (Bode) reveals features that are difficult to see on a linear scale, such as attenuation rates and operating bands.
- The output spectrum is obtained as the product of the input and system spectra, showing that the system acts as a spectral weighting function.
- In analog signal processing, Bode plots are widely used in amplifier and filter design to determine the passband, attenuation rate, and cutoff frequencies.
- Similarly, in digital signal processing, magnitude and phase plots are used to adjust filters, analyze stability, and examine frequency response.

Note: Examples marked with an asterisk (*) also appear in the theory lecture slides.