UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Signals and Systems

Lecture 13: Linear Filters for Frequency Selection

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Lecture Contents

1. Ideal linear frequency-selective filters
   (ideal low-pass, high-pass, band-pass and band-stop filters)
2. Properties of ideal filters
3. Practical filters
4. Analog low-pass filter prototypes

1. Linear Filters for Frequency Selection

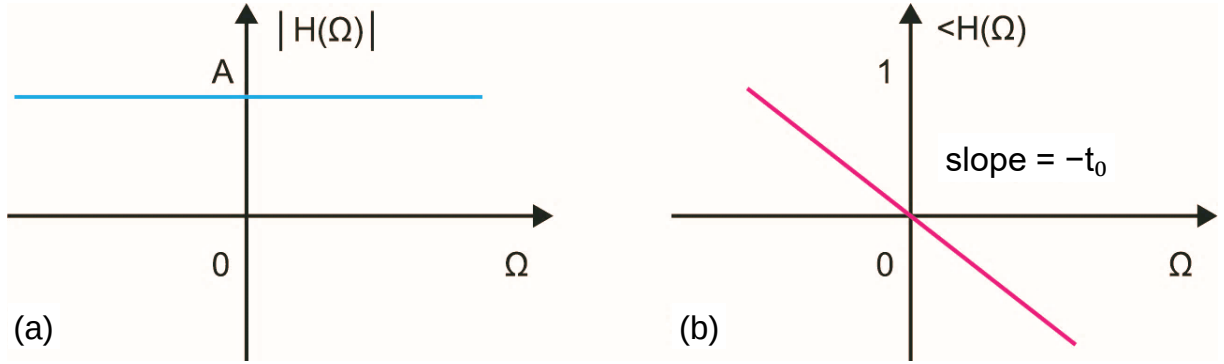
Ideal Filters

- By the eigenfunction property of LTI systems, the steady-state response to a sinusoidal input is also sinusoidal at the same frequency. Its **amplitude** and **phase** depend on the system frequency response.
- Fourier analysis represents a signal as a sum of simple sinusoids. We can therefore modify the magnitude and phase of its frequency components by choosing an appropriate LTI system frequency response.
- This process is called **filtering**. It has applications in telecommunications, signal processing, control systems and other fields.
- A filter is an LTI system whose magnitude response $|H(\Omega)|$ is negligible in certain frequency ranges. It removes selected frequency components from a signal or weights the components of its waveform.

All-Pass Filter

- An ideal all-pass filter has impulse response $h(t) = A\,\delta(t - t_0)$ and frequency response $H(\Omega) = A\,e^{-j\Omega t_0}$.
- Its magnitude spectrum (gain) is constant at all frequencies, while its phase spectrum is linear in frequency:

$$|H(\Omega)| = \mathrm{A}, \qquad \theta_H(\Omega) = -a\Omega,\ \ \alpha \in R$$



(a) Magnitude and (b) phase spectra of an ideal all-pass filter

- It introduces a constant group delay, equal for all signal frequencies.

Linear-Phase Filters

- A filter's phase shift arises from the negative complex exponent $e^{-j\Omega t_0}$ in its frequency response $H(\Omega)$. It represents a delay t_0 in the passage of signal frequencies, called **group delay**:

$$\tau(\Omega) = -\frac{d}{d\Omega}\{\theta_H(\Omega)\}$$

- In a linear-phase system, group delay is constant at all input frequencies. Thus, particularly in the passband, the output retains the input waveform and structure, differing only by a time shift and possibly a constant amplitude scale factor.
- For nonlinear phase, group delay varies with frequency. Different signal components experience different delays, distorting the output waveform.
- Although human hearing is not sensitive to shifts in the phases of a signal's harmonics, such changes can have serious effects when waveform shape matters, as in telecommunications, medical signals and images.

Ideal Linear Frequency-Selective Filters

Linear frequency-selective filters:

- Pass a selected range of signal frequencies and reject the remaining frequencies.
- Introduce the same group delay at all frequencies in the passband.

Types of ideal linear frequency-selective filters:

- Ideal low-pass filter (LPF)
- Ideal high-pass filter (HPF)
- Ideal band-pass filter (BPF)
- Ideal band-stop filter (BSF)

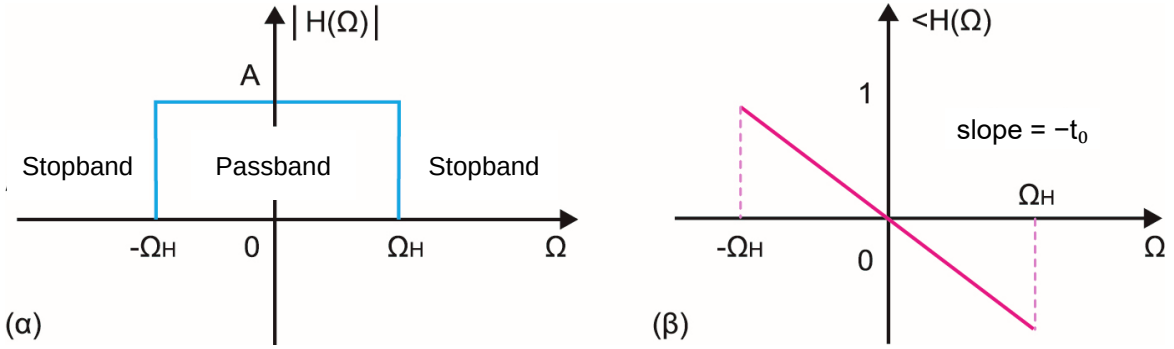
Ideal filters are not practically realizable: their impulse responses are **noncausal** and have **infinite duration**.

Ideal Low-Pass Filter

- An ideal low-pass filter (LPF) is an LTI system that acts as a distortionless filter when the input contains no frequencies above its cutoff frequency (Ω_H). Its frequency response is:

$$H(\Omega) = A\, e^{-j\Omega t_0}\, rect\left(\frac{\Omega}{2\Omega_H}\right) = \begin{cases} A\, e^{-j\Omega t_0}, & |\Omega| \le \Omega_H \\ 0, & |\Omega| > \Omega_H \end{cases}$$

- The factor A is the filter **gain**, usually expressed in dB. The cutoff frequency Ω_H separates the **passband** from the **stopband**.

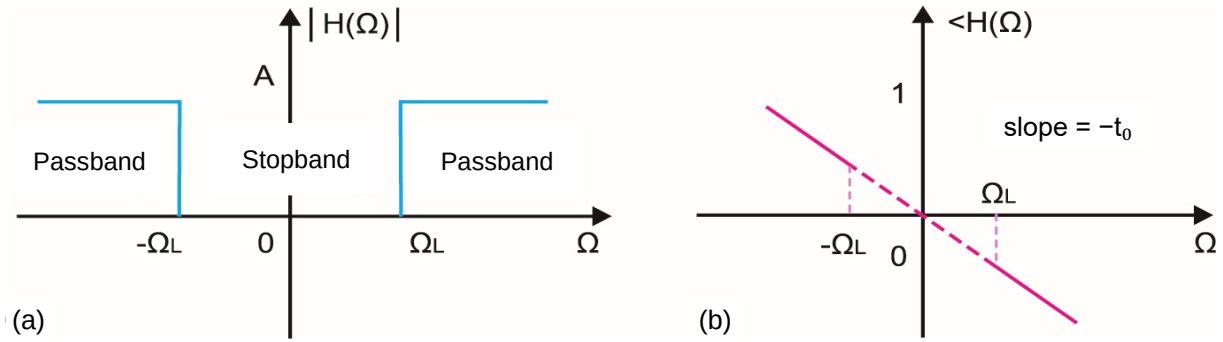


- An ideal LPF passes only low frequencies satisfying $\Omega \le \Omega_H$ and rejects the remaining, higher frequencies.

Ideal High-Pass Filter

- An ideal high-pass filter (HPF) is a linear system that acts as an ideal filter when the input contains no frequencies below its **cutoff frequency** Ω_L. Its frequency response is:

$$H(\Omega) = Ae^{-j\Omega t_0}\left(1 - rect\left(\frac{\Omega}{2\Omega_L}\right)\right) = \begin{cases} A\,e^{-j\Omega t_0}, & |\Omega| \geq \Omega_L \\ 0, & |\Omega| < \Omega_L \end{cases}$$



- An ideal HPF passes only high frequencies satisfying $\Omega \geq \Omega_L$ and rejects the remaining, lower frequencies.

Low-Pass and High-Pass Filter Comparison

- The frequency responses of an ideal high-pass filter $H_{HPF}(\Omega)$ and an ideal low-pass filter $H_{LPF}(\Omega)$ with the same cutoff frequency Ω_c, satisfy:

$$H_{HPF}(\Omega) = A\, e^{-j\Omega t_0} - H_{LPF}(\Omega)$$

For the special case $A = 1$ and $t_0 = 0$, this simplifies to:

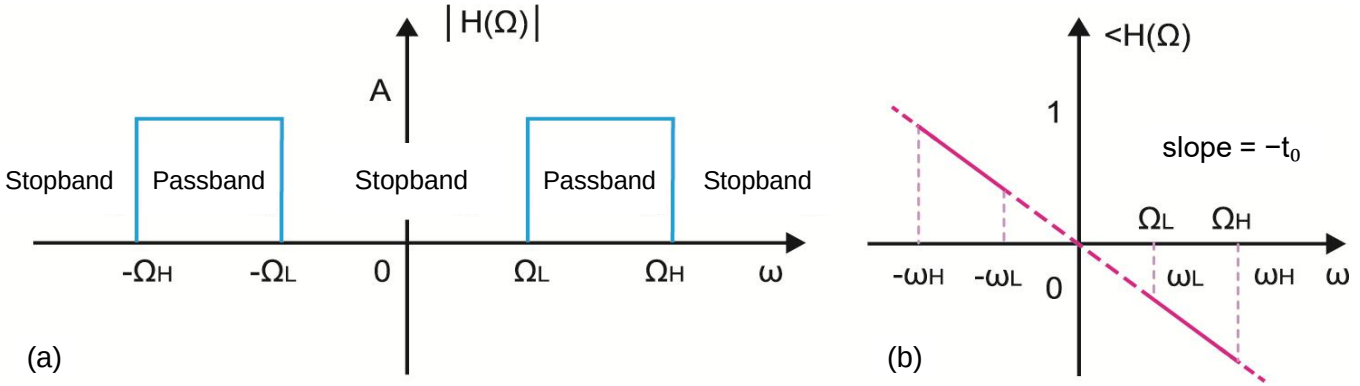
$$H_{HPF}(\Omega) = 1 - H_{LPF}(\Omega)$$

- An ideal all-pass filter is the **sum** of an ideal low-pass and an ideal high-pass filter with the same cutoff frequency, gain and linear phase.

Ideal Band-Pass Filter

- An ideal band-pass filter (BPF) is a linear system that acts as an ideal filter when the input contains no frequencies outside its passband $\Omega_L < \Omega < \Omega_H$. Its frequency response is:

$$H(\Omega) = \begin{cases} A\, e^{-j\Omega t_0}, & \Omega_L \le |\Omega| \le \Omega_H \\ 0, & \text{otherwise} \end{cases} = Ae^{-j\Omega t_0} rect\left[\left(\frac{\Omega - \frac{-\Omega_L - \Omega_H}{2}}{\Omega_H - \Omega_L}\right) + rect\left(\frac{\Omega - \frac{\Omega_L + \Omega_H}{2}}{\Omega_H - \Omega_L}\right)\right]$$

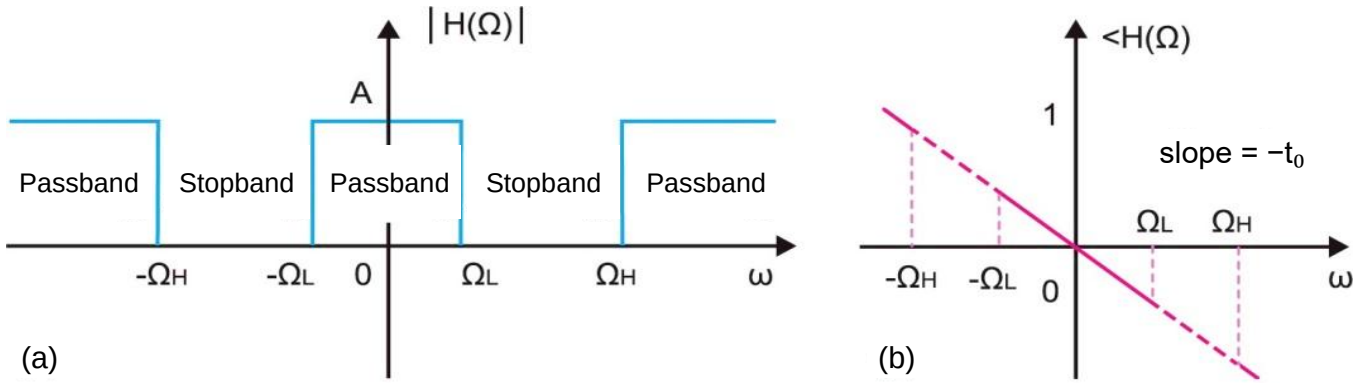


- An ideal BPF passes only frequencies within the passband, satisfying $\Omega_L \le \Omega \le \Omega_H$, and rejects all remaining signal frequencies.

Ideal Band-Stop Filter

- An ideal band-stop filter (BSF) is a linear system that acts as an ideal filter when the input contains no frequencies within its stopband $\Omega_L < \Omega < \Omega_H$. Its frequency response is:

$$H(\Omega) = \begin{cases} 0, & \Omega_L < |\Omega| < \Omega_H \\ A\, e^{-j\Omega t_0}, & \text{otherwise} \end{cases} = A\, e^{-j\Omega t_0} \left[rect\left(\frac{\Omega - \frac{-\Omega_L - \Omega_H}{2}}{\Omega_H - \Omega_L} \right) + rect\left(\frac{\Omega - \frac{\Omega_L + \Omega_H}{2}}{\Omega_H - \Omega_L} \right) \right]$$



- An ideal BSF rejects only frequencies within the stopband, satisfying $\Omega_L \leq \Omega \leq \Omega_H$, and passes all remaining signal frequencies.

2. Properties of Ideal Filters

Properties of Ideal Filters

1. Unity gain (1) in the passband and zero gain (0) in the stopband.
2. An instantaneous transition between passband and stopband.

In addition:

- The impulse response of an LPF is:

$$h(t) = A\frac{\Omega_H}{2\pi}sinc\left(\frac{\Omega_H(t-a)}{2\pi}\right)$$

(a) It is noncausal and (b) it has infinite duration.

- The impulse responses of the other ideal filters have similar properties. The HPF impulse response also contains a delta function. Thus, none of these ideal filters is practically realizable.
- Their noncausal impulse responses make ideal filters unsuitable for real-time applications, since future signal values would be required.

Causality Condition (Paley–Wiener) (1/3)

For a square-integrable function $|H(\Omega)|$ satisfying:

$$\int_{-\infty}^{\infty} |H(\Omega)|^2 d\Omega < \infty, \qquad (1)$$

a necessary and sufficient condition for it to be the magnitude spectrum of a causal function $h(t)$ is convergence of:

$$\int_{-\infty}^{+\infty} \frac{\left|log|H(\Omega)|\right|}{1+\Omega^2} d\Omega < \infty, \qquad (2)$$

- This statement does not fully solve the general problem of finding all conditions under which a complex function $H(\Omega)$ is the Fourier transform of a causal function $h(t)$.
- It assumes square integrability of $H(\Omega)$, as expressed by relation (1). This assumption corresponds to the requirement that the system pass finite energy.

Causality Condition (Paley–Wiener) (2/3)

- Condition (2) concerns the **magnitude** of the frequency response, rather than the full complex function $H(\Omega)$. Even if $|H(\Omega)|$ satisfies (2), an arbitrary phase choice need not yield a causal system.
- An appropriate phase must be found so that the resulting complex function $H(\Omega)$ corresponds to a causal impulse response.
- When relation (1) does not hold, condition (2) is neither necessary nor sufficient. Always apply the Paley–Wiener condition under the assumption that $|H(\Omega)|$ is square-integrable.

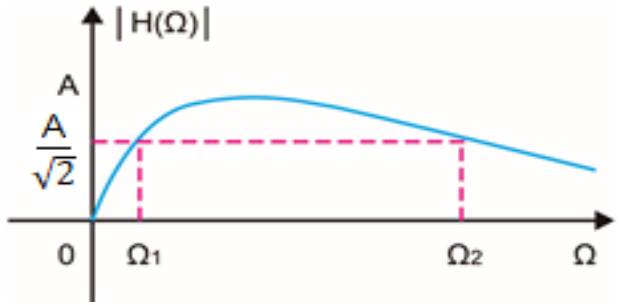
Causality Condition (Paley–Wiener) (3/3)

Summary:

- The Paley–Wiener condition requires the magnitude spectrum $|H(\Omega)|$ to be nonzero over every finite frequency band, although it may vanish at isolated frequencies.
- If $|H(\Omega)| = 0$ throughout a frequency interval, then $log|H(\Omega)|$ tends to $-\infty$, so the integral in condition (2) does not converge.
- Thus, ideal filters do not satisfy the Paley–Wiener condition and cannot be realized exactly in practice.
- They nevertheless provide useful theoretical prototypes for designing practical filters that approximate their behavior.

Filter Bandwidth

Bandwidth describes the significant frequency range of a filter's frequency response.



Consider a filter with transfer function $H(\Omega)$, where $|H(\Omega)|_{\max} = A$. Its **bandwidth** is the interval $BW = \Omega_2 - \Omega_1$ between frequencies Ω_1 and Ω_2, $(\Omega_1, \Omega_2 > 0)$, such that:

$$|H(\Omega_1)| = |H(\Omega_2)| = \frac{A}{\sqrt{2}}$$

Within this interval, the frequency response magnitude is at least $1/\sqrt{2} = 0{,}707$ of its maximum. This is the 3 dB bandwidth, since:

$$20log\frac{1}{\sqrt{2}} \approx -3\ dB$$

3. Practical Filters

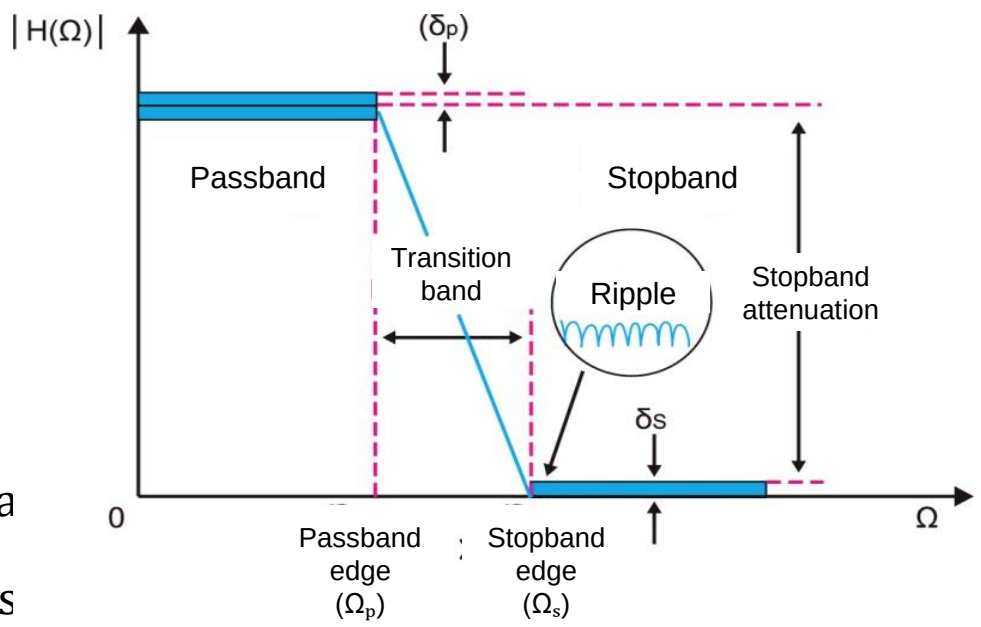
Practical Filters

- Ideal filter properties do not correspond to physically realizable structures:
  - Unity passband gain and zero stopband gain.
  - An instantaneous transition from passband to stopband.
- We approximate the ideal response with a nonideal, **practical** (physically realizable) filter.
- Practical filters do not have the two properties above.
- Many design methods exist for practical analog filters, including Butterworth, Chebyshev and Bessel.

Frequency Response of a Practical Low-Pass Filter

A practical filter has the following properties:

- A finite frequency interval, the **transition band**, separates the passband and stopband.
- In the transition band, gain changes gradually from one in the passband to zero in the stopband. The transition bandwidth is $\Omega_p - \Omega_s$.
- The passband response is not flat at unity (1): it ha
- The stopband response is not flat at zero (0): it has



Analog Filter Design

Frequency-domain filtering follows the Fourier convolution property:

$$Y(\Omega) = H(\Omega)\, X(\Omega)$$

The frequency response $H(\Omega)$ follows from the transfer function $H(s)$:

$$H(\Omega) = H(s)\Big|_{s=j\Omega}$$

The squared magnitude response of an analog low-pass filter is:

$$|H(\Omega)|^2 = \frac{1}{1 + f(\Omega^2)}$$

The following must hold:

- At low frequencies, $f(\Omega^2) \ll 1$ so that $|H(\Omega)|^2 \approx 1$ (passband).
- At high frequencies, $f(\Omega^2) \gg 1$ so that $|H(\Omega)|^2 \to 0$ (stopband).

The key steps in designing a practically realizable analog filter are:

- Choose an appropriate function $f(.)$.
- Find $H(s)$ from $|H(\Omega)|^2$, ensuring filter stability.

Practical Filter Specifications (1/3)

- The passband magnitude response remains near unity, with maximum allowed deviation $\pm\delta_p$ from unity gain:

$$1 - \delta_p < |H(\Omega)| \leq 1 + \delta_p, \qquad 0 \leq |\Omega| < \Omega_p$$

- The stopband magnitude response must be below δ_s:

$$0 \leq |H(\Omega)| \leq \delta_s, \qquad |\Omega| \geq \Omega_s$$

- A transition band of nonzero width must separate the passband and stopband:

$$\left|\Omega_s - \Omega_p\right| \neq 0$$

Practical Filter Specifications (2/3)

To specify a practical filter, define:

- $\boldsymbol{\Omega_p}$: passband cutoff frequency
- $\boldsymbol{\Omega_s}$: stopband cutoff frequency
- $\boldsymbol{\delta_p}$: passband magnitude deviation
- $\boldsymbol{\delta_s}$: stopband magnitude deviation
- $\left|\boldsymbol{\Omega_s - \Omega_p}\right|$: transition bandwidth

Practical Filter Specifications (3/3)

A logarithmic frequency response simplifies filter parameter calculations and improves visualization. Define the attenuation function $\alpha(\Omega)$ by:

$$\alpha(\Omega) = -10\log_{10}|H(\Omega)|^2 = -20\log_{10}|H(\Omega)| \qquad (dB)$$

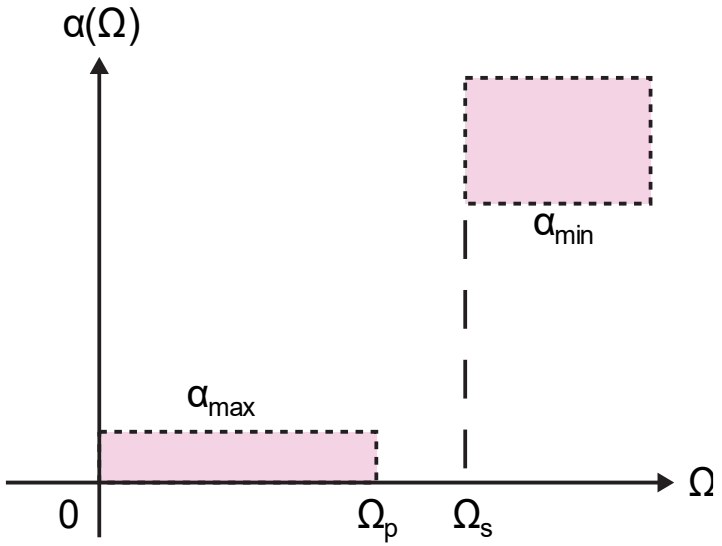
On a logarithmic scale, the specifications are:

- Passband: $0 \le \alpha(\Omega) \le a_{max}, \quad 0 \le |\Omega| < \Omega_p$
- Stopband: $\alpha(\Omega) \ge a_{min}, \quad |\Omega| \ge \Omega_s$

where:

- $a_{max} = -20\log_{10}|1 - \delta_p|$
- $a_{min} = -20\log_{10}(\delta_s)$

Reference level: 0 dB (unity gain).



Low-pass filter frequency response on a logarithmic scale

Realizable Filter Design

Given the spectral specifications (a) passband a_{max} and Ω_p, and (b) stopband a_{min} and Ω_s, design a practically realizable filter as follows:

- Choose an appropriate function $f(.)$, such as Butterworth or Chebyshev.
- Find the parameters of $f(.)$ that satisfy the specifications.
- Factor the squared magnitude function $|H(\Omega)|^2$. Select the poles in the left half-plane for stability to obtain the normalized prototype transfer function $H_N(s)$.
- Choose the gain K so that $KH_N(s)$ provides the required gain at zero frequency (DC gain).
- Denormalize frequency to obtain the desired band-edge frequencies. Use $S = s/\Omega_n$, where $\Omega_n = \Omega_{hp}$ for Butterworth and $\Omega_n = \Omega_p$ for Chebyshev.
- Verify that $KH_N(\Omega)$ satisfies the magnitude specifications. If required, also check that phase is approximately linear in the passband.

4. Analog Low-Pass Filter Prototypes

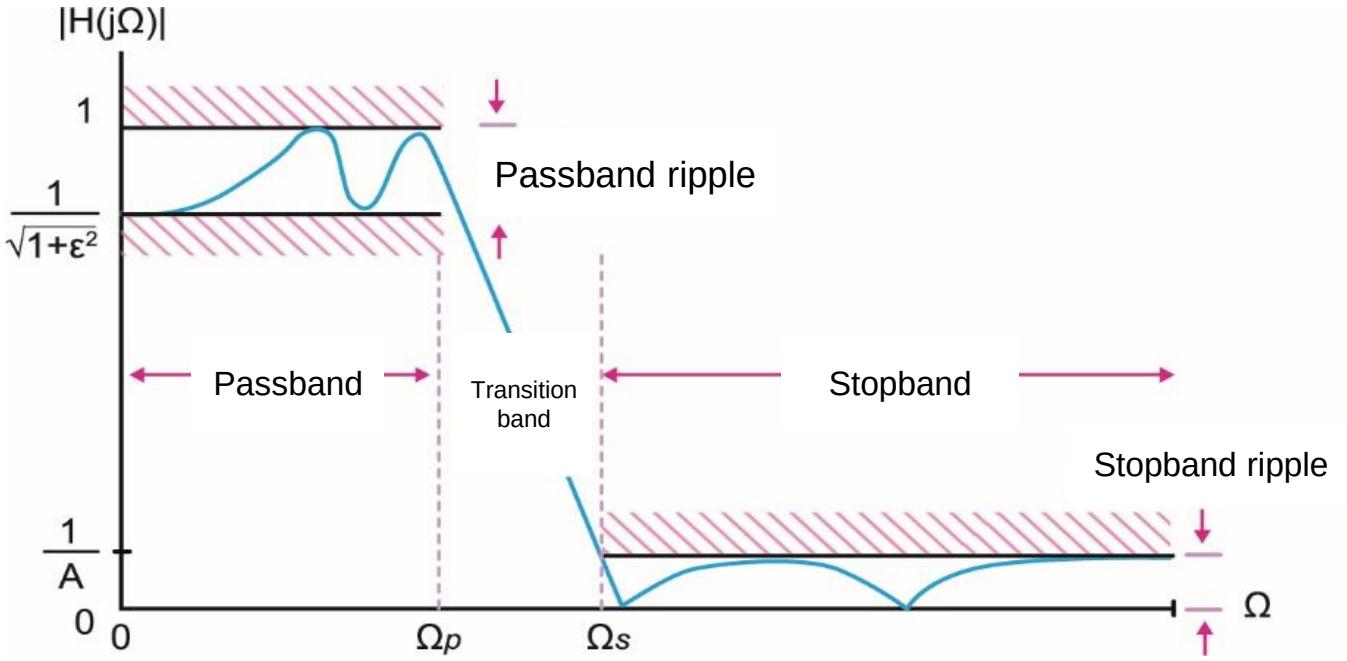
Analog Low-Pass Filter Prototype (1/2)

The squared magnitude response satisfies:

$$\frac{1}{1+\varepsilon^2} \le |H(j\Omega)|^2 \le 1, \qquad |\Omega| \le \Omega_p$$

$$0 \le |H(j\Omega)|^2 \le \frac{1}{A^2}, \qquad |\Omega| \ge \Omega_s$$

- ε: passband ripple
- $\boldsymbol{\Omega_p}$: passband cutoff frequency
- $\boldsymbol{A}$: stopband attenuation
- $\boldsymbol{\Omega_s}$: stopband cutoff frequency



Analog Low-Pass Filter Prototype (2/2)

Parameters ε and A relate to the ripple R_p and attenuation A_s parameters used for FIR filters through:

$$R_p = -10log_{10}\left(\frac{1}{1+\varepsilon^2}\right) \Rightarrow \varepsilon = \sqrt{10^{R_p/10}-1}$$

$$A_s = -10log_{10}\left(\frac{1}{A^2}\right) \Rightarrow A = 10^{A_s/10}$$

They also relate to magnitude deviation parameters δ_p and δ_s through:

$$\frac{1-\delta_p}{1+\delta_\pi} = \sqrt{\frac{1}{1+\varepsilon^2}} \Rightarrow \varepsilon = \frac{2\sqrt{\delta_p}}{1-\delta_p} \text{ and } \frac{\delta_s}{1+\delta_p} = \frac{1}{A} \Rightarrow A = \frac{1+\delta_p}{\delta_s}$$

At frequencies Ω_p and Ω_s,, the squared magnitude response is:

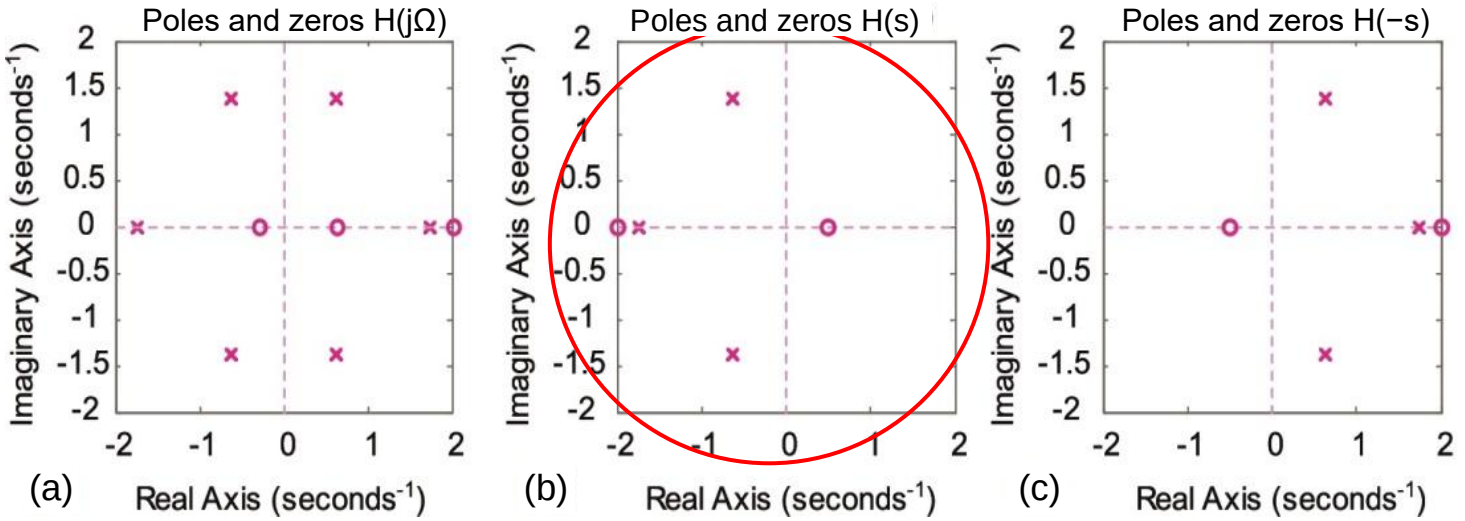
$$\left|H(j\Omega_p)\right|^2 = \frac{1}{1+\varepsilon^2} \text{ and } |H(j\Omega_s)|^2 = \frac{1}{A^2}$$

Pole-Zero Diagram of an IIR Filter

The squared magnitude response satisfies:

$$|H(j\Omega)|^2 = H(j\Omega)H(-j\Omega) = H(s)\,H(-s)\Big|_{s=j\Omega} \Rightarrow H(s)\mathrm{H}(-\mathrm{s}) = |\mathrm{H}(j\Omega)|^2\Big|_{\Omega=\mathrm{s/j}}$$

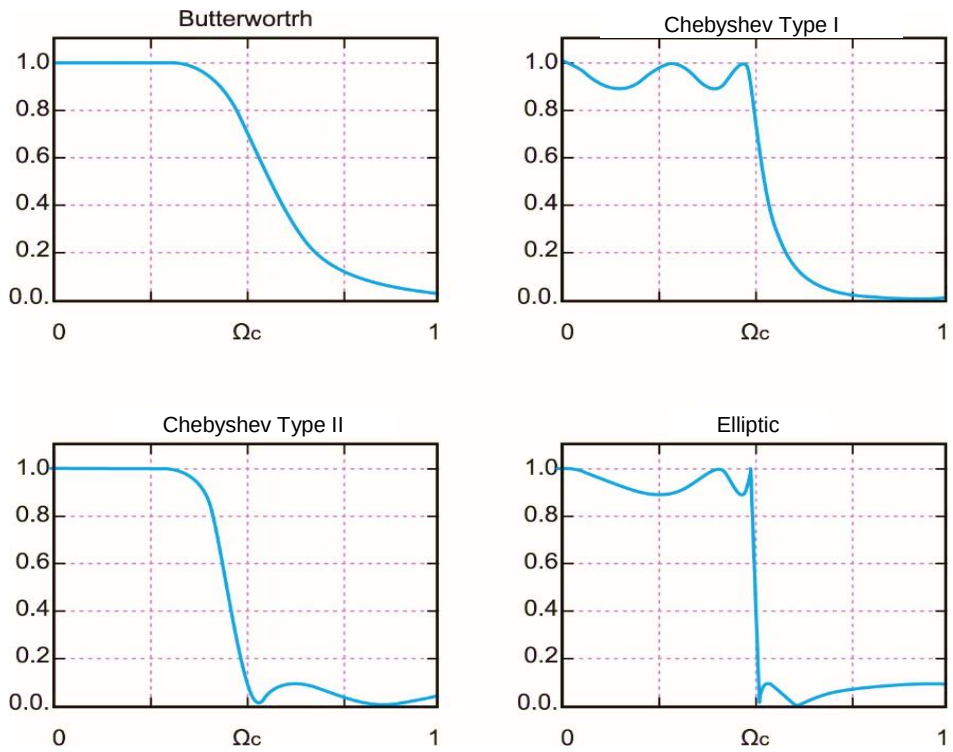
Thus, the poles and zeros of $|H(j\Omega)|^2$ are symmetric about the $j\Omega$ axis.



Pole-zero diagrams of (a) $|H(j\Omega)|^2 = \mathrm{H}(s)H(-s)$, (b) $H(s)$ and (c) $H(-s)$

Analog Low-Pass Filter Prototypes

- Butterworth low-pass filter
- Chebyshev Type I and Type II low-pass filters
- Elliptic low-pass filter



Frequency responses of analog low-pass prototypes: Butterworth, Chebyshev I, Chebyshev II and elliptic

Butterworth Low-Pass Filter (LPF)

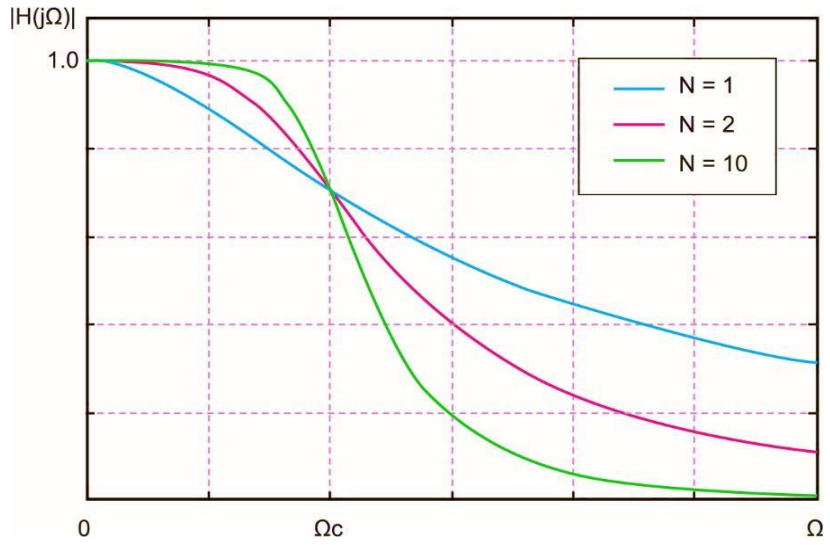
Squared magnitude frequency response:

$$|H(\Omega')|^2 = \frac{1}{1+(\Omega')^{2N}}, \qquad \text{where } \Omega' = \Omega/\Omega_c$$

- Ω_c: cutoff or half-power frequency (-3 dB)
- N: Butterworth filter order

The following hold:

- $|H(j0)|^2 = 1 = 0\text{dB}$ for every filter order N
- $|H(j\Omega_c)|^2 = 1/2 = 3dB$, independently of N
- For $\Omega_c = 1\ rad/sec$, we obtain the low-pass prototype.
- For $N \to \infty$, the response approaches an ideal LPF.



Butterworth magnitude responses for different filter orders

Butterworth LPF Design

Given the specifications:

- $\boldsymbol{\Omega_p}$: passband-edge cutoff frequency
- $\boldsymbol{\Omega_s}$: stopband-edge cutoff frequency
- Minimum passband response magnitude $1/\sqrt{1+\varepsilon^2}$
- Maximum stopband response magnitude $1/A$

Calculate:

- The minimum filter order:

$$N = \left\lceil \frac{\log_{10}\left[(A^2-1)/\varepsilon^2\right]}{2\log_{10}\left[\Omega_s/\Omega_p\right]} \right\rceil \text{ or } N = \left\lceil \frac{\log_{10}\left[(10^{Rp/10}-1)/(10^{As/10}-1)\right]}{2\log_{10}\left[\Omega_p/\Omega_s\right]} \right\rceil$$

- The ratio Ω_s/Ω_p is critical in determining the minimum filter order.
- The **cutoff frequency**:

$$\Omega_c = \frac{\Omega_p}{\sqrt[2N]{10^{Rp/10}-1}} \text{ or } \Omega_c = \frac{\Omega_s}{\sqrt[2N]{10^{As/10}-1}}$$

Key Properties of a Butterworth LPF

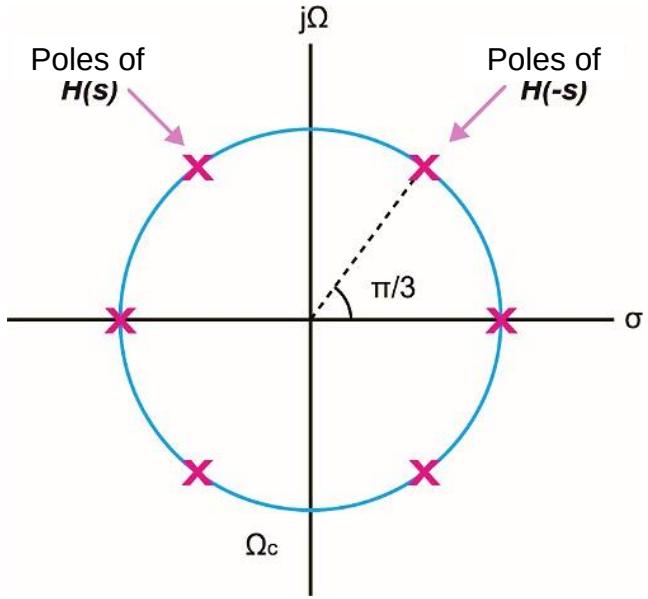
- A Butterworth filter has a **maximally flat** passband magnitude response, with no ripple.
- As order N increases, the passband-to-stopband transition becomes steeper and the response approaches the ideal low-pass response.
- The approximation to an ideal LPF improves when at least one of the following occurs:
  - Order N increases.
  - Transition bandwidth decreases.
  - Required stopband attenuation a_{min} increases.
  - Maximum allowed passband attenuation a_{max} decreases.
- Although many cutoff frequencies are acceptable, we usually choose:

$$\Omega_c = \frac{\Omega_p}{(10^{0.1a_{max}}-1)^{1/2N}} \text{ or } \Omega_c = \frac{\Omega_s}{\left(10^{0.1a_{min}}-1\right)^{1/2N}}$$

- For a required DC gain other than unity, scale the transfer function $H_N(s)$ by an appropriate gain K.

Poles and Zeros of a Butterworth LPF

- For filter order N, there are $2N$ poles.
- The poles lie on a circle of radius Ω_c with equal angular separation $2\pi/2N = \pi/N$.
- The filter is causal and stable when all poles lie in the left half-plane of frequency s.
- Left-half-plane poles correspond to $H(s)$; right-half-plane poles correspond to $H(-s)$.



Poles of $H(s)H(-s)$ for N = 3

Transfer Function of an Analog Butterworth LPF Prototype

Order (N)	Transfer function $H(s)$ (for $\Omega_c = 1 rad/sec$)
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+\sqrt{2}s+1}$
3	$\frac{1}{s^3+2s^2+2s+1}$
4	$\frac{1}{s^4+2.6131s^3+3.4142s^2++2.6131s+1}$

Butterworth LPF Design in MATLAB

Designing an analog Butterworth low-pass filter:

- [N, Wn] = buttord(Wp, Ws, Rp, As): Returns the minimum order N satisfying the specifications. Wn is the normalized cutoff frequency Ω_c, with $0 < W_n < 1$, where 1 corresponds to half the sampling frequency.
- [b, a] = butter(N, Wn): Returns transfer function coefficients $a[n], b[m]$ for order N and cutoff frequency Wn. Alternatively, [z, p, k] = butter(N, Wn) returns zeros, poles and gain.

Designing an analog Butterworth low-pass prototype:

- [z, p, k] = buttap(N): Returns zeros, poles and gain of a normalized analog Butterworth prototype of order N. It has N poles on the unit circle in the left half-plane and no zeros.

Chebyshev Type I LPF Prototype (1/2)

A Chebyshev Type I filter has only poles. It has equal ripple in the passband ($0 \leq \Omega < 1$) and decreases monotonically in the stopband ($\Omega > 1$). Its squared magnitude response is:

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \, T_N^2(\Omega/\Omega_c)}$$

- ε: passband ripple
- Ω_c: cutoff frequency

where $T_N(\Omega)$: Chebyshev polynomial of order N:

$$T_N(x) = \begin{cases} \cos(N \cos^{-1} x), & 0 \leq x \leq 1 \\ \cosh(N \cosh^{-1} x), & 1 < x < \infty \end{cases}, \qquad \text{where } x = \Omega/\Omega_c$$

Chebyshev Type I LPF Prototype (1/2)

The following hold:

- $\Omega = 0$ and N even: $|H(j0)|^2 = 1$
- $\Omega = 0$ and N odd: $|H(j0)|^2 = 1/(1+\varepsilon^2)$
- $\Omega = \Omega_c$: $|H(j\Omega_c)|^2 = 1/(1+\varepsilon^2)$, independently of N
- $0 \le \Omega \le \Omega_c$: $|H(j\Omega)|^2$ oscillates between 0 and $1/(1+\varepsilon^2)$
- $\Omega > \Omega_c$: $|H(j\Omega)|^2$ decreases monotonically.

Cutoff frequency Ω_c:

$$\Omega_c = \Omega_p \quad \text{and} \quad \Omega_r = \frac{\Omega_s}{\Omega_p}$$

Minimum order N:

$$N = \left\lceil \frac{\log_{10}\left[g + \sqrt{g^2 - 1}\right]}{\log_{10}\left[\Omega_r + \sqrt{\Omega_r^2 - 1}\right]} \right\rceil$$

Chebyshev I LPF Design in MATLAB

Designing an analog Chebyshev Type I low-pass filter:

- [N, Wn] = cheb1ord(Wp, Ws, Rp, Rs): Returns the minimum order N satisfying the specifications. Wn is the normalized cutoff frequency Ω_c.
- [b, a] = cheby1(N, R, Wn): Returns transfer function coefficients $a[n]$, $b[m]$ for order N, cutoff Wn and ripple R. Alternatively, [z, p, k] = butter(N, Wn) returns zeros, poles and gain.

Designing an analog Chebyshev Type I low-pass prototype:

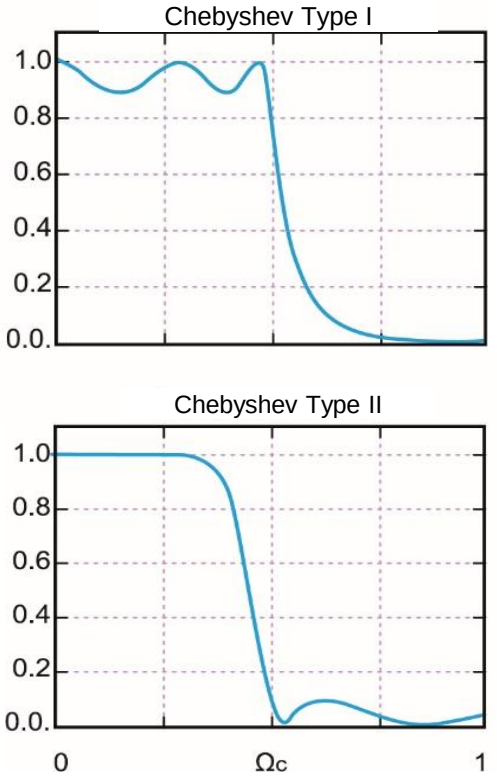
- [z, p, k] = cheb1app(N, Rp): Returns zeros, poles and gain of a normalized analog Chebyshev Type I prototype of order N with passband ripple Rp.

Chebyshev Type II LPF Prototype

- A Chebyshev Type II filter has poles and zeros. It has equal ripple in the stopband ($\Omega > 1$) and decreases monotonically in the passband ($0 \leq \Omega < 1$).
- Its zeros lie on the imaginary axis of the s-plane.
- Its squared magnitude response is:

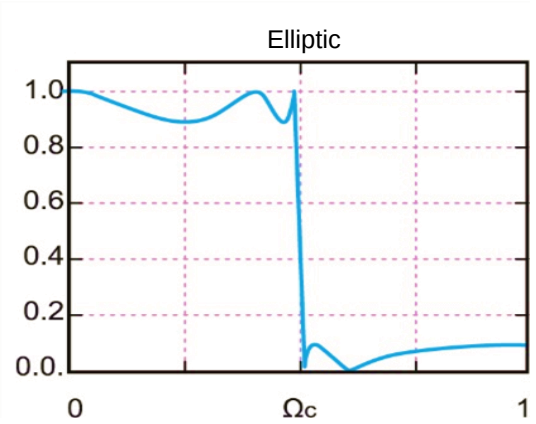
$$|H(j\Omega)|^2 = \frac{1}{1 + [\varepsilon^2 \, T_N^2(\Omega_c/\Omega)]^{-1}}$$

- It has more linear phase and improved group delay compared with a Chebyshev Type I filter.
- Implemented using cheb2ord(), cheby2() and cheb2ap().



Elliptic LPF Prototype

- An elliptic low-pass filter has equal ripple in both the passband and stopband, like an equiripple FIR filter.
- It requires a lower order than Butterworth or Chebyshev filters to satisfy the same specifications.
- However, its passband response has greater nonlinearity than Butterworth or Chebyshev filters.
- The squared magnitude response of an elliptic LPF is:



$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2\, U_N^2(\Omega/\Omega_c)}$$

- $\boldsymbol{\varepsilon}$: passband ripple, $\boldsymbol{\Omega_c}$: cutoff frequency, $\boldsymbol{U_N(\Omega)}$: Jacobi elliptic function of order N.
- MATLAB provides ellip() to design an analog elliptic LPF and ellipord() to calculate the optimum order. Use ellipap() to design an analog elliptic low-pass prototype.

Exercises

Exercise 1

A filter has impulse response $h(t) = e^{-at}u(t), a \in R$. Determine what type of filter it could be.

Answer: Calculate its frequency response:

$$H(\Omega) = \int_0^{+\infty} e^{-at}\, e^{-j\Omega t}\, dt = \int_0^{+\infty} e^{-(a+j\Omega)t}\, dt = -\frac{1}{a+j\Omega} e^{-(a+j\Omega)t}\Big|_0^{+\infty}$$

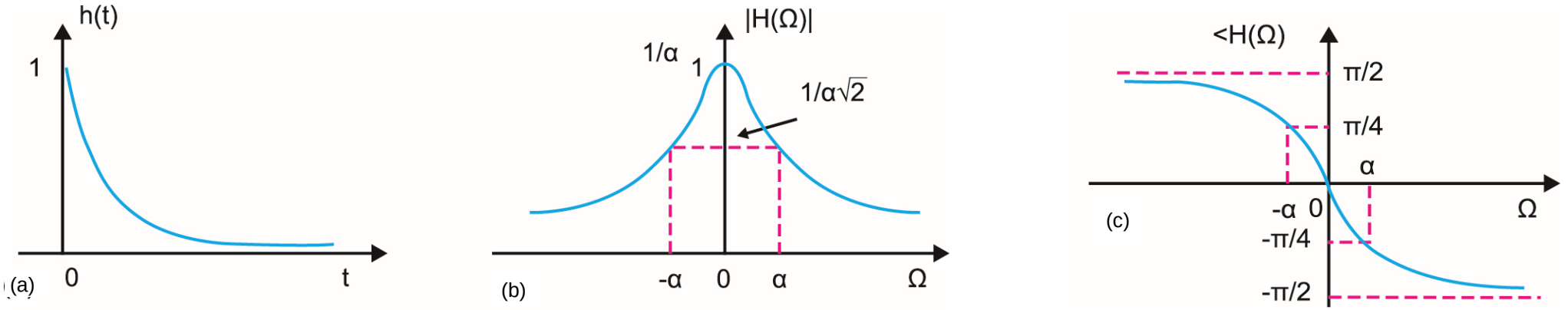
$$= -\frac{1}{a+j\Omega}\left[\lim_{t\to\infty} e^{-(a+j\Omega)t} - e^0\right] = -\frac{1}{a+j\Omega}\left[\lim_{t\to\infty}\left(e^{-at}e^{-j\Omega t}\right) - 1\right]$$

$$= -\frac{1}{a+j\Omega}\left[\lim_{t\to\infty} e^{-at}(\cos\Omega t - j sin\ \Omega t) - 1\right] = -\frac{1}{a+j\Omega}[0-1]$$

$$\text{Thus } H(\Omega) = \frac{1}{a+j\Omega}$$

Exercise 1 (continued)

The plots below show the impulse response $h(t)$, and the frequency response magnitude and phase.



(a) Impulse response $h(t)$, (b) magnitude spectrum, (c) phase spectrum

Magnitude decreases at high frequencies, $\lim_{|\Omega|\to\infty} |H(\Omega)| = 0$, so this could be a practical low-pass filter.

The decaying exponential in time suggests that the system smooths rapid changes. Its spectral behavior, passing low frequencies and attenuating high frequencies, follows directly from its time-domain form.

Exercise 2

A signal $x(t) = e^{-2t}u(t)$ passes through an ideal low-pass filter with cutoff frequency $\Omega_H = 1\ rad\ /$ sec. Find the ratio of output energy to input energy (assume A = 1).

Answer: The Fourier transform of $x(t)$ is:

$$X(\Omega) = \frac{1}{2 + j\Omega}$$

The energy spectral density of $x(t)$ is:

$$S_x(\Omega) = |X(\Omega)|^2 = \left|\frac{1}{2 + j\Omega}\right|^2 = \frac{1}{\Omega^2 + 4}$$

The magnitude of the given low-pass filter transfer function is:

$$|H(\Omega)| = \begin{cases} 1\,, & |\Omega| < \Omega_H = 1 \\ 0\,, & \text{otherwise} \end{cases}$$

Thus, the output energy spectral density is:

$$S_y(\Omega) = |X(\Omega)|^2\ |H(\Omega)|^2 = \begin{cases} \dfrac{1}{\Omega^2 + 4}\,, & |\Omega| < 1 \\ 0\,, & \text{otherwise} \end{cases}$$

Exercise 2 (continued)

By Parseval's theorem, the total output energy $y(t)$ is:

$$E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\Omega)\, d\Omega = \frac{1}{\pi}\int_{0}^{1} \frac{1}{\Omega^2 + 4}\, d\Omega = \frac{1}{2\pi}\tan^{-1}(0.5)$$

Similarly, the input energy $x(t)$ is:

$$E_x = \int_{-\infty}^{\infty} x^2(t)dt = \int_{0}^{\infty} e^{-4t}dt = 0.25$$

Dividing gives the required ratio:

$$\frac{E_y}{E_x} = \frac{\frac{1}{2\pi}\tan^{-1}(0.5)}{0.25} = \frac{2}{\pi}\tan^{-1}(0.5) \approx 0.295$$

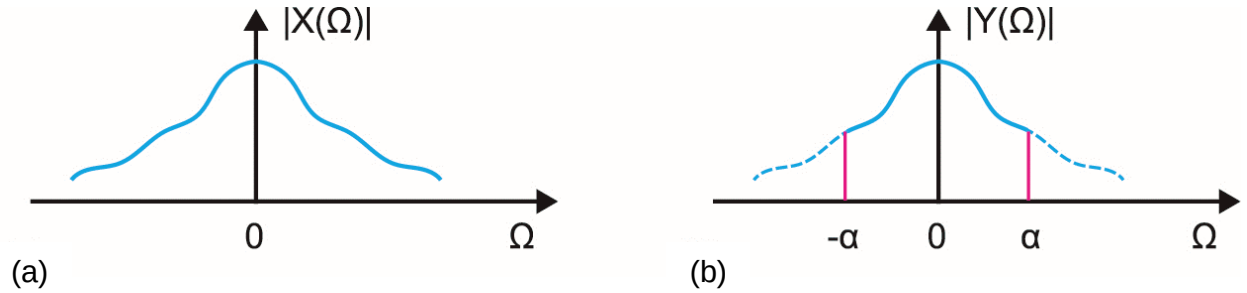
The ideal filter removes 70.5% of the input energy, and only 29.5% reaches the output.

Exercise 2 (continued)

- This example connects spectral analysis with signal energy. A filter changes the time-domain waveform and determines how much of the signal energy reaches the output.
- The spectrum fully explains the energy loss: components outside the passband are rejected, while those within it are retained.
- In telecommunications, bandwidth determines how much signal energy actually passes through a channel or filter.
- Choosing the cutoff frequency is a critical tradeoff. A narrow bandwidth reduces transmitted energy and may distort the signal. A wider bandwidth preserves more information but requires more spectral resources.

Exercise 3

A known signal $x(t)$ with magnitude spectrum $|X(\Omega)|$ enters an ideal LPF. It rejects high frequencies satisfying $|\Omega| > a$. The magnitude spectra $|X(\Omega)|$ and $|Y(\Omega)\}$ are shown below. Calculate the output $y(t)$.



Answer: Since the output magnitude spectrum $|Y(\Omega)|$ is known, find the output $y(t)$ by taking the inverse Fourier transform of $Y(\Omega)$. The ideal LPF has unity gain and zero phase.

Because the system is LTI, the output $y(t)$ is given by convolution, $y(t) = x(t) * h(t)$. By the Fourier convolution property:

$$Y(\Omega) = X(\Omega)\, H(\Omega)$$

Exercise 3 (continued)

From plot (b) and $|\Omega| > a$, the filter frequency response $H(\Omega)$ is rectangular:

$$H(\Omega) = rect\left(\frac{\Omega}{2\alpha}\right)$$

From Table 4.3, its impulse response $h(t)$ is:

$$h(t) = \frac{\sin(at)}{\pi t}$$

Thus, find the output $y(t)$ by convolving the input $x(t)$ with the impulse response $h(t)$:

$$y(t) = x(t) * \frac{\sin(at)}{\pi t}$$

Exercise 3 (continued)

If the ideal filter had linear phase:

$$H(\Omega) = \begin{cases} A\, e^{-j\Omega t_0}, & |\Omega| \le a \\ 0, & |\Omega| > a \end{cases}$$

its impulse response $h(t)$ would be:

$$h(t) = \frac{\sin\big(a(t - t_0)\big)}{\pi(t - t_0)}$$

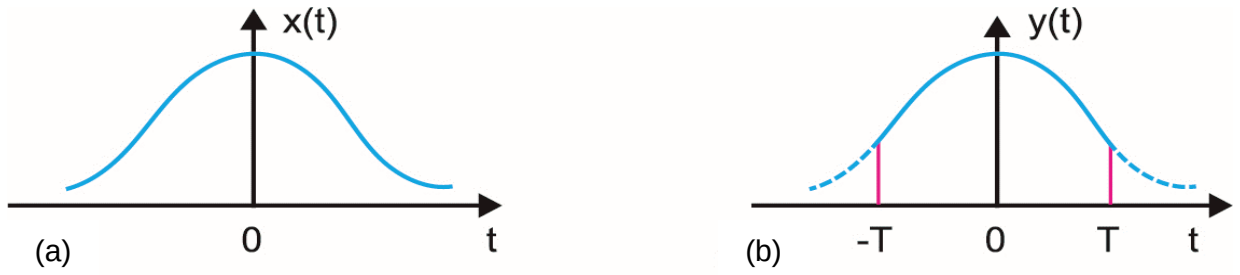
and the output would be:

$$y(t) = x(t) * \frac{\sin\big(a(t - t_0)\big)}{\pi(t - t_0)}$$

The filter acts selectively on the spectrum, retaining some frequencies and rejecting others. The output waveform changes because specific spectral components are missing.

Exercise 4

A known signal $x(t)$ enters a filter. At the output, $x(t)$ is removed for $|t| > T$, as shown below. Find the output Fourier transform $Y(\Omega)$.



Answer: The output retains $x(t)$ only over $[-T, T]$ and is zero for $|t| > T$, so:

$$y(t) = x(t)\, \Pi_T(t)$$

where $\Pi_T(t)$ is the rectangular pulse of duration:

$$\Pi_T(t) = \begin{cases} 1, & |t| < T \\ 0, & |t| > T \end{cases}$$

Exercise 4 (continued)

The known Fourier transform of $\Pi_T(t)$ is:

$$\Pi_T(t) \overset{F}{\longleftrightarrow} \frac{2\sin \Omega T}{\Omega}$$

Multiplication in time corresponds to convolution in frequency. Therefore:

$$Y(\Omega) = F\{y(t)\} = F\{\, x(t)\, \Pi_T(t)\, \} = \frac{1}{2\pi}\left[X(\Omega) * \frac{2\sin \Omega T}{\Omega}\right]$$

Using the Fourier transform convolution property in the frequency domain:

$$Y(\Omega) = \frac{1}{2\pi}\left[X(\Omega) * \frac{2\sin(\Omega T)}{\Omega}\right]$$

Exercise 4 (continued)

The required expression as a convolution integral is:

$$Y(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\nu) \, \frac{2 \sin[(\Omega - \nu)T]}{\Omega - \nu} \, d\nu$$

- Limiting a signal in time (windowing) changes its spectrum: the output spectrum is the convolution of the input spectrum with the window spectrum.
- A local operation in time has a broad effect in frequency. Each frequency component spreads into neighboring frequencies, causing spectral leakage of energy. Signal information is therefore distributed across the spectrum.
- In measurement systems such as oscilloscopes and spectrum analyzers, observing a signal over a limited interval distorts its spectrum. This explains why frequency measurements depend on the recording duration.

Exercise 5

A continuous-time signal described by:

$$x(t) = 2cos(20t) + 10sinc\left(\frac{10}{\pi}t\right) + 5sinc^2\left(\frac{5}{\pi}t\right)$$

passes through an ideal LPF with frequency response:

$$H(\Omega) = rect\left(\frac{\Omega}{32}\right)$$

(a) Calculate the spectrum $X(\Omega)$.

(b) Plot the frequency response $X(\Omega)$ and spectrum $X(\Omega)$.

(c) Calculate and plot the output magnitude spectrum $Y(\Omega)$.

(d) Calculate the time-domain signal $y(t)$.

(e) Calculate the time-domain signal $y(t)$ if the filter has frequency response:

$$H(\Omega) = 1 - rect\left(\frac{\Omega}{32}\right)$$

Exercise 5 (continued)

(a) Since $A\ cos(\Omega_0 t) \overset{F}{\longleftrightarrow} \pi A[\delta(\Omega+\Omega_0)+\delta(\Omega-\Omega_0)]$, the Fourier transform of the first term is:

$$2\ cos(20t) \overset{F}{\longleftrightarrow} 2\pi\ [\delta(\Omega+20)+\delta(\Omega-20)]$$

Since:

$$\left(\frac{B}{2\pi}\right) sinc\left(\frac{Bt}{2\pi}\right) \overset{F}{\longleftrightarrow} rect\left(\frac{\Omega}{B}\right)$$

the Fourier transform of the second term is:

$$10\ sinc\left(\frac{10}{\pi}t\right) = \pi\ \left(\frac{20}{2\pi}\right) sinc\left(\frac{20}{2\pi}t\right) \overset{F}{\longleftrightarrow} \pi\ rect\left(\frac{\Omega}{20}\right)$$

Finally, since:

$$\left(\frac{B}{2\pi}\right) sinc^2\left(\frac{Bt}{2\pi}\right) \overset{F}{\longleftrightarrow} tri\left(\frac{\Omega}{B}\right)$$

the Fourier transform of the third term is:

$$5sinc^2\left(\frac{5}{\pi}t\right) = \pi\left(\frac{10}{2\pi}\right) sinc^2\left(\frac{10t}{2\pi}\right) \overset{F}{\longleftrightarrow} \pi\ tri\left(\frac{\Omega}{10}\right)$$

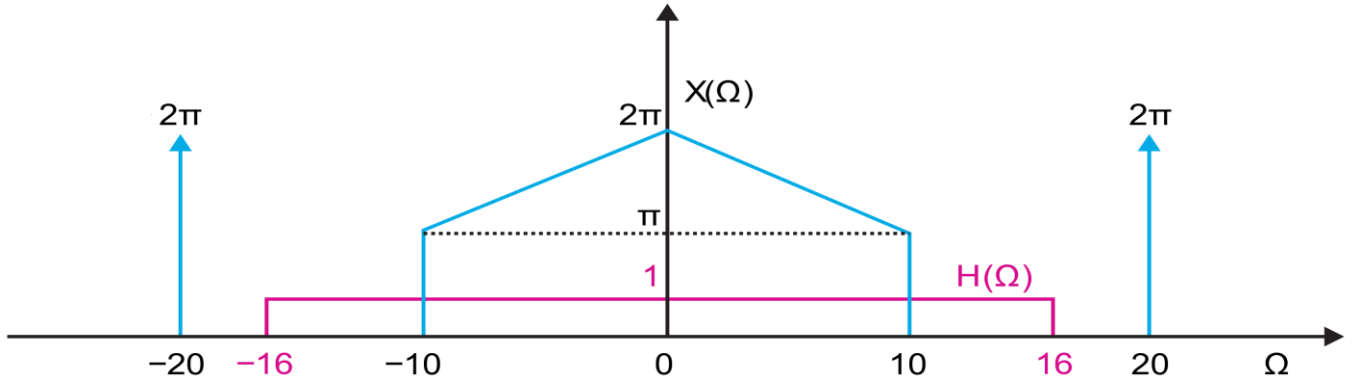
Exercise 5 (continued)

By Fourier transform linearity, the total spectrum $X(\Omega)$ is:

$$X(\Omega) = 2\pi\,[\delta(\Omega + 20) + \delta(\Omega - 20)] + \pi\, rect\left(\frac{\Omega}{20}\right) + \pi\, tri\left(\frac{\Omega}{10}\right)$$

(a) The LPF has cutoff frequency $\Omega_H = 16\ (rad/sec)$ and unity gain.

The spectra $X(\Omega)$ and $H(\Omega)$ are plotted below:



Spectra $X(\Omega)$ and $H(\Omega)$

Exercise 5 (continued)

(c) The output spectrum is:

$$Y(\Omega) = X(\Omega)H(\Omega)$$

From the preceding plot, the filter passes only frequencies satisfying $|\Omega| \leq 16$. The impulses at $\Omega = \pm 20$ are rejected, while both continuous components remain unchanged. Thus:

$$Y(\Omega) = \pi\, rect\left(\frac{\Omega}{20}\right) + \pi\, tri\left(\frac{\Omega}{10}\right)$$

The output magnitude spectrum is the sum of a rectangular and a triangular lobe over $[-10,10]$, without impulses at ± 20.

(d) From part (c), the LPF retains input frequencies in the passband and rejects those in the stopband. The frequency due to $2cos(20t)$ is therefore rejected, giving the time-domain output $y(t)$:

$$y(t) = 10sinc\left(\frac{10}{\pi}t\right) + 5sinc^2\left(\frac{5}{\pi}t\right)$$

Exercise 5 (continued)

(e) Here the filter is high-pass, with cutoff frequency $\Omega_L = 16\ (rad/sec)$, and passes only 20 rad/s. The time-domain output $y(t)$ is:

$$y(t) = 2cos(20t)$$

Comments:

- A filter selects frequencies: it retains some components and rejects others. The output can therefore differ qualitatively from the input, because specific frequency components are missing.
- Frequency-domain analysis predicts the output directly without convolution: we only need to know which frequencies the filter passes.

Exercise 6

A second-order Butterworth LPF, normalized in magnitude and frequency, has transfer function:

$$H(S) = \frac{1}{S^2 + \sqrt{2}S + 1}$$

Find the transfer function $H(s)$, with DC gain 10 and half-power frequency $\Omega_{hp} = 100$ rad/sec.

Answer: From the normalized prototype transfer function $H(S)$, the DC gain is:

$$H(0) = \frac{1}{0^2 + \sqrt{2 \cdot 0} + 1} = 1$$

The half-power frequency of $H(S)$ is:

$$H(j\Omega') = H(j1) = \frac{1}{j^2 + \sqrt{2}j + 1} = \frac{1}{-1 + j\sqrt{2} + 1} = \frac{1}{j\sqrt{2}}$$

Exercise 6 (continued)

Thus:

$$|H(j1)|^2 = \left|\frac{1}{j\sqrt{2}}\right| = \frac{1}{2}$$

Since $|H(0)|^2 = 1$, the normalized half-power frequency is:

$$\Omega'_{hp} = 1$$

To obtain half-power frequency $\Omega_{hp} = 100\ rad/s$, denormalize frequency by setting:

$$S = \frac{s}{100}$$

Finally, multiply $H(S)$ by 10 to obtain the required DC gain of 10.

$$H(s) = 10H\left(\frac{s}{100}\right) = \frac{10}{\left(\frac{s}{100}\right)^2 + \sqrt{2}\left(\frac{s}{100}\right) + 1} = \frac{10^5}{s^2 + 100\sqrt{2}s + 10^4}$$