

UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Signals and Systems

Lecture 12: LTI System Analysis Using the Fourier Transform

Michalis Paraskevas
Professor

Lecture Contents

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4. Linear differential equations using the Fourier transform
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1. Eigenvalue of an LTI System

Eigenvalue of an LTI System

Consider an LTI system with impulse response $h(t)$. Find the output $y(t)$ for input $x(t) = A\,e^{j\Omega_0 t}, -\infty < t < \infty$. Convolution gives:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau = A\int_{-\infty}^{\infty} h(\tau)\,e^{j\Omega_0(t-\tau)}\,d\tau = A\,e^{j\Omega_0 t}\int_{-\infty}^{\infty} h(\tau)\,e^{-j\Omega_0\tau}\,d\tau$$

Thus, the system output is:

$$y(t) = A\,e^{j\Omega_0 t}\,H(\Omega_0)$$

where $H(\Omega_0)$ is the **frequency response** evaluated at Ω_0, given by:

$$H(\Omega_0) = \int_{-\infty}^{\infty} h(\tau)\,e^{-j\Omega_0\tau}\,d\tau$$

The output $y(t)$ equals the periodic input $Ae^{j\Omega_0 t}$ multiplied by the Fourier transform of the impulse response $H(\Omega_0)$, evaluated at Ω_0.

The complex function $H(\Omega_0)$ is an **eigenvalue** of the system.

Eigenvalue of an LTI System

Signals of the form $x(t) = A\, e^{j\Omega_0 t}$ are **eigenfunctions**: they retain their form as they pass through the system, changing only in amplitude and phase. The frequency response in polar form is:

$$H(\Omega_0) = |H(\Omega_0)|\, e^{j\theta_H(\Omega_0)}$$

Here, $|H(\Omega_0)|$ is the magnitude and $\theta_H(\Omega_0)$ is the phase shift of the system frequency response, evaluated at the input frequency Ω_0. Thus, the output is:

$$y(t) = A|H(\Omega_0)|\, e^{j[\Omega_0 t + \theta_H(\Omega_0)]}$$

- The output $y(t)$ has amplitude $A|H(\Omega_0)|$, equal to the input amplitude multiplied by the frequency response magnitude $|H(\Omega_0)|$.
- The output phase is shifted by $\theta_H(\Omega_0)$ relative to the input phase $x(t)$.

This is an important LTI system property for a complex exponential input (or a combination of a cosine and a sine) at a given frequency.

These results apply more generally, even when the LTI system input is not an eigenfunction.

2. Frequency Response of an LTI System

Frequency Response of an LTI System

An LTI system initially at rest is completely described by its impulse response $h(t)$. Its output $y(t)$ for any input $x(t)$ can be calculated by convolution:

$$y(t) = x(t) * h(t)$$

By the Fourier transform convolution property:

$$Y(\Omega) = X(\Omega)\, H(\Omega)$$

Solving for $H(\Omega)$:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

The function $H(\Omega)$ is the system **frequency response**, the Fourier transform (FT) of the impulse response $h(t)$:

$$H(\Omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\Omega t}\, dt$$

LTI Output Calculation in the Frequency Domain

For an LTI system initially at rest, working in the frequency domain with transforms is computationally simpler than using convolution in the time domain.

- Given the impulse response $h(t)$ and input $x(t)$, calculate the output $y(t)$ more simply as follows:
  - Find $H(\Omega)$ by taking the FT of $h(t)$.
  - Find $Y(\Omega)$ using $Y(\Omega) = X(\Omega)\, H(\Omega)$.
  - Find the output $y(t)$ by taking the inverse Fourier transform of $Y(\Omega)$.
- Given the input $x(t)$ and output $y(t)$, calculate the impulse response $h(t)$ more simply as follows:
  - Find $X(\Omega)$ and $Y(\Omega)$.
  - Find $H(\Omega)$ using $H(\Omega) = Y(\Omega)/X(\Omega)$.
  - Find the impulse response $h(t)$ by taking the inverse FT of $H(\Omega)$.

LTI Output Calculation in the Frequency Domain

- Given the frequency response $H(\Omega)$, express it in rational form $Y(\Omega)/X(\Omega)$, determine the relation between $X(\Omega)$ and $Y(\Omega)$, then use the inverse Fourier transform to obtain the linear differential equation with constant coefficients describing the system.
- Given the system differential equation, take the Fourier transform of both sides. Use the Fourier transform differentiation property to form an algebraic equation and find the frequency response $H(\Omega)$ or the impulse response $h(t)$.

More generally, the same approach applies when any two of $x(t)$, $y(t)$ and $h(t)$ are known and the third is required.

Magnitude and Phase Representation

The frequency response $H(\Omega)$ of an LTI system is complex and can be written in **polar form**:

$$H(\Omega) = |H(\Omega)|\, e^{j\theta_{\mathrm{H}}(\Omega)}$$

Here, $|H(\Omega)|$ is the system **magnitude or gain** and $\theta_{\mathrm{H}}(\Omega)$ is its **phase**. The plots of $|H(\Omega)|$ and $\theta_{\mathrm{H}}(\Omega)$ are the **magnitude response** and **phase response**, respectively. It is easy to show that:

$$|Y(\Omega)| = |H(\Omega)|\,|X(\Omega)|$$

$$\theta_Y(\Omega) = \theta_H(\Omega) + \theta_X(\Omega)$$

- The output magnitude spectrum is the product of the input magnitude spectrum and the system magnitude response. Thus, the system magnitude response **multiplies** the input magnitude spectrum.
- The output phase spectrum is the sum of the input phase spectrum and the system phase response. Thus, the system phase response **adds** to the input phase spectrum.

These results hold for any input: periodic, nonperiodic, energy or power signals.

Magnitude and Phase Representation

If the frequency response $h(t)$ of an LTI system is a real, periodic or nonperiodic function, it has the Fourier transform's **spectral symmetry** properties.

Specifically, the magnitude $|H(\Omega)|$ and real part $Re\{H(\Omega)\}$ are **even functions** of frequency Ω, while the phase $\measuredangle X(\Omega)$ and imaginary part $Im\{H(\Omega)\}$ are **odd**:

$$|H(\Omega)| = |H(-\Omega)|$$

$$Re\{H(\Omega)\} = Re\{H(-\Omega)\}$$

$$\measuredangle H(\Omega) = -\measuredangle H(-\Omega)$$

$$Im\{H(\Omega)\} = -Im\{H(-\Omega)\}$$

3. LTI System Frequency Response to Periodic and Nonperiodic Inputs

LTI Frequency Response to a Periodic Input

For an LTI system with impulse response $h(t)$, the output $y(t)$ for a periodic input of period T_0, expanded in an exponential Fourier series, is:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{jk\Omega_0 t}$$

The output is also periodic, with the same period:

$$y(t) = \sum_{k=-\infty}^{\infty} H(k\Omega_0) X_k \, e^{jk\Omega_0 t}$$

The signals $x(t)$ and $y(t)$ have the same form, except that the complex Fourier coefficients X_k are multiplied by the frequency response $H(\Omega)$ at the harmonic frequencies $k\Omega_0$.

The output exponential Fourier series is:

$$y(t) = X_0 + 2\sum_{k=1}^{\infty} |H(k\Omega_0)| \, |X_k| \cos\big(\Omega_0 t + \theta(k\Omega_0) + \theta_H(k\Omega_0)\big)$$

- θ_k: phase of the coefficient X_k
- $\theta_H(k\Omega_0)$: phase of the frequency response at frequency $k\Omega_0$.

LTI Frequency Response to a Periodic Input

Each output harmonic has the same frequency as the corresponding input harmonic, with modified amplitude and phase.

Specifically, each harmonic amplitude is multiplied by $|H(k\Omega_0)|$, and its phase is shifted by $\theta_H(k\Omega_0)$.

For a periodic input, an LTI system acts separately on each harmonic component. Each input term $e^{jk\Omega_0 t}$ is multiplied by $H(k\Omega_0)$. The output therefore remains periodic with the same period, provided the corresponding frequency response values are well defined.

The relation:

$$y(t) = \sum_{k=-\infty}^{\infty} H(k\Omega_0) X_k \, e^{jk\Omega_0 t}$$

shows that the system **changes** the amplitude and phase of existing harmonics **without creating new frequencies**. The periodic signal's spectral structure **remains**, while each output harmonic's relative weight and time shift change.

LTI Frequency Response to a Nonperiodic Input

For an arbitrary nonperiodic input to an LTI system, calculate the output by convolution $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$:

$$Y(\Omega) = X(\Omega)\, H(\Omega) \qquad \text{or equivalently: } H(\Omega) = \frac{Y(\Omega)}{X(\Omega)},\ X(\Omega) \neq 0$$

After all possible simplifications, the rational form of $H(\Omega)$ is:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{N(\Omega)}{D(\Omega)}$$

For simple denominator roots, expand the frequency response in partial fractions:

$$H(\Omega) = \frac{N(\Omega)}{D(\Omega)} = \sum_{k=1}^{M} \frac{A_k}{a_k + j\Omega_0 t}$$

Thus, the impulse response is:

$$h(t) = \sum_{k=1}^{M} A_k\, e^{-a_k t}\, u(t)$$

LTI Frequency Response to a Nonperiodic Input

- This procedure is particularly useful when the frequency response is known and the impulse response is required.
- Given the impulse response, take its Fourier transform to find the frequency response and fully study the system behavior in the frequency domain.
- For a **nonperiodic input**, frequency-domain analysis uses:

$$Y(\Omega) = X(\Omega)\, H(\Omega)$$

This is the counterpart of the Fourier series for general signals. The frequency response fully describes how the system changes the magnitude and phase of each input frequency component, without directly calculating convolution in time.

- Partial fraction expansion of $H(\Omega)$ simplifies the transition from frequency to time. It directly gives the impulse response $h(t)$, expressed through exponential terms, and a clear picture of the system's dynamic behavior.

4. Linear Differential Equations Using the Fourier Transform

Differential Equations Using Fourier (1/2)

Convolution gives an **indirect description** of LTI system behavior and **cannot** describe the internal system structure.

We often use a **linear differential equation with constant coefficients** to **relate** the input and output:

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m y(t)}{dt^m}$$

where α_n, b_m are real constants and the initial conditions are zero.

Apply the FT differentiation property, $F\{x^{(n)}(t)\} \overset{F}{\longleftrightarrow} (j\Omega)^n X(\Omega)$, to each side:

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} \overset{F}{\longleftrightarrow} \left[\sum_{n=0}^{N} a_n (j\Omega)^n\right] Y(\Omega)$$

$$\sum_{m=0}^{M} b_m \frac{d^m y(t)}{dt^m} \overset{F}{\longleftrightarrow} \left[\sum_{m=0}^{M} b_m (j\Omega)^m\right] X(\Omega)$$

Differential Equations Using Fourier (2/2)

$$\left[\sum_{n=0}^{N} a_n (j\Omega)^n\right] Y(\Omega) = \left[\sum_{m=0}^{M} b_m (j\Omega)^m\right] X(\Omega)$$

Dividing both sides gives $H(\Omega)$:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{m=0}^{M} b_m (j\Omega)^m}{\sum_{n=0}^{N} a_n (j\Omega)^n}$$

or equivalently:

$$H(\Omega) = \frac{b_M (j\Omega)^M + b_{M-1} (j\Omega)^{M-1} + \cdots + b_1 (j\Omega) + b_0}{a_N (j\Omega)^N + a_{N-1} (j\Omega)^{N-1} + \cdots + a_1 (j\Omega) + a_0}$$

This derivation assumes initial rest, or zero initial conditions. Thus, H(Ω) depends only on the differential equation coefficients and expresses the system's steady-state frequency behavior.

This also connects with the Laplace transform, used for a more general treatment of initial conditions.

5. Inverse Systems

Inverse Systems

For an LTI system with impulse response $h(t)$, the **inverse system** has impulse response $g(t)$ such that:

$$h(t) * g(t) = \delta(t)$$

The inverse system frequency response is:

$$H(\Omega)\, G(\Omega) = 1 \Longrightarrow G(\Omega) = \frac{1}{H(\Omega)}$$

After finding the inverse system, check its **causal**ity and stability for physical realizability. A mathematically defined inverse may be impractical because it requires noncausal operation or is not BIBO **stable.** In the latter case, small errors or noise can be amplified without bound.

Exercises

Exercise 1

Find the output of an LTI system with impulse response $h(t) = e^{-150t}u(t)$ for a periodic square-wave input of amplitude ± 1 and fundamental angular frequency $\Omega_0 = 100\ rad/s$. Use a Fourier series and approximate the output with the first three odd harmonics.

Answer: For a square wave of amplitude ± 1, the known expansion is:

$$x(t) = \frac{4}{\pi}\left[sin(\Omega_0 t) + \frac{1}{3}sin(3\Omega_0 t) + \frac{1}{5}sin(5\Omega_0 t) + \cdots\right]$$

The system frequency response is:

$$H(j\Omega) = \frac{1}{150 + j\Omega}$$

Its magnitude and phase are:

$$|H(j\Omega)| = \frac{1}{\sqrt{150^2 + \Omega^2}} \quad \text{and} \quad \varphi_k = -tan^{-1}\left(\frac{\Omega}{150}\right)$$

The system multiplies each harmonic by the complex gain $H(jk\Omega_0)$, scaling its amplitude by $|H(jk\Omega_0)|$ and shifting its phase by φ_k.

Exercise 1 (continued)

Evaluate at the harmonic frequencies $k = 1,3,5$:

- $|H(j100)| = \frac{1}{\sqrt{150^2+100^2}} = \frac{1}{180.28} \approx 0.00555$
- $\varphi_1 = -tan^{-1}\left(\frac{100}{150}\right) \approx -0.588\, rad$
- $|H(j300)| = \frac{1}{\sqrt{150^2+300^2}} = \frac{1}{335.41} \approx 0.00298$
- $\varphi_3 = -tan^{-1}\left(\frac{300}{150}\right) \approx -1.107\, rad$
- $|H(j500)| = \frac{1}{\sqrt{150^2+500^2}} = \frac{1}{522.02} \approx 0.00192$
- $\varphi_5 = -tan^{-1}\left(\frac{500}{150}\right) \approx -1.279\, rad$

Exercise 1 (continued)

For each term of the form $A_k sin(k\Omega_0 t)$, the output is $A_k|H(jk\Omega_0)|\ sin(k\Omega_0 t + \phi_k)$.

Find the steady-state periodic output, assuming that the periodic input has been applied long enough for the system's transient response to decay:

$$y(t) \approx \frac{4}{\pi}\left[0.00555\ sin(100t - 0.588) + \frac{1}{3}0.00298\ sin(300t - 1.107) + \frac{1}{5}0.00192\ sin(500t - 1.279)\right]$$

Evaluating gives:

$$y(t) \approx 0.00707\ sin(100t - 0.588) + 0.00126\ sin(300t - 1.107) + 0.00049\ sin(500t - 1.279)$$

Higher harmonics are attenuated much more strongly than the fundamental. Thus, the output is much smoother than the square-wave input and approaches a sinusoid.

Exercise 1 (continued)

- A square wave is ideal here because of its rich harmonic content. It shows that a low-pass system acts selectively on harmonic order, beyond merely changing the overall amplitude.
- The filter with $H(j\Omega) = 1/(150 + j\Omega)$ is low-pass. It therefore passes more of the fundamental harmonic and attenuates higher harmonics more strongly.
- The spectrum explains the smoother output: sharp square-wave transitions require higher harmonics, which the system strongly suppresses.
- Fourier analysis directly gives the steady-state periodic response without time-domain convolution. This is especially useful for periodic inputs and systems readily described in the frequency domain.

Exercise 2

Calculate the output of an LTI system with impulse response $h(t) = 2e^{-3t}u(t)$ for input $x(t) = (2e^{-t} + e^{-4t})\,u(t)$.

Answer: The system frequency response is:

$$H(\Omega) = \frac{2}{3 + j\Omega}$$

The input Fourier transform is:

$$X(\Omega) = \frac{2}{1 + j\Omega} + \frac{1}{4 + j\Omega}$$

The output Fourier transform is:

$$Y(\Omega) = H(\Omega)X(\Omega) = \frac{2}{3 + j\Omega}\left(\frac{2}{1 + j\Omega} + \frac{1}{4 + j\Omega}\right) = \cdots = \frac{6(3 + j\Omega)}{(1 + j\Omega)(3 + j\Omega)(4 + j\Omega)}$$

$$= \frac{6}{(1 + j\Omega)(4 + j\Omega)} = \frac{C_1}{1 + j\Omega} + \frac{C_2}{4 + j\Omega}$$

Exercise 2 (continued)

Calculate the coefficients C_1 and C_2:

$$C_1 = \frac{6}{(1+j\Omega)(4+j\Omega)}(1+j\Omega)\Big|_{j\Omega=-1} = 2$$

$$C_2 = \frac{6}{(1+j\Omega)(4+j\Omega)}(4+j\Omega)\Big|_{j\Omega=-4} = -2$$

Thus, the output Fourier transform is:

$$Y(\Omega) = \frac{2}{1+j\Omega} - \frac{2}{4+j\Omega}$$

The output signal $y(t)$ is:

$$y(t) = (2e^{-t} - 2e^{-4t})\,u(t)$$

The system changes the input components according to its frequency response, demonstrating the usefulness of frequency response in LTI system analysis.

Exercise 3

Find the impulse response of the system described by:

$$\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 15y(t) = 15\,x(t) - 2\frac{dx(t)}{dt}$$

Answer: Solving in the time domain gives:

$$h(t) = \frac{1}{2}(21e^{-3t} - 25e^{-5t})u(t)$$

Taking the Fourier transform of both sides and using linearity and differentiation gives:

$$F\left\{\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 15y(t)\right\} = F\left\{15x(t) - 2\frac{dx(t)}{dt}\right\} \Rightarrow$$

$$F\left\{\frac{d^2y(t)}{dt^2}\right\} + 8F\left\{\frac{dy(t)}{dt}\right\} + 15F\{y(t)\} = 15F\{x(t)\} - 2F\left\{\frac{dx(t)}{dt}\right\} \Rightarrow$$

$$(j\Omega)^2Y(\Omega) + 8j\Omega\,Y(\Omega) + 15\,Y(\Omega) = 15\,X(\Omega) - 2j\Omega\,X(\Omega) \Rightarrow$$

$$Y(\Omega)(-\Omega^2 + 8j\Omega + 15) = X(\Omega)(15 - 2j\Omega) \Rightarrow$$

Exercise 3 (continued)

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{15 - 2j\Omega}{-\Omega^2 + 8j\Omega + 15} = \frac{15 - 2j\Omega}{(3 + j\Omega)(5 + j\Omega)}$$

The impulse response $h(t)$ is the inverse Fourier transform of the frequency response $H(\Omega)$, which can be written as:

$$H(\Omega) = \frac{C_1}{(3 + j\Omega)} + \frac{C_2}{(5 + j\Omega)}$$

Calculate C_1 and C_2:

$$C_1 = \frac{15 - 2j\Omega}{(3 + j\Omega)\,(5 + j\Omega)}(3 + j\Omega)\Big|_{j\Omega=-3} = \cdots = \frac{21}{2}$$

$$C_2 = \frac{15 - 2j\Omega}{(3 + j\Omega)\,(5 + j\Omega)}(5 + j\Omega)\Big|_{j\Omega=-5} = \cdots = -\frac{25}{2}$$

Thus, the frequency response in partial fraction form is:

$$H(\Omega) = \frac{21}{2}\frac{1}{(3 + j\Omega)} - \frac{25}{2}\frac{1}{(5 + j\Omega)}$$

Exercise 3 (continued)

Thus, the impulse response is:

$$h(t) = \frac{1}{2}(21e^{-3t} - 25e^{-5t})u(t)$$

This gives the same result as solving the differential equation, using a simpler method.

Calculating the frequency response $H(\Omega)$ and expanding it in partial fractions is the key step: it allows direct application of the inverse Fourier transform and straightforward calculation of the impulse response, which completely describes the LTI system in time.

Exercise 4

An LTI system receives input $x(t) = e^{-3t}u(t)$ and produces output $y(t) = t\,e^{-3t}u(t)$.
Find its frequency response and impulse response.

Answer: (a) The input and output Fourier transforms are:

$$X(\Omega) = \frac{1}{3 + j\Omega}$$

$$Y(\Omega) = \frac{1}{(3 + j\Omega)^2}$$

The frequency response is:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\frac{1}{(3 + j\Omega)^2}}{\frac{1}{3 + j\Omega}} = \frac{1}{3 + j\Omega}$$

Thus, the impulse response is:

$$h(t) = e^{-3t}u(t)$$

Exercise 5

Find the magnitude and phase of an LTI system with frequency response:

$$H(\Omega) = \frac{1 - j\Omega}{1 + j\Omega}$$

Answer: From the definition of the frequency response, its magnitude and phase are:

$$|H(\Omega)| = \frac{|1 - j\Omega|}{|1 + j\Omega|} = \frac{\sqrt{1 + \Omega^2}}{\sqrt{1 + \Omega^2}} = 1$$

$$\measuredangle H(\Omega) = \measuredangle H\{1 - j\Omega\} - \measuredangle H\{1 + j\Omega\} = \tan^{-1}(-\Omega) - \tan^{-1}(\Omega) = 2\tan^{-1}(\Omega)$$

Exercise 6

An LTI system has frequency response:

$$H(\Omega) = \frac{5 + j\Omega}{6 + 5j\Omega - \Omega^2}$$

(a) Find the differential equation describing the system.

(b) Find the system impulse response $h(t)$.

(c) Find the system output for input $x(t) = e^{-5t}u(t)$.

Answer: (a) The frequency response is:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

By the frequency response definition:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{5 + j\Omega}{6 + 5j\Omega - \Omega^2} \Rightarrow$$

Exercise 6 (continued)

$$\Rightarrow Y(\Omega)(6 + 5j\Omega - \Omega^2) = X(\Omega)(5 + j\Omega) \Rightarrow$$

$$\Rightarrow (j\Omega)^2 Y(\Omega) + 5j\Omega\, Y(\Omega) + 6Y(\Omega) = j\Omega X(\Omega) + 5X(\Omega)$$

Convert this relation to a differential equation, using the Fourier transform differentiation property:

$$F\left\{\frac{d^2y(t)}{dt^2}\right\} = (j\Omega)^2 Y(\Omega), \qquad F\left\{\frac{dy(t)}{dt}\right\} = (j\Omega)Y(\Omega), \qquad F\left\{\frac{dx(t)}{dt}\right\} = (j\Omega)X(\Omega)$$

We obtain:

$$F\left\{\frac{d^2y(t)}{dt^2}\right\} + 5F\left\{\frac{dy(t)}{dt}\right\} + 6y(t) = F\left\{\frac{dx(t)}{dt}\right\} + 5\,x(t)$$

Exercise 6 (continued)

Thus, the differential equation describing the system is:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 5\,x(t)$$

(b) Find the impulse response $h(t)$ by taking the inverse Fourier transform of the given frequency response $H(\Omega)$. Partial fraction expansion gives:

$$H(\Omega) = \frac{5 + j\Omega}{6 + 5j\Omega - \Omega^2} = \frac{5 + j\Omega}{(2 + j\Omega)(3 + j\Omega)} = \cdots = \frac{3}{2 + j\Omega} - \frac{2}{3 + j\Omega}$$

Using tables of elementary transform pairs:

$$h(t) = F^{-1}\{H(\Omega)\} = 3\,F^{-1}\left\{\frac{1}{2 + j\Omega}\right\} - 2\,F^{-1}\left\{\frac{1}{3 + j\Omega}\right\} = (3e^{-2t} - 2e^{-3t})\,u(t)$$

Exercise 6 (continued)

(c) To find the output $y(t)$, work in the frequency domain for computational simplicity. First, take the FT of the input $X(\Omega)$:

$$X(\Omega) = F\{e^{-5t}u(t)\} = \frac{1}{5 + j\Omega}$$

Then find the Fourier transform of the output $Y(\omega)$:

$$Y(\Omega) = H(\Omega)\, X(\Omega) = \frac{5 + j\Omega}{(2 + j\Omega)(3 + j\Omega)} \cdot \frac{1}{(5 + j\Omega)} = \frac{1}{(2 + j\Omega)(3 + j\Omega)} = \frac{1}{(2 + j\Omega)} - \frac{1}{(3 + j\Omega)}$$

Finally, recover the output $y(t)$ using the inverse Fourier transform:

$$y(t) = F^{-1}\{Y(\Omega)\} = \cdots = (e^{-2t} - e^{-3t})\, u(t)$$

Exercise 7

Calculate and plot the magnitude and phase responses of the LTI system with impulse response $h(t) = u(t-2) - u(t+1)$.

Answer: Take the Fourier transform of the impulse response to find the frequency response:

$$H(\Omega) = \int_{-\infty}^{\infty} h(t)e^{-j\Omega t}dt = \int_{-\infty}^{\infty} u(t-2)e^{-j\Omega t}dt - \int_{-\infty}^{\infty} u(t+1)e^{-j\Omega t}dt$$

Since the step functions $u(t-2)$ and $u(t+1)$ are defined as:

$$u(t-2) = \begin{cases} 1 \text{ for } t > 2 \\ 0 \text{ for } t < 2 \end{cases} \quad and \quad u(t+1) = \begin{cases} 1 \text{ for } t > -1 \\ 0 \text{ for } t < -1 \end{cases}$$

we obtain:

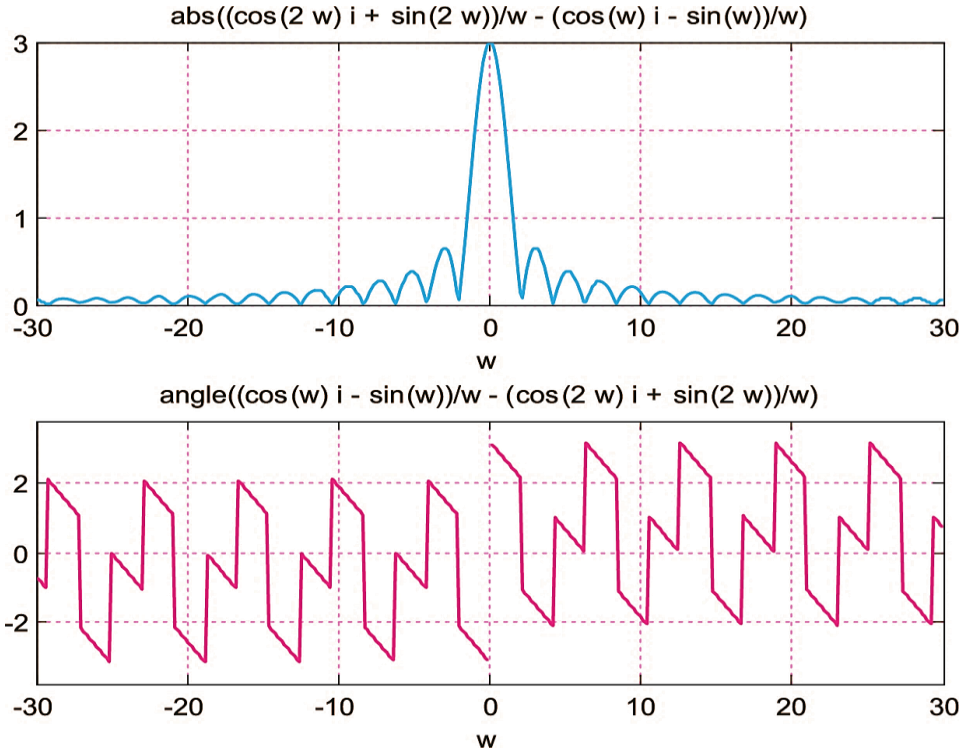
$$H(\Omega) = \int_{2}^{\infty} e^{-j\Omega t}dt - \int_{-1}^{\infty} e^{-j\Omega t}dt = -e^{-j\Omega/2}\,\frac{2\sin(3\Omega/2)}{\Omega} = -3e^{-j\Omega}\,\mathrm{sinc}\left(\frac{3\Omega}{2\pi}\right)$$

Exercise 7 (continued)

Thus, the magnitude and phase are:

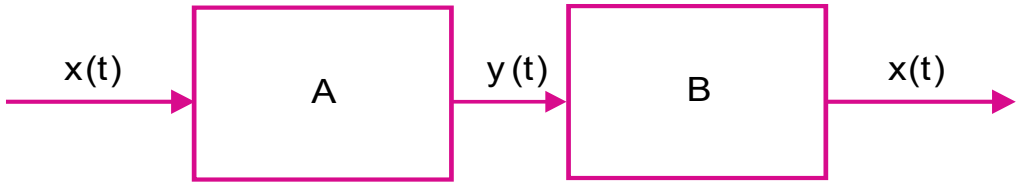
$$|H(\Omega)| = 3\left|\text{sinc}\left(\frac{3\Omega}{2\pi}\right)\right|$$

$$\measuredangle H(\Omega) = -\frac{\Omega}{2} = -\pi f$$



Exercise 8

Consider the cascade connection of continuous-time systems A and B shown below. System A receives $x(t)$, and system B produces the same signal $x(t)$. If A is LTI, determine whether B is also LTI. What other conditions must B satisfy?



Answer: For $x(t)$ to appear at the output of B, the overall equivalent impulse response must be:

$$h_{eq}(t) = \delta(t)$$

Since the systems are connected in cascade, their impulse responses must satisfy:

$$h_A(t) * h_B(t) = \delta(t) \qquad (1)$$

Exercise 8 (continued)

System B must be LTI for its output to be calculated by convolution:

$$x(t) = y(t) * h_B(t) = \big(x(t) * h_A(t)\big) * h_B(t) = x(t) * \big(h_A(t) * h_B(t)\big) = x(t) * \delta(t) = x(t)$$

Thus, system B undoes the effect of system A. By relation (1), B is described by the impulse response $h_B(t)$ and is therefore also LTI. In the frequency domain, relation (1) becomes:

$$H_A(s)\, H_B(s) = 1$$

so:

$$H_B(s) = \frac{1}{H_A(s)}$$

Thus, the frequency response of system B is the reciprocal of that of system A.

Exercise 8 (continued)

In the Laplace domain, the corresponding relation is:

$$H_B(s) = \frac{1}{H_A(s)}$$

Thus, the poles of system A become zeros of system B, and the zeros of A become poles of B.

For system B to be **practically realizable**, it must be **causal** and **stable** as well as the mathematical inverse of A. From:

$$H_B(s) = \frac{1}{H_A(s)}$$

inverse system stability requires the zeros of A to lie in the left half of the complex plane.

A **system** satisfying this condition is a **minimum-phase** system and can be inverted. Its inverse is also minimum-phase.

The causal realizability of $1/H_A(s)$ must also be checked.