UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

SIGNALS AND SYSTEMS

Solved Examples – Tutorial Sheet 11

Instructor: Michalis Paraskevas, Professor

ANALYSIS OF LTI SYSTEMS USING THE LAPLACE TRANSFORM

- Solving linear differential equations with constant coefficients using the Laplace transform
- System transfer function
- Step response of an LTI system
- State-space analysis of systems

1. Solving Linear Differential Equations with Constant Coefficients Using the Laplace Transform

📖 Example 1

Using the Laplace transform, calculate the impulse and step responses of the LTI system described by the linear differential equation with constant coefficients:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t), \quad \text{initial conditions } y(0^-) = 1, y'(0^-) = 0$$

Answer: We first calculate a general solution independent of the applied input, then find the impulse and step responses separately. We begin by taking the Laplace transform of both sides of the differential equation and solving for $Y(s)$:

$$[s^2Y(s) - sy(0^-) - y'(0^-)] + 3[sY(s) - y(0^-)] + 2Y(s) = X(s)$$

$$\Rightarrow s^2Y(s) - s + 3sY(s) - 3 + 2Y(s) = X(s)$$

$$\Rightarrow Y(s)[s^2 + 3s + 2] - (s + 3) = X(s)$$

$$\Rightarrow Y(s)(s + 1)(s + 2) = X(s) + (s + 3)$$

$$\Rightarrow Y(s) = \frac{X(s)}{(s + 1)(s + 2)} + \frac{s + 3}{(s + 1)(s + 2)}$$

(a) For the input $x(t) = u(t)$, we have:

$$X(s) = \frac{1}{s}, \qquad Re(s) > 0$$

Therefore:

$$Y(s) = \frac{1}{s(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)}$$
$$= \frac{1+3s+s^2}{s(s+1)(s+2)}$$
$$= \frac{R_1}{s} + \frac{R_2}{s+1} + \frac{R_3}{s+2}$$

We calculate the residues R_1, R_2, R_3 using:

$$R_1 = \frac{1+3s+s^2}{(s+1)(s+2)}\Big|_{s=0} = \cdots = \frac{1}{2}$$

$$R_2 = \frac{1+3s+s^2}{s(s+2)}\Big|_{s=-1} = \cdots = 1$$

$$R_3 = \frac{1+3s+s^2}{s(s+1)}\Big|_{s=-2} = \cdots = -\frac{1}{2}$$

Thus:

$$Y(s) = \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{s+1} - \frac{1}{2}\cdot\frac{1}{s+2}$$

and since the region of convergence extends to the right, the step response $s(t)$ is:

$$s(t) \equiv y(t) = \left(\frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t}\right)u(t)$$

The step response has two components: the transient response:

$$s_{ss}(t) = \left(e^{-t} - \frac{1}{2}e^{-2t}\right)u(t)$$

and the steady-state response:

$$s_{ts}(t) = \frac{1}{2}u(t)$$

(b) We can calculate the impulse response either by setting $x(t) = \delta(\tau)$ and solving the differential equation again, or by taking the first derivative of the step response. We choose the second approach, which is simpler:

$$h(t) = \frac{ds(t)}{dt}$$
$$= \frac{d}{dt}\left[\left(\frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t}\right)u(t)\right]$$
$$= \cdots = (-e^{-t} + e^{-2t})u(t)$$

2. Analysis of Causality and Stability in the Frequency Domain

📖 Example 2

An LTI system has impulse response $h(t) = \delta(t) - 2e^{-t}\,u(t)$. How many different inputs $x(t)$ can produce the output $y(t) = e^{-2t}\,u(t)$, and what are they?

Answer: We work in the complex-frequency domain s and calculate the system transfer function $H(s)$ from the Laplace transform of its impulse response $h(t)$:

$$H(s) = 1 - \frac{2}{s+1} = \frac{s-1}{s+1} \qquad \text{with region of convergence} \qquad Re(s) > -1$$

The Laplace transform $Y(s)$ of the output $y(t) = e^{-2t}u(t)$ is:

$$Y(s) = \frac{1}{s+2}, \qquad Re(s) > -2$$

To calculate the input signal $x(t)$, we find $X(s)$ using:

$$X(s) = \frac{Y(s)}{H(s)} = \frac{\frac{1}{s+2}}{\frac{s-1}{s+1}}$$

$$= \frac{s+1}{(s+2)(s-1)} =$$

$$\frac{1}{3}\cdot\frac{1}{s+2} + \frac{2}{3}\cdot\frac{1}{s-1}$$

The function $X(s)$ has two poles, at $s = -2$ and $s = 1$. Their locations determine the following possible regions of convergence:

$$Re(s) < -2, \qquad -2 < Re(s) < 1, \qquad Re(s) > 1$$

(a) For $Re(s) < -2$, both terms are left-sided:

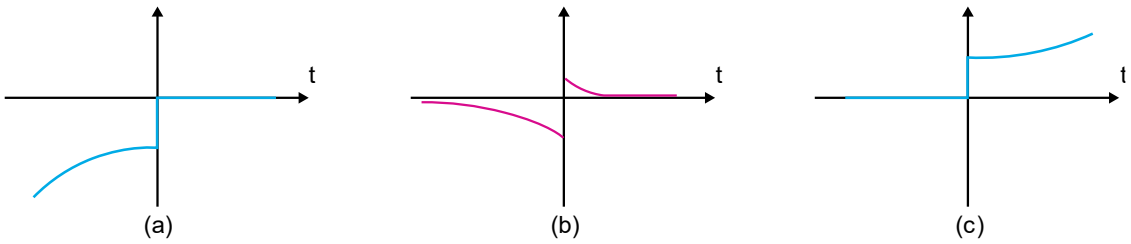
$$x_1(t) = -\frac{1}{3}\,e^{-2t}u(-t) - \frac{2}{3}\,e^{t}u(-t)$$

(b) For $-2 < Re(s) < 1$, the first term is right-sided and the second is left-sided:

$$x_2(t) = \frac{1}{3}\,e^{-2t}u(t) - \frac{2}{3}\,e^{t}u(-t)$$

(c) For $Re(s) > 1$, both terms are right-sided:

$$x_3(t) = \frac{1}{3}\,e^{-2t}u(t) + \frac{2}{3}\,e^{t}u(t)$$



Forms of $x(t)$ for cases (a), (b), and (c) above

To determine which signals are valid system inputs, we must examine the compatibility of their regions of convergence with the transfer function $H(s)$, whose region of convergence is:

$$Re(s) > -1$$

- Since $X_1(s)$ converges only for $Re(s) < -2$, there is no common region of convergence, so $x_1(t)$ is not a valid input.
- The region of convergence of $X_2(s)$ is $-2 < Re(s) < 1$, which intersects that of $H(s)$. Therefore, $x_2(t)$ is a valid input.
- The region of convergence of $X_3(s)$ is $Re(s) > 1$, which also intersects that of $H(s)$. Therefore, $x_3(t)$ is also a valid input.

Thus, two different inputs can produce the output $y(t) = e^{-2t}u(t)$:

$$x_2(t) = \frac{1}{3}\, e^{-2t}u(t) - \frac{2}{3}\, e^{t}u(-t)$$

and

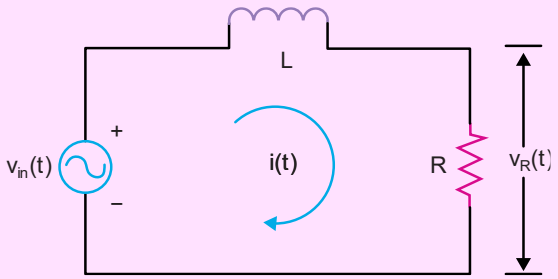
$$x_3(t) = \frac{1}{3}\, e^{-2t}u(t) + \frac{2}{3}\, e^{t}u(t)$$

Comments:

- This example shows that, in the Laplace domain, the algebraic form of a function is insufficient to fully specify the corresponding signal. The same expression for $X(s)$ can correspond to different time-domain signals, depending on the region of convergence.
- The region of convergence is an essential part of the Laplace transform because it determines whether the signal is left-sided or right-sided. In this example, this information leads to more than one possible input.
- A candidate input is valid only if both the algebraic relation $X(s) = Y(s)/H(s)$ and the compatibility of the regions of convergence of $X(s)$ and $H(s)$ hold. A common region of convergence is required for the input to be physically and mathematically valid.

Example 3

Find the transfer function $H(s)$ and impulse response $h(t)$ of the electrical circuit below, taking the source voltage $v_{in}(t)$ as the input and the voltage $v_R(t)$ across the resistor as the output. Assume the system is initially at rest.



Answer: Applying Kirchhoff's law gives the following differential equation:

$$v_{in}(t) - L\frac{di(t)}{dt} - R\,i(t) = 0$$

Since: $v_R = i(t)\,R$ we have: $\dfrac{dv_R(t)}{dt} = R\dfrac{di(t)}{dt}$

Using this relation, we rewrite the original differential equation in terms of the source voltage $v_{in}(t)$ (input) and the required resistor voltage $v_R(t)$ (output):

$$\frac{dv_R(t)}{dt} + \frac{R}{L}v_R(t) = \frac{R}{L}v_{in}(t)$$

Since the system is at rest, all initial conditions are zero. We can therefore calculate the transfer function by applying the Laplace transform to both sides of the differential equation:

$$s\,V_R(s) + \frac{R}{L}V_R(s) = \frac{R}{L}V_{in}(s)$$

Thus:

$$V_R(s)\left(s + \frac{R}{L}\right) = \frac{R}{L}V_{in}(s)$$

so the transfer function is:

$$H(s) = \frac{V_R(s)}{V_{in}(s)} = \frac{R/L}{s + R/L}$$

Since the system is also causal, the region of convergence is:

$$Re(s) > -\frac{R}{L}$$

Applying the inverse Laplace transform gives the system impulse response:

$$h(t) = \frac{R}{L}e^{-(R/L)\,t}\,u(t)$$

Comments:

- This example shows how a simple circuit becomes a mathematical model in the frequency domain. The transfer function summarizes the physical operation of the circuit.
- The system has low-pass behavior: slow input variations pass to the output, while rapid variations are suppressed.
- The circuit's dynamic behavior is determined by a single parameter, the time constant, making it a typical first-order system.

3. System Transfer Function

Example 4 (*)

An LTI system with impulse response $h(t) = e^{-2t}u(t)$ receives the input signal $x(\mathrm{t}) = e^{-t}u(t)$. Find:

(a) The Laplace transforms $H(s)$ and $X(s)$.

(b) The Laplace transform $Y(s)$ of the output $y(t)$.

(c) The output $y(t)$, using the inverse Laplace transform of $Y(s)$.

(d) The output $y(t)$, using convolution, $y(t) = h(t) * x(t)$. Does this verify the result of the previous part?

Answer: (a) From the definition of the Laplace transform, we have:

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st} dt =$$

$$\int_0^{\infty} e^{-(s+2)t} u(t) dt = \left[\frac{e^{-(s+2)t}}{-(s+2)}\right]_0^{\infty} = \frac{1}{s+2}, \qquad Re(s) > -2$$

and

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt =$$

$$\int_0^{\infty} e^{-(s+1)t}\, u(t) dt = \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^{\infty}$$

$$= \frac{1}{s+1}, \qquad Re(s) > -1$$

(b) The system output is the convolution of the impulse response with the input signal, $y(t) = h(t) * x(t)$. Applying the convolution theorem for the Laplace transform gives:

$$Y(s) = H(s)\, X(s) = \frac{1}{(s+1)(s+2)}$$

with region of convergence $Re(s) > -1$, the intersection of the regions of convergence of $H(s)$ and $X(s)$, corresponding to a right-sided signal.

(c) Using partial-fraction decomposition of $Y(s)$, we have:

$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

and the inverse Laplace transform is the sum of the inverse transforms of the individual fractions:

$$y(t) = e^{-t}u(t) - e^{-2t}u(t) = (e^{-t} - e^{-2t})\,u(t)$$

We used the region of convergence $Re(s) > -1$ to determine that the individual components of $y(t)$ are right-sided.

(d) Using convolution, we obtain:

$$y(t) = h(t) * x(t) =$$
$$\int_{-\infty}^{\infty} h(t-\tau)\,x(\tau)\,d\tau = \int_{-\infty}^{\infty} e^{-2(t-\tau)}\,u(t-\tau)\,e^{-\tau}\,u(\tau)\,d\tau$$

The step functions restrict the integral to $0 \le \tau \le t$, so for $t \ge 0$, we have:

$$y(t) = \int_0^t e^{-2t}\,e^{2\tau}e^{-\tau}d\tau = e^{-2t}\int_0^t e^{\tau}d\tau = e^{-2t}[e^{\tau}]_0^t$$
$$= e^{-2t}(e^t - 1) = (e^{-t} - e^{-2t})\,u(t)$$

This verifies the result in part (c).

Comments:

- This example illustrates the fundamental relation $Y(s) = H(s)\,X(s)$, which holds for LTI systems and turns the calculation of the output from a time-domain convolution into a simple algebraic product in the Laplace domain.
- The region of convergence of $Y(s)$ determines the time-domain form of the output and must always accompany the algebraic expression of the transform. Here, $Re(s) > -1$ indicates a right-sided output, which therefore includes the factor $u(t)$ in the time domain.
- Partial-fraction decomposition of $Y(s)$ allows direct calculation of the inverse Laplace transform and hence the output $y(t)$. This technique is especially useful when $Y(s)$ is rational and can be factored.
- Verifying the same result using the inverse Laplace transform and direct convolution confirms both the calculations and the equivalence of these approaches to LTI system analysis.

4. Step Response of an LTI System

📖 Example 5

A feedback system has forward-path transfer function $G(s) = K/(s-1)$ and negative unity feedback. For $K = 3$, calculate the equivalent transfer function $T(s)$, determine the values of K for which the closed-loop system is stable, and calculate its step response.

Answer: For negative unity feedback, the transfer function is:

$$T(s) = \frac{G(s)}{1 + G(s)}$$

Thus:

$$T(s) = \frac{\frac{K}{s-1}}{1 + \frac{K}{s-1}} = \frac{K}{s + K - 1}$$

The system pole is:

$$s_p = 1 - K$$

Stability requires $Re(s_p) < 0$:

$$1 - K < 0 \Rightarrow K > 1$$

For $K = 3$, the transfer function is:

$$T(s) = \frac{3}{s + 2}$$

For a unit step input, the Laplace transform of the output is:

$$Y(s) = \frac{1}{s} T(s) = \frac{3}{s(s+2)}$$

Using partial fractions:

$$Y(s) = \frac{3}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)$$

Therefore, the step response is:

$$y(t) = \frac{3}{2}(1 - e^{-2t})\, u(t)$$

As K increases, the closed-loop pole $s_p = 1 - K$ moves left along the real axis. For $K = 3$, the pole is at $s_p = 1 - 3 = -2$, making the closed-loop system stable.

Comments:

- Feedback is a mechanism for moving poles. The open-loop pole moves from the right half-plane to the left when the gain exceeds a critical threshold.
- The condition $K > 1$ shows that feedback alone does not guarantee stability. Stability depends on the locations of the closed-loop poles, and therefore on the gain K.
- For $K = 3$, the response is first-order, with exponential term e^{-2t}, and time constant τ=1/2. The final value is 3/2, showing that feedback affects stability, response speed, and the static gain of the closed-loop system.

📖 Example 6

Two stable LTI systems have the same poles but different zeros:

$$H_1(s) = \frac{1}{(s+1)^2} \quad \text{and} \quad H_2(s) = \frac{s+2}{(s+1)^2}$$

Calculate their step responses and discuss the role of the zero in the transient response.

Answer: For a unit step input, $X(s) = 1/s$. For the first system, we have:

$$Y_1(s) = H_1(s)X(s) = \frac{1}{s(s+1)^2}$$

Using partial fractions:

$$\frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Thus, the response is:

$$y_1(t) = [1 - e^{-t} - te^{-t}]u(t) = [1 - (1+t)e^{-t}]u(t)$$

For the second system, we have:

$$Y_2(s) = H_2(s)X(s) = \frac{s+2}{s(s+1)^2}$$

Using partial fractions:

$$\frac{s+2}{s(s+1)^2} = \frac{2}{s} - \frac{2}{s+1} - \frac{1}{(s+1)^2}$$

Thus, the response is:

$$y_2(t) = [2 - 2e^{-t} - te^{-t}]u(t) = [2 - (2+t)e^{-t}]u(t)$$

The two responses have the same poles, so their exponential decay has the same form. However, the zero of $H_2(s)$ changes the coefficients of the exponential terms and noticeably changes the transient waveform. The DC gains also give:

$$\lim_{t\to\infty} y_1(t) = H_1(0) = 1 \quad \text{and} \quad \lim_{t\to\infty} y_2(t) = H_2(0) = 2$$

The zeros do not determine the BIBO stability of a causal rational system; this is determined solely by its poles. However, zeros affect the coefficients of the natural response modes and can significantly change the transient waveform. In this example, the zero of $H_2(s)$ also changes the DC gain, so the final value is 2 rather than 1.

Comments:

- This example shows the effect of zeros while keeping the poles fixed. The poles determine the types of dynamic modes, while the zeros shape how those modes combine at the output. The poles determine stability, while the zeros determine the output waveform.
- The two responses have the same exponential decay rate, since they share a double pole at $s = -1$. Nevertheless, the second system responds differently to a step input because its zero changes the partial-fraction residues.

5. State-Space Analysis of Systems

📖 Example 7

Analyze the state-space system $x'(t) = Ax(t) + Bu(t),\ y(t) = Cx(t)$, with matrices:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = [1,1], \qquad D = 0$$

Calculate the system transfer function, then show that although it indicates stability in the classical sense, with all its poles in the left half of the complex plane, the system itself is internally unstable for certain initial conditions.

Answer: We first calculate the transfer function:

$$H(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$$

Using the given data:

$$s\boldsymbol{I} - \boldsymbol{A} = \begin{bmatrix} s+1 & 0 \\ 0 & s-1 \end{bmatrix}$$

Thus:

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s-1} \end{bmatrix}$$

Multiplying by B:

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} \\ 0 \end{bmatrix}$$

$$H(s) = [1 \quad 1] \begin{bmatrix} \dfrac{1}{s+1} \\ 0 \end{bmatrix} = \frac{1}{s+1}$$

The transfer function $H(s)$ has only one pole, at $s = -1$, so its external input–output description appears stable. However, the eigenvalues of A are:

$$\lambda_1 = -1, \qquad \lambda_2 = +1$$

Since one eigenvalue has a positive real part, the system is internally unstable. To demonstrate this, set $u(t) = 0$ and initial state $x(0) = [0,1]^T$. The system becomes:

$$\frac{dx_1(t)}{dt} = -x_1(t), \qquad \frac{dx_2(t)}{dt} = x_2(t)$$

Therefore:

$$x_1(t) = 0, \qquad x_2(t) = e^t$$

the output is:

$$y(t) = Cx(t) = x_1(t) + x_2(t) = e^t$$

Thus, with zero input but a nonzero initial condition, the output diverges. This clearly shows internal instability, since the internal states do not decay even without external excitation. The instability is absent from the transfer function because the input does not excite it through B, but it remains in the initial conditions. Therefore, the external description $H(s) = 1/(s + 1)$ is insufficient for a complete analysis of the internal dynamics. State analysis is also required to identify unobservable or uncontrollable unstable modes.

Comments:

- This example shows that internal stability is stronger than BIBO stability of the external description. Here, the transfer function is stable, but the state-space system has an unstable eigenvalue. This occurs because the transfer function describes the input–output relation under zero initial conditions and can therefore conceal internal modes that are not excited by the input.
- The internal stability of a state-space system is determined by the eigenvalues of A. Here, the eigenvalue $+1$ gives the unstable mode e^t, so the system is internally unstable even though $H(s)$ is stable.
- The unstable mode e^t, corresponding to the eigenvalue $+1$, is uncontrollable from the input $u(t)$, because the second element of B is zero. Therefore, it does not appear in $H(s)$. However, because C includes the second state, this mode appears at the output when that state has a nonzero initial condition, that is, when $x_2(0) \neq 0$.

Note: Examples marked with an asterisk (*) also appear in the theory lecture slides.