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DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Signals and Systems

Lecture 11: LTI System Analysis Using the Laplace Transform

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Lecture Contents

1. Solving linear differential equations using the Laplace transform
2. System transfer function
3. System stability and causality in the frequency domain
4. System connections in the frequency domain
5. Step response of an LTI system
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1. Solving Linear Differential Equations Using the Laplace Transform

LTI System Analysis Using Laplace (1/3)

Calculate the response of a stable, causal LTI system, whether or not it is initially at rest. The system is described by the linear differential equation with constant coefficients:

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}, \qquad N > M$$

where $x(t)$ and $y(t)$ are the system input and output, and the initial conditions are:

$$y(0), \left.\frac{d^k y(t)}{dt^k}\right|_{t=0} \quad \text{for } k = 1 \ldots N-1$$

The notation $dy^k(t)/dt$ denotes the kth derivative of the signal $y(t)$. Under the assumption:

$$\left[\sum_{n=0}^{N} a_n s^n\right] Y(s) = \left[\sum_{m=0}^{M} b_m s^m\right] X(s) + \sum_{n=1}^{N} a_n \left(\sum_{k=0}^{n-1} s^{n-1-k} \left.\frac{d^k y(t)}{dt^k}\right|_{t=0-} \right)$$

where $Y(s) = L\{y(t)\},\ X(s) = L\{x(t)\}$.

LTI System Analysis Using Laplace (2/3)

Make the following substitutions:

$$A(s) = \sum_{n=0}^{N} a_n s^n, \qquad a_N = 1, \qquad B(s) = \sum_{m=0}^{M} b_m s^m$$

$$I(s) = \sum_{n=1}^{N} a_n \left(\sum_{k=0}^{n-1} s^{n-1-k} \left. \frac{d^k y(t)}{dt^k} \right|_{t=0-} \right), \qquad a_N = 1$$

The preceding equation then becomes a polynomial relation:

$$A(s)\, Y(s) = B(s)\, X(s) + I(s)$$

Solving for $Y(s)$ gives:

$$Y(s) = \frac{B(s)}{A(s)} X(s) + \frac{1}{A(s)} I(s)$$

In the above relation, define:

$$H(s) = \frac{B(s)}{A(s)} \quad \text{and} \quad H_1(s) = \frac{1}{A(s)}$$

LTI System Analysis Using Laplace (3/3)

We obtain:

$$Y(s) = H(s)\,X(s) + H_1(s)\,I(s)$$

Taking the inverse Laplace transform gives the total output $y(t)$:

$$y(t) = y_{zs}(t) + y_{zi}(t)$$

where:

- $y_{zs}(t) = L\{H(s)X(s)\}$ is the **zero-state response**, and
- $y_{zi}(t) = L\{H(s)I(s)\}$ is the **zero-input response**.

Using convolution integrals, these expressions become:

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau + \int_0^t i(\tau)h_1(t-\tau)d\tau$$

where $h(t) = L^{-1}\{H(s)\}$ and $h_1(t) = L^{-1}\{H_1(s)\}$, and

$$i(t) = L^{-1}\{I(s)\} = \sum_{n=1}^{N} a_k \left(\sum_{k=0}^{n-1} \delta^{(n-1-k)}(t)\ y^{(k)}(0^-) \right)$$

The term $\delta^{(i)}(t)$ denotes the ith derivative of the impulse function. We have $\delta^{(0)}(t) = \delta(t)$.

Observations

- By superposition, an LTI system response splits into two components: the **zero-input response** and the **zero-state response**. This separation allows us to study the effects of the initial conditions and external excitation independently.
- The distinction between **zero-input and zero-state response** differs from the distinction between **transient and steady-state response**. The former concerns the origin of the response, while the latter concerns its behavior over time.
- The Laplace transform simplifies LTI system analysis by converting the linear differential equation into an **algebraic relation** in the complex frequency domain. Solving the differential equation therefore reduces to algebraic operations on polynomials.

Observations

- Nonzero initial conditions appear in the Laplace domain through the term $I(s)$, which acts as an **equivalent excitation term**. In this sense, the initial conditions can be treated as an additional system "input."
- The assumption $N > M$ means that the system is **strictly proper**. This has physical significance: the response contains no impulses or impulse derivatives, which would represent unrealistic instantaneous transmission from input to output.
- The total response can be written as the sum of two convolution integrals: one for the ordinary input $x(t)$, and one for the equivalent input $i(t)$ representing the initial conditions. Thus, even the effect of initial conditions can be interpreted through convolution.

2. System Transfer Function

System Transfer Function (1/3)

The Laplace transform (LT) provides a simple way to calculate a system's **transfer function.**

Consider an LTI system described by the linear differential equation with constant coefficients:

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_0 x(t)$$

where $\alpha_n, n = 1,2, \ldots, N,\ \ b_m, m = 1,2, \ldots, M$ are real constants.

For an LTI system initially at rest, taking the LT of both sides gives:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{n=0}^{N} a_n s^n} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

System Transfer Function (2/3)

- The generally complex function $H(s)$ is the system **transfer function**. It depends only on the parameters $\alpha_n, n = 1,2, \dots, N$ and $b_m, m = 1,2, \dots, M$, the coefficients of the differential equation.
- It relates the Laplace transforms of the system output and input, provided the system is initially at rest (zero initial conditions).
- The transfer function is **independent** of the input signal applied to the system.
- These results hold for an LTI system **initially at rest**, that is, with zero initial conditions.
- With nonzero **initial conditions**, the output-to-input ratio would depend on those conditions as well as the input and output signals.
- With nonzero initial conditions, the output has two components: the zero-state response due to the input, and the zero-input response due to the system's stored initial energy.

System Transfer Function (3/3)

The output of an LTI system is the convolution of the input and impulse response:

$$y(t) = \int_0^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Assuming the LTI system is also causal [that is, $h(t) = 0,\ t < 0$], the LT time-convolution property gives:

$$Y(s) = H(s)\, X(s), \qquad Re(s) > \sigma_0$$

or

$$H(s) = \frac{Y(s)}{X(s)}$$

where $H(s) = L\{h(t)\}$. This leads to an important result: the **transfer function** is the Laplace transform of the impulse response $h(t)$.

The function $H(\Omega) = H(s)|_{s=j\Omega}$, if it exists, is the system **frequency response**: it describes system behavior in the frequency domain. The jΩ axis must **lie within** the region of convergence of $H(s)$.

Discussion (1/2)

For a rational transfer function, $\boldsymbol{M < N}$ must hold for the function to be **strictly proper.** Otherwise, the system is BIBO **unstable.**

Indeed, if $M \geq N$, polynomial division of $H(s)$ gives:

$$H(s) = c_k s^k + c_{k-1} s^{k-1} + \cdots + c_1 s + c_0 + \frac{Q(s)}{X(s)} = R(s) + \frac{Q(s)}{X(s)}$$

where $R(s)$ is a polynomial of degree $M - N$, given by:

$$R(s) = c_k s^k + c_{k-1} s^{k-1} + \cdots + c_1 s + c_0$$

The degree of $Q(s)$ is less than the degree of $X(s)$, so the ratio $Q(s)/\mathrm{X(s)}$ is strictly proper.

Apply a bounded input, say $x(t) = u(t)$, with LT $X(s) = 1/s$. The LT of the output is:

Discussion (2/2)

$$Y(s) = H(s)X(s) = \frac{H(s)}{s} = c_k s^{k-1} + c_{k-1} s^{k-2} + \cdots + c_1 + \frac{c_0}{s} + \frac{Q(s)}{sX(s)}$$

Thus, the output $y(t)$ is given by the inverse Laplace transform:

$$y(t) = c_k L^{-1}\{s^{k-1}\} + c_{k-1} L^{-1}\{s^{k-2}\} + \cdots + L^{-1}\{c_1\} + c_0 L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{Q(s)}{sX(s)}\right\}$$

The term $L^{-1}\{c_1\}$ produces a delta function, since:

$$L^{-1}\{c_1\} = L^{-1}\{c_1 1\} = c_1 L^{-1}\{1\} = c_1 \delta(t)$$

The system output contains a **delta** function and is therefore unbounded. Thus, the system is BIBO **unstable.**

Hence, BIBO stability requires $\boldsymbol{M < N}$.

3. System Stability and Causality in the Frequency Domain

Real Poles and Stability

For a real pole $p_k = \lambda k_p$, the corresponding component is: $y_k(t) = c_k e^{\lambda_k t}$

- $\boldsymbol{p_k = -\xi,\ \xi > 0}$**:**

  $$y_k(t) = c_k e^{-\xi t}$$

  The response decays exponentially: a **stable** system. Moving the pole farther left makes the decay faster.

- $\boldsymbol{p_k = +\xi,\ \xi > 0}$**:**

  $$y_k(t) = c_k e^{\xi t}$$

  The response grows exponentially: an **unstable** system.

- $\boldsymbol{p_k = 0}$**:**

  $$y_k(t) = c_k$$

  The response remains constant: **marginal stability**.

- **Repeated poles at** $\boldsymbol{p = 0}$**:**

  $$L^{-1}\left\{\frac{1}{s^r}\right\} = \frac{t^r - 1}{(r-1)!}u(t)$$

  Polynomial growth: **instability**.

Complex Poles and Oscillatory Response

For a conjugate pole pair $p_{1,2} = \sigma + j\Omega$, the corresponding output component is:

$$y_k(t) = C\ e^{\sigma t}\sin(\Omega t + \varphi)$$

- Ω: oscillation frequency
- φ: phase
- $Ce^{\sigma t}$: varying amplitude

The value of σ determines stability:

- $\sigma < 0$ Decaying oscillation Stable
- $\sigma = 0$ Sustained oscillation Marginally stable
- $\sigma > 0$ Growing oscillation Unstable

LTI System Stability (Summary)

The following **stability** conditions apply to LTI systems:

- If all system poles lie in the **left half-plane**, with negative real parts, the system is **asymptotically stable.**
- If at least one **pole** lies in the **right half-plane**, the system is **unstable.**
- If **simple poles lie on the imaginary axis** and all remaining poles lie in the **left half-plane**, the system is **marginally stable.**
- If **repeated poles lie on the imaginary axis**, the system is **unstable.**

LTI System Causality

- An LTI system is **causal** when its impulse response is **causal, or right-sided**:

$$h(t) = 0 \text{ for } t < 0$$

The region of convergence of $H(s)$ then lies to the **right of the rightmost pole**. For a rational transfer function, the region of convergence is:

$$Re(s) > \sigma_{max}$$

where σ_{max} is the **largest real part** among the poles of $H(s)$.

- An LTI system is **anticausal** when its impulse response is **left-sided**:

$$h(t) = 0 \text{ for } t > 0$$

The region of convergence of $H(s)$ then lies to the **left of the leftmost pole**. For a rational transfer function, the region of convergence is:

$$Re(s) < \sigma_{min}$$

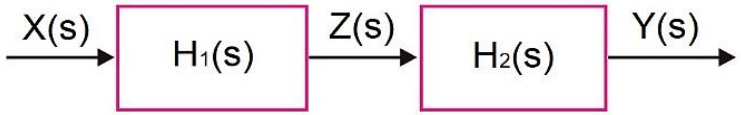
where σ_{min} is the **smallest real part** among the poles of $H(s)$.

- Finally, if the region of convergence lies **between two poles**, the impulse response is **two-sided**, and the system is **neither causal nor anticausal**.

4. System Connections in the Frequency Domain

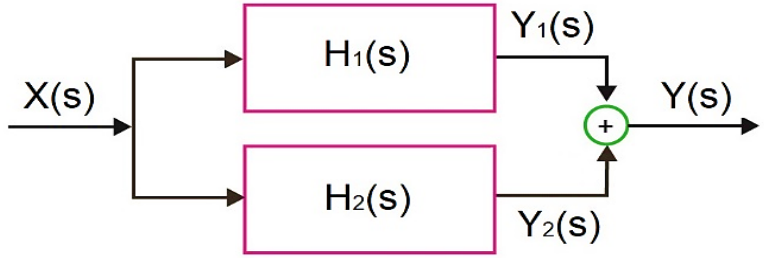
System Connections

Cascade connection: Two LTI systems are connected in cascade when the output of the first becomes the input to the second.



- Equivalent impulse response: $h_{eq}(t) = h_1(t) * h_2(t)$
- Equivalent transfer function: $H_{eq}(s) = H_1(s)H_2(s) = H_2(s)H_1(s)$

Parallel connection: Two LTI systems are connected in parallel when they share an input and their outputs are added at a common node.



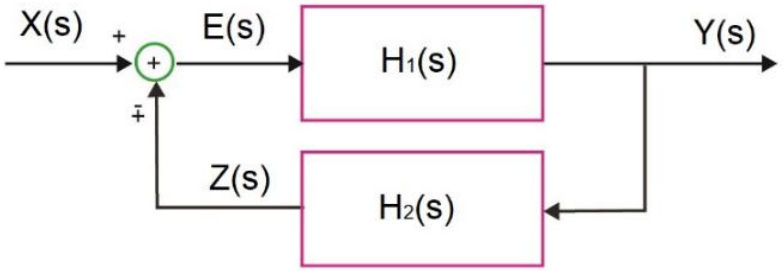
- Equivalent impulse response: $h_{eq}(t) = h_1(t) + h_2(t)$
- Equivalent transfer function: $H_{eq}(s) = H_1(s) + H_2(s) = H_2(s) + H_1(s)$

System Connections

Feedback connection: Two LTI systems are connected in feedback when the output of the first feeds the second, whose output is fed back to the first. It is added to the input for positive feedback or subtracted for negative feedback.

- Equivalent transfer function:

$$H_{eq}(s) = \frac{H_1(s)}{1 \pm H_1(s)H_2(s)}$$



The signal $e(t)$ is the error signal, given by:

- Negative feedback: $e(t) = x(t) - h_2(t) * y(t)$
- Positive feedback: $e(t) = x(t) + h_2(t) * y(t)$
- Equivalent impulse response:

$$h_{eq}(t) = L^{-1}\left\{\frac{H_1(s)}{1 \pm H_1(s)H_2(s)}\right\}$$

5. System Step Response

LTI System Step Response (1/4)

The step response of an LTI system initially at rest, with transfer function $H(s)$, is its output for the unit step input $u(t)$.

Express the LTI system transfer function $H(s)$ as a sum of partial fractions:

$$H(s) = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \cdots + \frac{C_N}{s - p_n} = \sum_{k=1}^{N} \frac{C_k}{s - p_k}$$

For the unit step input $x(t) = u(t)$, the input Laplace transform is $X(s) = 1/s$. Thus, the Laplace transform of the output, in partial fraction form, is:

$$Y(s) = X(s)\, H(s) = \frac{H(s)}{s} = \frac{1}{s} \sum_{k=1}^{N} \frac{C_k}{s - p_k} = \frac{A}{s} + \frac{1}{s} \sum_{k=1}^{N} \frac{A_k}{s - p_k}$$

where A and A_k are suitable constants.

LTI System Step Response (2/4)

Taking the inverse Laplace transform gives:

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{A}{s} + \frac{1}{s}\sum_{k=1}^{N}\frac{A_k}{s-p_k}\right\} = L^{-1}\left\{\frac{A}{s}\right\} + L^{-1}\left\{\sum_{k=1}^{N}\frac{A_k}{s-p_k}\right\} = A\,u(t) + \sum_{k=1}^{N} A_k\, e^{jp_k t}$$

$$= y_{ss}(t) + y_{ts}(t)$$

Here, $y_{ss}(t)$ and $y_{ts}(t)$ are the steady-state and **transient response**s, respectively, given by:

$$y_{ss}(t) = A\,u(t)$$

$$y_{ts}(t) = \sum_{k=1}^{N} A_k\, e^{jp_k t}$$

LTI System Step Response (3/4)

Since the system is stable, all poles have negative real parts, so the transient response terms decay exponentially as $t \to \infty$. For large times, only the steady-state component remains.

The steady-state value equals the transfer function at frequency $s = 0$, the system's DC gain. Indeed, using the final value theorem and $X(s) = 1/s$, we obtain:

$$y_{ss}(t) = \lim_{t\to\infty} y(t) = \lim_{s\to 0}[s\,Y(s)] = \lim_{s\to 0}\left[s\,H(s)\frac{1}{s}\right] = \lim_{s\to 0}[H(s)] = H(0)$$

Thus, for a stable LTI system, the step response approaches the constant $H(0)$, and the steady-state response is:

$$y_{ss}(t) = H(0)\,u(t)$$

LTI System Step Response (4/4)

- The transfer function **poles** determine the **transient response**: each pole p_k contributes a term of the form $e^{p_k t}$. In a stable system, all poles have negative real parts, so these exponential terms decay over time. This explains why transient behavior is **temporary**: its effect decreases exponentially as time increases, until it vanishes.
- The **steady-state component** is the final value to which the system converges. It equals $H(0)$, the transfer function at zero frequency. This quantity is the **DC gain** and characterizes the system response to constant or very slowly varying inputs.

5. State-Space Description of Systems

Introduction to State Space

The **transfer function** is an external representation of an LTI system. It directly relates input to output, facilitating the study of dynamic behavior, stability and frequency response.

It is defined for **zero initial conditions**, so it does not directly describe the effect of the initial state or the internal evolution of the system.

With **pole-zero cancellation**, some internal modes may disappear from the transfer function while remaining part of the system's physical dynamics.

By contrast, the state-space description, directly linked to the system differential equation:

- is unaffected by pole-zero cancellations,
- provides a more complete representation of the system's dynamic behavior,
- readily extends the analysis to systems with multiple inputs and multiple outputs.

State-Space Description of Systems (1/3)

State-space analysis studies the system's internal structure and allows conclusions about stability through a pair of **dynamic equations**:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}w(t)$$

$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t), \qquad t > 0$$

The first is the **state equation**, and the second is the **output equation**.

We seek the complete solution $y(t), t > 0$. Take the Laplace transform of each dynamic equation:

$$s\boldsymbol{I}\boldsymbol{X}(s) - x(0) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{b}W(s) \Rightarrow (s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X}(s) = x(0) + \boldsymbol{b}W(s)$$

$$Y(s) = c^T\boldsymbol{X}(s)$$

State-Space Description of Systems (2/3)

Solve the first equation for $X(s)$:

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}[x(0) + \boldsymbol{b}W(s)]$$

Substitute into the second equation:

$$Y(s) = \boldsymbol{c}^T\boldsymbol{X}(s) = \boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}[x(0) + \boldsymbol{b}W(s)]$$

To find the output $y(t)$, take the inverse Laplace transform of $Y(s)$. The matrix $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$ plays a central role. Formally, it can be written as:

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{1}{s}\left(\boldsymbol{I} - \frac{\boldsymbol{A}}{s}\right)^{-1} = \boldsymbol{I}s^{-1} + \boldsymbol{A}s^{-2} + \boldsymbol{A}^2s^{-3} + \cdots$$

This is the geometric series of the matrix $\left(\boldsymbol{I} - \frac{\boldsymbol{A}}{s}\right)^{-1}$, when it converges. Indeed:

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1}(s\boldsymbol{I} - \boldsymbol{A}) = \boldsymbol{I} + \boldsymbol{A}s^{-1} + \boldsymbol{A}^2s^{-2} + \cdots - \boldsymbol{A}s^{-1} - \boldsymbol{A}^2s^{-2} - \boldsymbol{A}^3s^{-3} - \cdots = \boldsymbol{I}$$

State-Space Description of Systems (3/3)

The inverse Laplace transform of $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$ is:

$$L^{-1}\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\} = \left(\boldsymbol{I} + \boldsymbol{A}t + \boldsymbol{A}^2\frac{t^2}{2!} + \boldsymbol{A}^3\frac{t^3}{3!} + \cdots\right)u(t) = e^{\boldsymbol{A}t}u(t)$$

The matrix e^{At} is the **state transition matrix**, a fundamental tool in state-space system analysis.

The inverse Laplace transform of $Y(s)$ follows from the above relation and the fact that the inverse transform of a product is a convolution integral:

$$y(t) = \boldsymbol{c}^T e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \boldsymbol{c}^T\int_0^\tau e^{\boldsymbol{A}(t-t)}\boldsymbol{b}\, w(\tau)\, d\tau, \qquad t \geq 0$$

The first component is the response due to initial conditions, and the second is the response due to the external input.

For a delta function input, the impulse response $h(t)$ is the response for $w(t) = \delta(t)$ with zero initial conditions:

$$h(t) = \boldsymbol{c}^T\int_0^t e^{\boldsymbol{A}(t-t)}\boldsymbol{b}\,\delta(\tau)\, d\tau = \boldsymbol{c}^T e^{\boldsymbol{A}t}\boldsymbol{b}\, u(t)$$

State-Space Analysis Summary

- A key advantage of state space is that it naturally includes **initial conditions**. The total response is analyzed as the combination of the free state evolution and the response driven by the external input.
- State-space descriptions are unaffected by **pole-zero cancellations**, which can hide internal dynamic modes in a transfer function. They therefore give a more complete picture of the system's physical dynamics.
- The matrix e^{At} is the **state transition matrix.** It plays a central role in state-space analysis, describes how the state evolves over time and directly links the system solution to the properties of matrix A.
- The final solution:

$$y(t) = \boldsymbol{c}^T e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \boldsymbol{c}^T \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}\, w(\tau)\, d\tau\,, \qquad t \geq 0$$

shows that the output is the sum of two terms: one due to **initial conditions** and one due to the **input**. This fundamental separation in system theory corresponds to the zero-input and zero-state responses.

Exercises

Exercise 1

Find the impulse and step responses of the system described by the following differential equation, with initial condition $y(0^-) = 0$.

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

Answer: (a) Input $x(t) = \delta(t)$. Taking the Laplace transform of both sides of the differential equation gives:

$$s\,Y(s) - y(0^-) + a\,Y(s) = L\{\delta(t)\} \Longrightarrow s\,Y(s) + a\,Y(s) = 1, \qquad Re(s) > 0$$

Solving for $Y(s)$:

$$Y(s) = \frac{1}{s + a}$$

Since the given region of convergence is right-sided, the impulse response is:

$$h(t) \equiv y(t) = e^{-at}u(t)$$

Exercise 1 (continued)

(b) Input $x(t) = u(t)$. Taking the Laplace transform of both sides of the differential equation gives:

$$s\,Y(s) - y(0^-) + a\,Y(s) = L\{u(t)\} \Longrightarrow s\,Y(s) + a\,Y(s) = \frac{1}{s}, \qquad Re(s) > 0$$

Solving for $Y(s)$:

$$Y(s) = \frac{1}{s(s+a)}$$

Since the given region of convergence is right-sided, the step response is:

$$s(t) \equiv y(t) = \frac{1}{a}(1 - e^{-at})\,u(t)$$

- The relation between the step and impulse responses shows that the step response contains information about the system's dynamics and can also be used to characterize it.
- The response form (exponential recovery) shows that the system has a particular temporal "memory," determined by its poles.

Exercise 1 (continued)

Note on the initial condition:

- If the input contains no impulse $\delta(t)$, the initial values on either side of zero are equal: $y(0^-) = y(0^+) = y(0)$. In this case, specifying either $y(0)$ or $y(0^+)$ gives the same initial condition, since both equal $y(0^-)$.
- If the right-hand side contains $\delta(t)$, the initial condition $y(0^-)$ must be specified, rather than $y(0)$ or $y(0^+)$. The presence of $\delta(t)$ imposes significant changes at time zero, so $y(0^-) \neq y(0^+)$.

Exercise 2

An LTI system is described by the following differential equation, where $x(t)$ is the input and $y(t)$ is the output:

$$\frac{d^3y(t)}{dt^3} - \frac{dy(t)}{dt} = x(t) \equiv sint$$

For initial conditions $y(0^-) = 2,\ \ y'(0^-) = 0$ and $y''(0^-) = -1$, find the system output.

Answer: First, take the unilateral LT of the differential equation:

$$s^3Y(s) - s^2y(0^-) - sy'(0^-) - y''(0^-) - sY(s) + y(0^-) = L\{sint\} \Rightarrow$$

$$\Rightarrow s^3\,Y(s) - 2s^2 + 1 - s\,Y(s) + 2 = \frac{1}{s^2+1} \Rightarrow Y(s)(s^3 - s) - 2s^2 + 3 = \frac{1}{s^2+1}$$

Solving for $Y(s)$:

$$Y(s) = \frac{2s^2 - 3}{s^3 - s} + \frac{1}{(s^3 - s)(s^2 + 1)} = \frac{2s^4 - s - 2}{s(s^2 + 1)(s^2 - 1)}$$

Exercise 2 (continued)

Express $Y(s)$ as a sum of partial fractions:

$$Y(s) = \frac{2}{s} - \frac{1}{4(s-1)} - \frac{1}{4(s+1)} + \frac{s}{2(s^2+1)}$$

Thus, the output $y(t)$ is:

$$y(t) = \left[2 - \frac{1}{4}e^{t} - \frac{1}{4}e^{-t} + \frac{1}{2}cos\ t\right]u(t)$$

Checking the initial conditions:

- $y(0) = 2 - \frac{1}{2} + \frac{1}{2} = 2$
- $y'(0) = 0$
- $y''(0) = -\frac{1}{2} - \frac{1}{2} = -1$

Thus, the solution satisfies both the differential equation and the given initial conditions.

Exercise 3

Determine when the ratio $Y(s)/X(s)$ equals the transfer function of an LTI system. Consider the system:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

with initial condition $y(0^-) = 3$ and input $x(t) = u(t)$. Calculate:

(a) the transfer function $H(s)$,

(b) the Laplace transform of the output $Y(s)$,

(c) the ratio $Y(s)/X(s)$, and explain why it differs from $H(s)$. Find the output $y[n]$.

Answer: (a) The transfer function is defined for zero initial conditions. Taking the Laplace transform of the given differential equation under initial rest gives:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+2}$$

For the given input:

$$X(s) = \frac{1}{s}$$

Exercise 3 (continued)

(b) With nonzero initial condition $y(0^-) = 3$, the Laplace transform of the differential equation gives:

$$sY(s) - 3 + 2Y(s) = X(s)$$

thus

$$(s+2)Y(s) = 3 + \frac{1}{s}$$

and therefore

$$Y(s) = \frac{3 + \frac{1}{s}}{s+2} = \frac{3s+1}{s(s+2)}$$

(c) The ratio of the output and input Laplace transforms is:

$$\frac{Y(s)}{X(s)} = s \cdot \frac{3s+1}{s(s+2)} = \frac{3s+1}{s+2}$$

Exercise 3 (continued)

The ratio $Y(s)/X(s)$ differs from the transfer function $H(s)$: the ratio $Y(s)/X(s)$ incorporates the initial state, while the transfer function assumes zero initial conditions. Splitting the output into zero-state and zero-input responses gives:

$$Y(s) = Y_{zs}(s) + Y_{zi}(s)$$

where:

$$Y_{zs}(s) = H(s)X(s) = \frac{1}{s} \cdot \frac{1}{(s+2)} = \frac{1}{s(s+2)} \quad \text{and} \quad Y_{zi}(s) = \frac{3}{s+2}$$

Therefore:

$$Y(s) = \frac{1}{s(s+2)} + \frac{3}{s+2} = \cdots = \frac{1}{2} \cdot \frac{1}{s} + \frac{5}{2} \cdot \frac{1}{s+2}$$

Taking the inverse Laplace transform:

$$y(t) = \left(\frac{1}{2} + \frac{5}{2}e^{-2t}\right)u(t)$$

Exercise 3 (continued)

Thus, the additional output component comes from the system's initial energy.

Observations:

- The transfer function $H(s)$ is an intrinsic system property, defined under zero initial conditions. It must therefore be distinguished from the algebraic ratio of total output to input.
- With nonzero initial conditions, the output has two components: the zero-state response due to the input, and the zero-input response due to the system's stored initial energy.
- The ratio $Y(s)/X(s)$ can misrepresent the system if the initial conditions are ignored. Here, the additional term $3/(s+2)$ changes the ratio without changing the transfer function $H(s) = 1/(s+2)$, since it comes from the initial state rather than the applied input.

Exercise 4

An LTI system has impulse response $h(t) = \delta(t) - 2e^{-t}\,u(t)$. How many different inputs $x(t)$, and which ones, can produce the output $y(t) = e^{-2t}\,u(t)$?

Answer: Calculate the transfer function $H(s)$ from the LT of the impulse response $h(t)$:

$$H(s) = 1 - \frac{2}{s+1} = \frac{s-1}{s+1}, \qquad Re(s) > -1$$

The Laplace transform $Y(s)$ of the output $y(t) = e^{-2t}u(t)$ is:

$$Y(s) = \frac{1}{s+2}, \qquad Re(s) > -2$$

To find the input $x(t)$, determine $X(s)$ from:

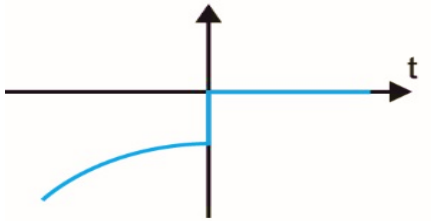
$$X(s) = \frac{Y(s)}{H(s)} = \frac{\frac{1}{s+2}}{\frac{s-1}{s+1}} = \frac{s+1}{(s+2)(s-1)} = \frac{1}{3}\cdot\frac{1}{s+2} + \frac{2}{3}\cdot\frac{1}{s-1}$$

Exercise 4 (continued)

The function $X(s)$ has two real poles ($s = -2$ and $s = 1$), whose positions give the possible regions of convergence: $\mathrm{Re}(s) < 1$, $-2 < Re(s) < 1$ and $Re(s) > 1$.

- For $Re(s) < -2$, both terms are left-sided:

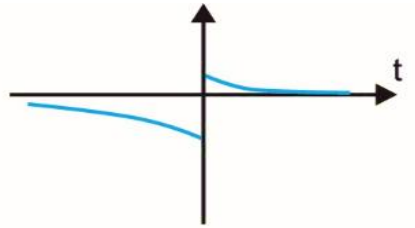
$$x_1(t) = -\frac{1}{3}\, e^{-2t}u(-t) - \frac{2}{3}\, e^{t}u(-t)$$



(a)

- For $-2 < Re(s) < 1$, the first term is right-sided and the second is left-sided:

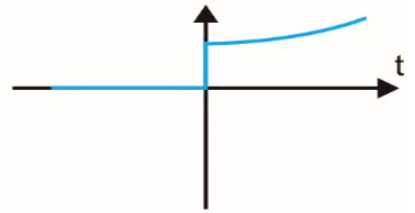
$$x_2(t) = \frac{1}{3}\, e^{-2t}u(t) - \frac{2}{3}\, e^{t}u(-t)$$



(b)

- For $Re(s) > 1$, both terms are right-sided:

$$x_3(t) = \frac{1}{3}\, e^{-2t}u(t) + \frac{2}{3}\, e^{t}u(t)$$



(c)

Exercise 4 (continued)

To determine which of these signals could produce the given output, consider the regions of convergence. The system $H(s)$ converges for $Re(s) > -1$.

- Since $X_1(s)$ converges only for $Re(s) < -2$, there is no common region of convergence, so $x_1(t)$ is not an admissible input.
- The region of convergence of $X_2(s)$ is $-2 < Re(s) < -1$, which intersects the region of convergence of $H(s)$. Thus, $x_2(t)$ is an admissible input.
- The region of convergence of $X_3(s)$ is $Re(s) > 1$, which also intersects the region of convergence of $H(s)$. Thus, $x_3(t)$ is also an admissible input.

Therefore, the **admissible inputs** are $x_2(t)$ and $x_3(t)$.

Exercise 5

Find the transfer function and impulse response of an LTI system that, for input $x(t) = e^{-t}u(t)$, produces output $y(t) = 10e^{-t}\cos(4t)\,u(t)$.

Answer: Since no initial conditions are given, assume the system is initially at rest. The Laplace transform of the input is:

$$X(s) = \frac{1}{s+1}$$

Using the known transform:

$$e^{-at}\cos(\Omega t)\,u(t) \xrightarrow{L} \frac{s+a}{(s+a)^2+\Omega^2}$$

for $\alpha = -1$ and $\Omega = 4$, we obtain:

$$Y(s) = \frac{10(s+1)}{(s+1)^2+16}$$

Thus, the transfer function $H(s)$ is:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{10(s+1)^2}{(s+1)^2+16} = 10\frac{s^2+2s+1}{s^2+2s+17}$$

Exercise 5 (continued)

The function $H(s)$ has a double zero at $s = -1$ and two simple complex conjugate poles at $s_1 = -1 + 4i$ and $s_2 = -1 - 4i$.

The impulse response $h(t)$ is the inverse LT of the transfer function $H(s)$:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{10(s+1)^2}{(s+1)^2 + 16} = 10 - 40\frac{4}{(s+1)^2 + 4^2}$$

The impulse response is:

$$h(t) = L^{-1}\{H(s)\} = 10L^{-1}\{1\} - 40L^{-1}\left\{\frac{4}{(s+1)^2 + 4^2}\right\} = 10\delta(t) - 40e^{-4t}\sin(4t)\,u(t)$$

which is the required result.

To obtain this expression, we used the inverse transforms:

$$1 \xrightarrow{L^{-1}} \delta(t) \quad \text{and} \quad \frac{\omega}{(s+a)^2 + \Omega^2} \xrightarrow{L^{-1}} e^{-at}\sin(\Omega t)\,u(t)$$

Exercise 6

Find the transfer function and impulse response of the system described by:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$$

Answer: Take the LT of both sides and use the complex frequency differentiation property:

$$L\left\{\frac{d^2y(t)}{dt^2}\right\} + 5L\left\{\frac{dy(t)}{dt}\right\} + 6L\{y(t)\} = L\left\{\frac{dx(t)}{dt}\right\} + 4L\{x(t)\} \Rightarrow$$

$$s^2\,Y(s) + 5s\,Y(s) + 6\,Y(s) = s\,X(s) + 4\,X(s) \Rightarrow$$

$$Y(s)\,[s^2 + 5s + 6] = X(s)\,(s + 4) \Rightarrow$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+4}{s^2+5s+6} = \frac{s+4}{(s+2)(s+3)}$$

Exercise 6 (continued)

The function $H(s)$ has one zero at $s = -4$ and two simple real poles at $s_1 = -2$ and $s_2 = -3$.

Both poles lie in the **left half-plane**, so the system is **stable.**

The impulse response is the inverse Laplace transform of $H(s)$. Partial fraction expansion gives:

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\left\{\frac{s+4}{(s+2)(s+3)}\right\} = 2e^{-2t}u(t) - e^{-3t}u(t) \Rightarrow$$

$$h(t) = (2e^{-2t} - e^{-3t})\, u(t)$$

Exercise 7

An LTI system with impulse response $h(t) = e^{-2t}u(t)$ receives the input $x(t) = e^{-t}u(t)$.

(a) Find the Laplace transforms $H(s)$ and $X(s)$.

(b) Find the Laplace transform $Y(s)$ of the output $y(t)$.

(c) Find the output $y(t)$ using the inverse LT of $Y(s)$.

(d) Find the output $y(t)$ using convolution $y(t) = h(t) * x(t)$, verifying the preceding result.

Answer: (a) By the LT definition:

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st} dt = \int_{0}^{\infty} e^{-(s+2)t} u(t)\, dt = \left[\frac{e^{-(s+2)t}}{-(s+2)}\right]_{0}^{\infty} = \frac{1}{s+2}, \qquad Re(s) > -2$$

Exercise 7 (continued)

and:

$$\mathrm{X}(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt = \int_{0}^{\infty} e^{-(s+1)t}\, u(t) dt = \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_{0}^{\infty} = \frac{1}{s+1}, \qquad Re(s) > -1$$

(b) The system output is the convolution of the impulse response and input: $y(t) = h(t) * x(t)$.

Applying the LT convolution theorem gives:

$$Y(s) = H(s)\, X(s) = \frac{1}{(s+1)(s+2)}, \qquad Re(\mathrm{s}) > -1$$

The region of convergence is the intersection of those of $H(s)$ and $X(s)$.

Exercise 7 (continued)

(c) Expanding $Y(s)$ in partial fractions gives:

$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

The inverse LT is the sum of the inverse transforms of the individual fractions:

$$y(t) = e^{-t}u(t) - e^{-2t}u(t) = [e^{-t} - e^{-2t}]u(t)$$

The region of convergence $Re(s) > -1$ determines that the individual components of $y(t)$ are right-sided.

(d) Using convolution:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-2(t-\tau)}\, u(t-\tau)\, e^{-t}\, u(\tau)\, d\tau$$

$$= \int_{0}^{t} e^{-2t}\, e^{2\tau} e^{-\tau} d\tau = e^{-2t} \int_{0}^{t} e^{\tau} d\tau = e^{-2t}[e^{\tau}]_{0}^{t} = e^{-2t}(e^{t} - 1) = (e^{-t} - e^{-2t})\, u(t)$$

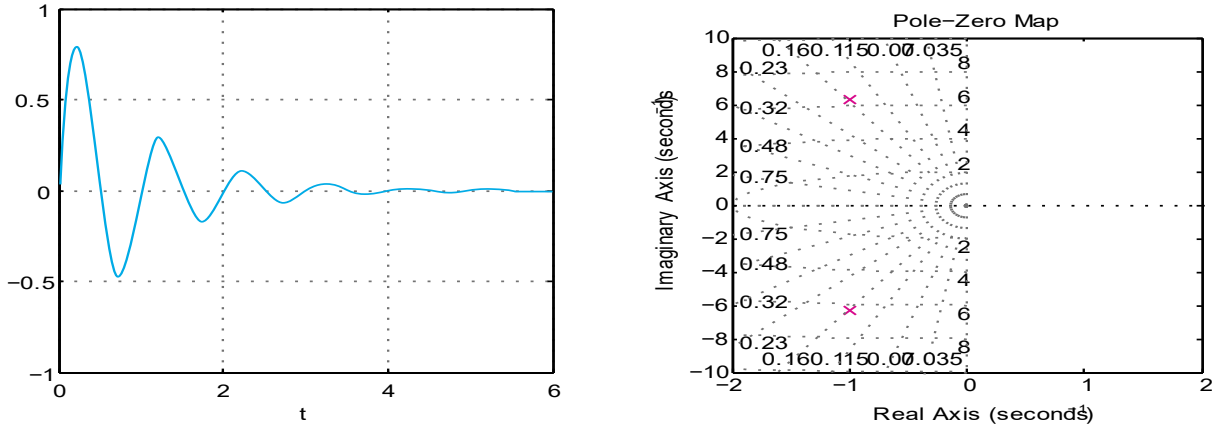
This gives the same result as part (c).

Exercise 8

Study the stability of an LTI system with impulse response $h(t) = e^{-at}sin(2\pi t)u(t)$ for different values of α.

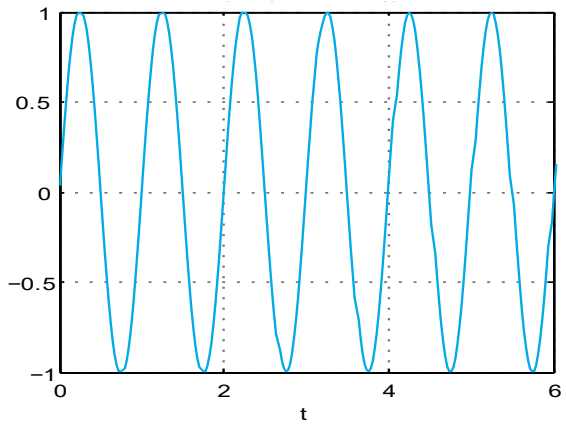
Answer: The transfer function $H(s)$, in rational form, is:

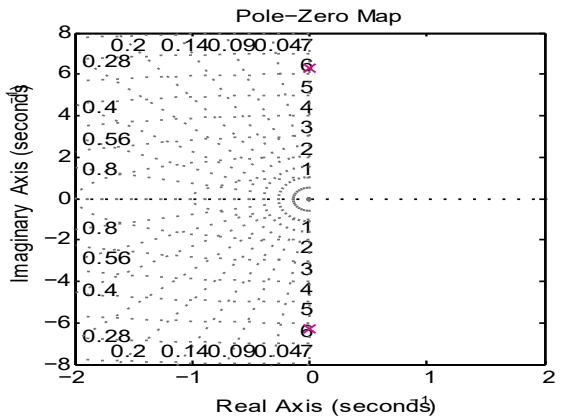
$$H(s) = \frac{2\,\pi}{(s\ +\ 1)^2 +\ 4\,\pi^2} = \frac{2\,\pi}{s^2\ + 2s + (1 +\ 4\,\pi^2)}$$



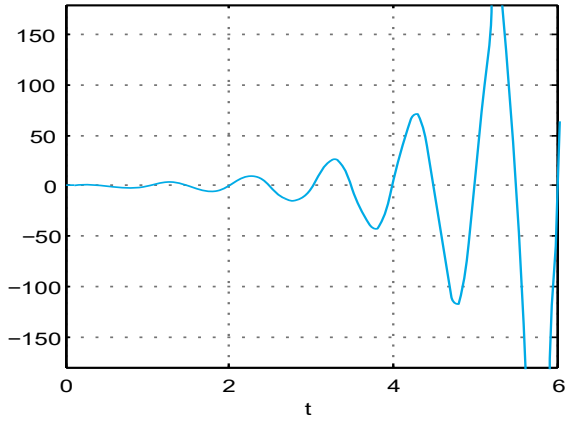
(a) Impulse response, (b) Pole-zero map, for $\alpha = 1$.

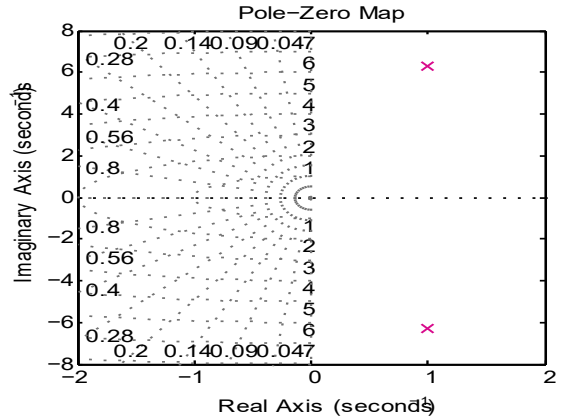
Exercise 8 (continued)





(a) Impulse response, (b) Pole-zero map, for $\alpha = 0$.





(a) Impulse response, (b) Pole-zero map, for $\alpha = -1$.

Exercise 9

Calculate and plot the impulse and step responses of the LTI system with transfer function:

$$H(S) = \frac{s+2}{s^2+3s+3}$$

Answer: The Laplace transform of the output for a step input is:

$$Y(s) = X(s)\,H(s) = \frac{1}{s}H(s) = \frac{s+2}{s(s^2+3s+3)}$$

Expand $Y(s)$ in partial fractions. Set:

$$\frac{s+2}{s(s^2+3s+3)} = \frac{A}{s} + \frac{Bs+C}{s^2+3s+3}$$

Multiplying by $s(s^2+3s+3)$ gives:

$$s+2 = A(s^2+3s+3) + s(Bs+C) = (A+B)s^2 + (3A+C)s + 3A$$

Equating coefficients gives:

$$A+B=0, \qquad 3A+C=1, \qquad 3A=2$$

Exercise 9 (continued)

Solving these equations gives:

$$A = \frac{2}{3}, \qquad B = -\frac{2}{3}, \qquad C = -1$$

Therefore:

$$Y(s) = \left(\frac{2}{3}\right)\frac{1}{s} + \frac{-\frac{2}{3}s - 1}{s^2 + 3s + 3}$$

Observe that:

$$-\frac{2}{3}s - 1 = -\frac{2}{3}\left(s + \frac{2}{3}\right)$$

The denominator can also be written as:

$$s^2 + 3s + 3 = \left(s + \frac{2}{3}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Exercise 9 (continued)

Thus:

$$Y(s) = \left(\frac{2}{3}\right)\frac{1}{s} - \left(\frac{2}{3}\right)\frac{\left(s+\frac{2}{3}\right)}{\left(s+\frac{2}{3}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

Using the known transform:

$$L^{-1}\left\{\frac{s+a}{(s+a)^2+\Omega^2}\right\} = e^{-at}cos\,(\Omega t)$$

Taking the inverse Laplace transform gives the **step response**:

$$s(t) = y(t) = L^{-1}\{Y(s)\} = \left[\frac{2}{3} - \frac{2}{3}e^{-\frac{3t}{2}}cos\left(\frac{\sqrt{3}}{2}t\right)\right]u(t)$$

or equivalently:

$$s(t) = \frac{2}{3}\left[1 - e^{-\frac{3t}{2}}cos\left(\frac{\sqrt{3}}{2}t\right)\right]u(t)$$

Exercise 9 (continued)

The **transient** response is:

$$s_{ts}(t) = y_{ts}(t) = -\left(\frac{2}{3}\right)e^{-\frac{3t}{2}}cos\left(\frac{\sqrt{3}}{2}t\right)u(t)$$

The **steady-state** response is:

$$s_{ss}(t) = y_{ss}(t) = \left(\frac{2}{3}\right)u(t)$$

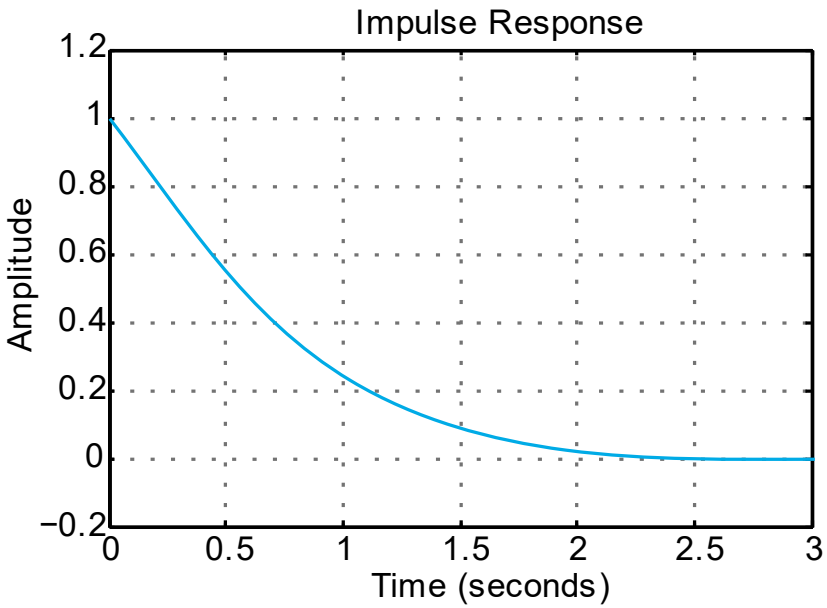
The steady-state response can also be calculated directly using the final value theorem:

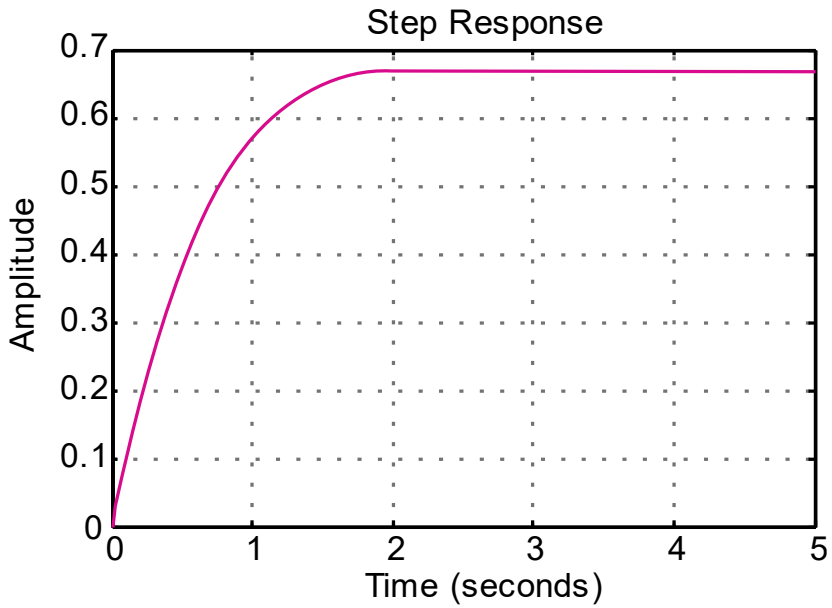
$$s_{ss}(t) = y_{ss}(t) = \lim_{s\to 0} H(s) = \lim_{s\to 0} \frac{s+2}{s^2+3s+3} = \frac{2}{3}$$

The impulse response is the first derivative of the step response:

$$h(t) = \frac{dy(t)}{dt} = \cdots = e^{-\frac{3t}{2}}\left[cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}sin\left(\frac{\sqrt{3}}{2}t\right)\right]u(t)$$

Exercise 9 (continued)





(a) Impulse response, (b) Step response

Exercise 10

An LTI system is described by the following dynamic (state and output) equations:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \end{bmatrix} w(t)$$

$$y(t) = [1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

(a) Find the zero-input response for initial conditions $x_1(0) = 1$ and $x_2(0) = 0$.

(b) Find the zero-state response for the step input $w(t) = u(t)$.

(c) Find the system impulse response $h(t)$.

Answer: Taking the Laplace transform of the state equation with nonzero initial conditions gives:

$$\begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) + 8W(s) \end{bmatrix}$$

$$Y(s) = X_1(s)$$

Exercise 10 (continued)

The output Laplace transform $Y(s)$ shows that $X_1(s)$ must be calculated from the state equation:

$$Y(s) = X_1(s) = \frac{\det\begin{bmatrix} x_1(0) & -1 \\ x_2(0) + 8W(s) & s+6 \end{bmatrix}}{\det\begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix}} = \frac{x_1(0)(s+6) + x_2(0) + 8W(s)}{s^2 + 6s + 8}$$

$$= \frac{s+6}{s^2+6s+8} + \frac{8}{s(s^2+6s+8)}$$

This expression directly separates the zero-input and zero-state responses.

Here, we used initial conditions $x_1(0) = 1$ and $x_2(0) = 0$, and $W(s) = 1/s$ because the input is the unit step $u(t)$.

Exercise 10 (continued)

(a) The first fraction is the Laplace transform of the zero-input response:

$$Y_{zi}(s) = \frac{s+6}{s^2+6s+8} = \frac{s+6}{(s+2)(s+4)} = \cdots = \frac{2}{s+2} - \frac{1}{s+4}$$

Thus, the **zero-input response** is:

$$y_{zi}(t) = (2e^{-2t} - e^{-4t})\, u(t)$$

(b) The second fraction is the Laplace transform of the zero-state response. Use zero initial conditions $x_1(0) = 0,\ x_2(0) = 0$ and a step input:

$$w(t) = u(t) \Longrightarrow W(s) = \frac{1}{s}$$

Therefore:

$$Y_{zs}(s) = \frac{8W(s)}{(s^2+6s+8)} = \frac{8}{s(s+2)(s+4)}$$

Expanding in partial fractions:

$$Y_{zs}(s) = \frac{8}{s(s^2+6s+8)} = \frac{8}{s(s+2)(s+4)} = \cdots = \frac{1}{s} - \frac{2}{s+2} + \frac{1}{s+4}$$

Exercise 10 (continued)

Thus, the **zero-state response** is:

$$y_{zs}(t) = (1 - 2e^{-2t} + e^{-4t})\, u(t)$$

For the given initial conditions and a step input, the total response is:

$$y(t) = y_{zi(t)} + y_{zs(t)} = (2e^{-2t} - e^{-4t})\, u(t) + (1 - 2e^{-2t} + e^{-4t})\, u(t) = u(t)$$

(c) To find the impulse response, set $w(t) = \delta(t)$, so $W(s) = 1$, with zero initial conditions $x_1(0) = 1$, $x_2(0) = 0$ and $Y(s) = H(s)$:

$$H(s) = \frac{8}{s^2 + 6s + 8} == \frac{8}{(s+2)(s+4)} = \cdots = \frac{4}{s+2} - \frac{4}{s+4}$$

Thus, the impulse response is:

$$h(t) = (4e^{-2t} - 4e^{-4t})\, u(t)$$