UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

SIGNALS AND SYSTEMS
Solved Examples – Tutorial Sheet 10

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INVERSE FOURIER TRANSFORM – FOURIER TRANSFORM OF POWER SIGNALS

- Inverse Fourier transform
- Fourier transform of power signals
- Correlations (autocorrelation, cross-correlation)
- Spectral densities

1. Inverse Fourier Transform

📖 Example 1

Find the inverse Fourier transform of the function:

$$X(\Omega) = [\delta(\Omega - 2) + \delta(\Omega + 2)] + 3j[\delta(\Omega - 1) - \delta(\Omega + 1)]$$

Answer: We know the following Fourier transform pairs:

$$\cos(at) \overset{F}{\longleftrightarrow} \pi[\delta(\Omega - \alpha) + \delta(\Omega + \alpha)]$$

$$\sin(\beta t) \overset{F}{\longleftrightarrow} \frac{\pi}{j}[\delta(\Omega - \beta) - \delta(\Omega + \beta)]$$

Therefore, for $\alpha = 2$ and $\beta = 1$, we have:

$$[\delta(\Omega - 2) + \delta(\Omega + 2)] \overset{F^{-1}}{\longleftrightarrow} \frac{1}{\pi}\, cos(2t)$$

$$3j[\delta(\Omega - 1) - \delta(\Omega + 1)] \overset{F^{-1}}{\longleftrightarrow} -\frac{3}{\pi} sin(t)$$

Therefore:

$$x(t) = \frac{1}{\pi}\, cos(2t) - \frac{3}{\pi} sin(t)$$

Comments:

- A sum of symmetric impulses in the spectrum, $\delta(\Omega - \alpha) + \delta(\Omega + \alpha)$, corresponds to a cosine in the time domain.
- A difference of symmetric impulses, $\delta(\Omega - \beta) - \delta(\Omega + \beta)$, corresponds to a sine. The coefficient j determines the sign and amplitude of the sine term.
- The positions of the impulses δ(t) in frequency determine the frequencies of the sinusoidal terms in time. Thus, the impulses at ± 2 give $cos(2t)$, while those at ± 1 give $sin(t)$.

📖 Example 2

Find the inverse Fourier transform of the function:

$$X(\Omega) = \frac{4 + 10j\Omega}{-\Omega^2 + 6j\Omega + 8}$$

Answer: We factor the denominator, split the expression into two partial fractions, and apply the solution method of the previous example:

$$X(\Omega) = \frac{4 + 10j\Omega}{(2 + j\Omega)(4 + j\Omega)} = -\frac{8}{2 + j\Omega} + \frac{18}{4 + j\Omega}$$

Since:

$$Ae^{-\alpha t}u(t) \overset{F}{\longleftrightarrow} \frac{A}{a + j\Omega}$$

we obtain:

$$x(t) = F^{-1}\{X(\Omega)\}$$
$$= F^{-1}\left\{-\frac{8}{2 + j\Omega}\right\} + F^{-1}\left\{\frac{18}{4 + j\Omega}\right\}$$
$$= -8\, e^{-2t}u(t) + 18\, e^{-4t}u(t)$$
$$= [18\, e^{-4t} - 8\, e^{-2t}]\, u(t)$$

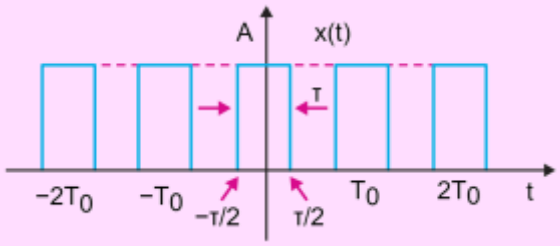
Comments:

- The output signal $x(t) = [18e^{-4t} - 8e^{-2t}]u(t)$ consists of two competing exponential terms:
  - The term $18e^{-4t}$ starts at a higher value (18), but decays very rapidly (time constant $\tau = 0.25\ s$).
  - The term $-8e^{-2t}$ starts at a lower magnitude (8), but decays at half the rate ($\tau = 0.5\ s$).
- This combination gives the signal a characteristic transient-response curve, often encountered during charging and discharging in automatic control systems and telecommunications networks.

2. Fourier Transform of Power Signals

📖 Example 3 (*)

Calculate the Fourier transform of the periodic rectangular signal $x(t)$, with pulse duration τ and period T_0:



Answer: The rectangular pulse of duration τ is centered at zero within each period. We follow the same solution method as in the previous example. The periodic rectangular signal has the Fourier

series expansion:

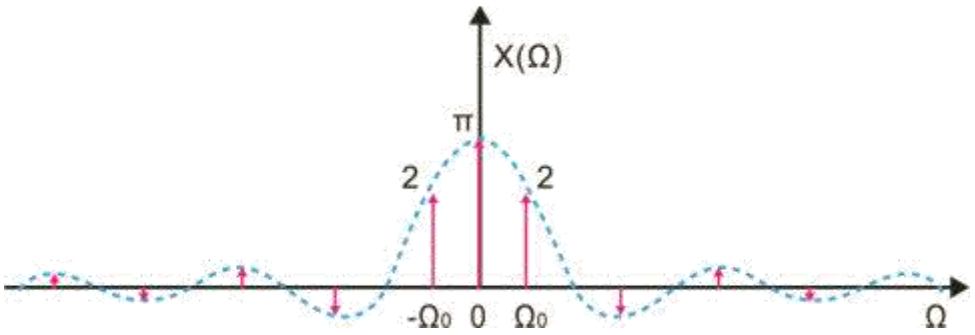
$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{jk\Omega_0 t}$$

where:

$$X_k = \frac{\tau}{T_0} \frac{sin\left(\frac{k\Omega_0 \tau}{2}\right)}{\left(\frac{k\Omega_0 \tau}{2}\right)} = \frac{\tau \Omega_0}{2\pi} sinc\left(\frac{k\Omega_0 \tau}{2}\right)$$

By linearity of the Fourier transform, and since $e^{jk\Omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\delta(\Omega - k\Omega_0)$, the Fourier transform of the periodic signal is:

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} X_k \, \delta(\Omega - k\Omega_0)$$

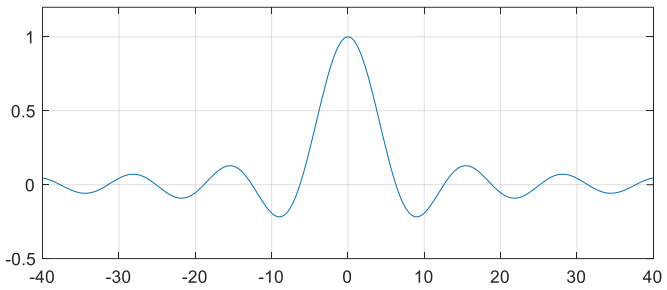


Spectrum of the periodic rectangular signal.

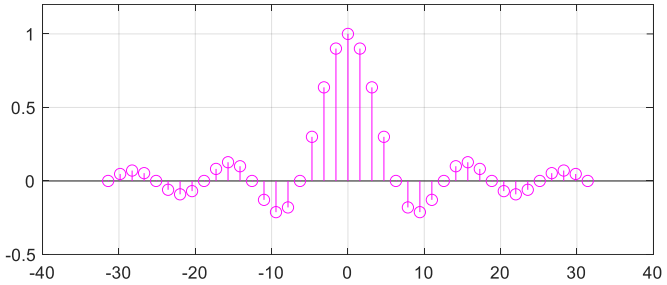
Comments:

- The Fourier transform of the periodic rectangular pulse train is a train of impulses at the harmonic frequencies $k\Omega_0$.
- The amplitudes of the frequency-domain impulses are $2\pi X_k$, where X_k are the exponential Fourier series coefficients of the periodic signal.
- Equivalently, the line spectrum can be viewed as samples of the Fourier transform of the truncated signal, with sampling interval $\Omega_0 = 2\pi/T_0$.
- The sinc envelope of the periodic signal's spectrum coincides with the spectrum of the signal truncated to one period. It arises because each period contains a rectangular pulse of duration τ. The shorter the pulse duration, the more slowly the spectral lines decay.
- The spacing between spectral lines is $\Omega_0 = 2\pi/T_0$. Thus, the time-domain period directly determines the density of the lines in the frequency domain.

(a)



(b)



(a) Fourier transform of the truncated signal $x_T(t)$

(b) Fourier transform of the periodic signal $x(t)$

3. Correlations (Autocorrelation and Cross-Correlation)

📖 Example 4

Calculate the autocorrelation function and energy spectral density of a rectangular pulse with amplitude A and duration T:

$$x(t) = A\ rect\left(\frac{t}{T}\right)$$

Answer: The given signal is an energy signal. We calculate its autocorrelation function:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\ x(t+\tau)\ dt = \int_{-\infty}^{\infty} A\ rect\left(\frac{t}{T}\right) A\ rect\left(\frac{t+\tau}{T}\right) dt$$

Because the rectangular pulse is symmetric, its autocorrelation equals its convolution, which we know is a triangular pulse. Let us calculate the autocorrelation analytically.

The position of the pulse $A\ rect\left(\frac{t+\tau}{T}\right)$ on the time axis is determined by the lag τ. The integral above is zero when the two pulses do not overlap. From figure (a) below, this occurs when:

$$-\tau + \frac{T}{2} < -\frac{T}{2} \Rightarrow -\tau < -T \Rightarrow \tau > T$$

and from figure (b), when:

$$-\tau - \frac{T}{2} > \frac{T}{2} \Rightarrow -\tau > T \Rightarrow \tau < -T$$

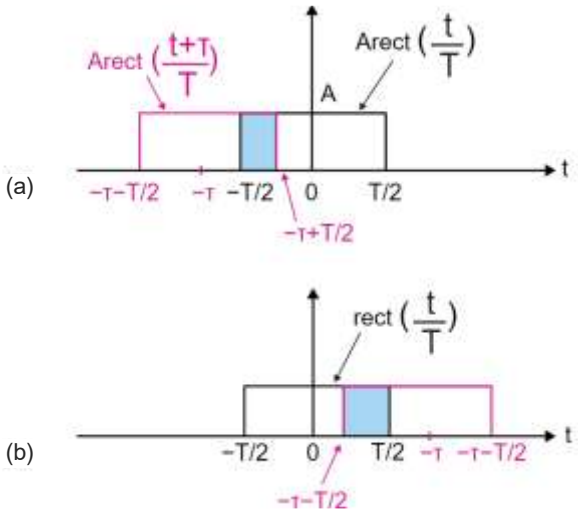
Therefore, the autocorrelation is zero for lags satisfying $\tau > T$ and $\tau < -T$. We distinguish two cases of the lag τ for which the rectangular pulses overlap, producing a nonzero result.

The first case is shown in figure (a) and is described by:

$$-\frac{T}{2} < -\tau + \frac{T}{2} < \frac{T}{2} \Longrightarrow \ldots \Longrightarrow 0 < \tau < T$$

In this case, the autocorrelation integral is:

$$R_{xx}(\tau) = \int_{-T/2}^{-\tau+T/2} A^2 dt = A^2 t|_{-T/2}^{-\tau+T/2} = \cdots = A^2(T-\tau)$$



Autocorrelation calculation for lag: (a) $0 < \tau < T$, (b) $-T < \tau < 0$.

The second case of the lag τ that gives a nonzero result is shown in figure (b) and is described by:

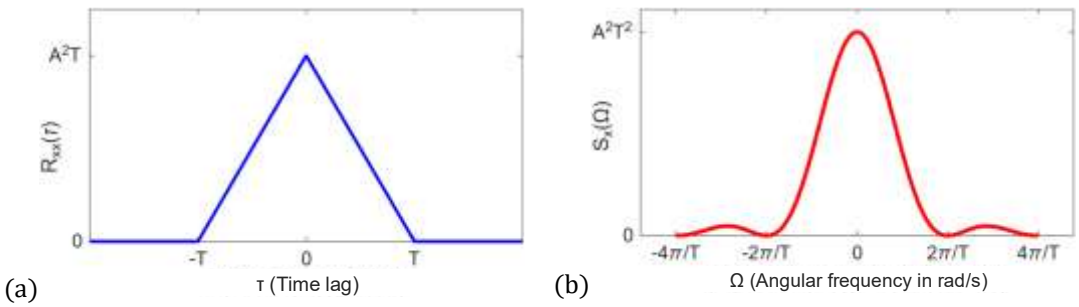
$$-\frac{T}{2} < -\tau - \frac{T}{2} < \frac{T}{2} \Longrightarrow \ldots \Longrightarrow -T < \tau < 0$$

In this case, the autocorrelation integral is:

$$R_{xx}(\tau) = \int_{-\tau-T/2}^{T/2} A^2 dt = A^2 t|_{-\tau-T/2}^{T/2} = \cdots = A^2(T+\tau)$$

We summarize the result as:

$$R_{xx}(\tau) = \begin{cases} 0, & \tau > T \text{ and } \tau < -T \\ A^2(T+\tau), & -T < \tau < 0 \\ A^2(T-\tau), & 0 < \tau < T \end{cases}$$



(a) (b)

(a) Autocorrelation $R_{xx}(\tau)$, (b) Energy spectral density $S_x(\Omega)$

The result is a triangular pulse centered at zero, with duration $2T$ and peak amplitude A^2. The energy spectral density is given by:

$$S_x(\Omega) = |X(\Omega)|^2$$

Since the Fourier transform of the rectangular pulse $A\ rect(t/T)$ is:

$$X(\Omega) = AT\ sinc\left(\frac{\Omega T}{2\pi}\right)$$

the energy spectral density is:

$$S_x(\Omega) = |X(\Omega)|^2 = A^2T^2\ sinc^2\left(\frac{\Omega T}{2\pi}\right)$$

Comments:

- The autocorrelation $R_{xx}(\tau)$ measures the similarity between a signal and a copy shifted in time by τ. For $\tau = 0$, the similarity is greatest and equals the total signal energy, $R_{xx}(0) = A^2T$. As the copy moves, the overlap decreases linearly, forming a triangular pulse.
- The energy spectral density $S_x(\Omega)$ shows how the energy of the rectangular pulse is distributed across frequencies. Because of its $sinc^2$ form, most energy is concentrated at low frequencies, while the lobe amplitudes decrease farther from zero. The spectrum vanishes at $\Omega = 2k\pi/T$.
- We know that the Fourier transform of a triangular pulse in time gives a $sinc^2$ function in frequency. Here, starting from the autocorrelation, which is triangular in time, we obtain the same mathematical structure $sinc^2$ for the energy density.

4. Spectral Densities

Example 5

Calculate the energy in the frequency band from $\Omega_1 = 0$ to $\Omega_2 = 20\pi$ for the signal $x(t) = e^{-t}u(t)$, and the percentage of the total signal energy contained in this band.

Answer: The Fourier transform of $x(t)$ is:

$$X(\Omega) = \frac{1}{1 + j\Omega}$$

Therefore, the energy spectral density is:

$$S_x(\Omega) = |X(\Omega)|^2 = \frac{1}{1+\Omega^2}$$

The energy in the frequency interval $[0, 20\pi]$ is given by:

$$E_x(0, 20\pi) = \frac{1}{\pi}\int_0^{20\pi} \frac{1}{1+\Omega^2}\, d\Omega = \frac{1}{\pi}\tan^{-1}(\Omega)\Big|_0^{20\pi} = \frac{1}{\pi}\tan^{-1}(20\pi) \approx 0.495\, Joules$$

The total signal energy can be calculated in either the time or frequency domain, respectively:

$$E_x = \int_0^{\infty} (e^{-t})^2\, dt = \frac{1}{2}\, Joule \qquad \text{or} \qquad E_x = \frac{1}{\pi}\int_0^{\infty} \frac{1}{1+\Omega^2}\, d\Omega = \frac{1}{2}\, Joule$$

Therefore, the percentage of energy in the frequency interval [0, 20π] relative to the total energy is approximately 98.99% (=0.495/0.5).

Comments:

- This example confirms energy conservation between the time and frequency domains. The total energy $E_x = 0.5\, Joule$ is obtained either by integrating the squared signal in time $(e^{-2t}dt)$ or by integrating the energy spectral density over the entire frequency spectrum $(1/\pi)\int_0^{\infty} S_x(\Omega)d\Omega$.
- Although the signal $e^{-t}u(t)$ theoretically has infinite spectral width, it behaves in practice as a band-limited signal, since 99% of its total energy lies between 0 and 20π rad/s.
- If we pass this signal through an ideal low-pass filter with cutoff frequency $\Omega_c = 20\pi\, rad/s$, the output signal loses only 1% of its energy. This approximation can be used to determine the required bandwidth of a communication channel or a recording system, such as one for biomedical signals, saving spectrum without appreciably altering the information.

📖 **Example 6**

Given the signal $x(t) = u(t)$, show that it is a power signal, then calculate its power, time-averaged autocorrelation function, and power spectral density.

Answer: Its energy is:

$$E_x = \int_{-\infty}^{\infty} |u(t)|^2 dt = \int_0^{\infty} 1\, dt = \infty$$

Its average power is:

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |u(t)|^2\, dt = \lim_{T\to\infty}\frac{1}{T}\int_0^{T/2} 1\, dt = \lim_{T\to\infty}\frac{1}{T}\cdot\frac{T}{2} = \frac{1}{2}$$

Thus, it is a power signal. We calculate its time-averaged autocorrelation (equation 4.125):

$$R_{xx}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} u(t)\, u(t+\tau)\, dt$$

For $x(t) = u(t)$, the product $u(t)u(t+\tau)$ depends on the sign of the lag τ:

- If $\boldsymbol{\tau \geq 0}$: The function $u(t)$ equals one for $t \geq 0$. The function $u(t+\tau)$ equals one for $t \geq -\tau$. Therefore, the product $u(t)u(t+\tau)$ is nonzero (and equals 1) only when both conditions hold, that is, for $t \geq 0$. The integral becomes:

$$R_{xx}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^{T/2} 1\cdot 1\, dt = \lim_{T\to\infty}\frac{1}{T}\left(\frac{T}{2} - 0\right) = \frac{1}{2}$$

- If $\boldsymbol{\tau < 0}$: The function $u(t+\tau)$ equals one for $t \geq -\tau$, which is now positive. The product of the two functions equals 1 only for $t \geq -\tau$. The integral becomes:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{0}^{T/2} 1 \cdot 1 \, dt = \lim_{T\to\infty} \frac{1}{T}\left(\frac{T}{2} - (-\tau)\right) = \lim_{T\to\infty} \frac{1}{T}\left(\frac{T}{2} + \tau\right)$$

  As $T \to \infty$, the fixed shift τ becomes negligible relative to the infinite size of T. Thus:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T}\left(\frac{T}{2} + \frac{\tau}{T}\right) = \frac{1}{2}$$

Therefore, for every finite value of τ, we have rigorously shown that:

$$R_{xx}(\tau) = \frac{1}{2}$$

Since the power spectral density is the Fourier transform of the time-averaged autocorrelation function, we have:

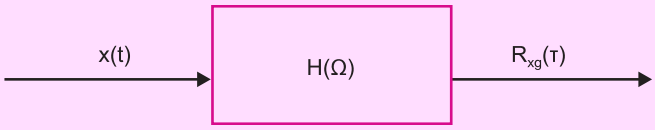
$$S_x(\Omega) = \mathcal{F}\{R_{xx}(\tau)\} = \mathcal{F}\left\{\frac{1}{2}\right\} = \pi\delta(\Omega)$$

Comments:

- The fact that $R_{xx}(\tau)$ equals $1/2$ for every time shift τ shows that $u(t)$ retains a permanent, long-term correlation with itself. This occurs because, as $t \to \infty$, the signal remains nonzero, occupying half of the entire time axis.
- According to the Wiener–Khintchine theorem, the power spectral density is the Fourier transform of the autocorrelation. Since the autocorrelation is a constant DC value of $1/2$, its transform is $\pi\delta(\Omega)$. Therefore, all the signal power ($P_x = 1/2\ W$) is concentrated in the DC component.
- Since the Fourier transform of $u(t)$ contains a $1/j\Omega$ term in addition to the impulse, why does this term not appear in the power spectral density? During time averaging over an infinite interval, $\lim_{T\to\infty}(1/T)$, the transient terms that carry energy but no power vanish. Therefore, the power spectral density retains only components with permanent power over an infinite interval.

Example 7

Design a cross-correlator for the signals $x(t)$ and $g(t) = e^{at}u(-t)$, as shown in the figure.



The signal $x(t)$ is applied to the system input, and the output is the cross-correlation of $x(t)$ and $g(t)$.

(a) What should the impulse response $h(t)$ of the system be?

(b) What is the frequency response $H(\Omega)$ of the system?

Answer: (a) Since the impulse response $h(t)$ is the system's response to an impulse input $\delta(t)$, producing the cross-correlation $R_{xg}(\tau)$ at the output requires $h(t) = g^*(-t)$. This follows from the reasoning below:

- The output of an LTI system with input $x(t)$ and impulse response $h(t)$ is given by convolution:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

- The required cross-correlation between $x(t)$ and $g(t)$, with respect to the time shift t, is defined as:

$$R_{xg}(t) = \int_{-\infty}^{\infty} x(\tau)g^*(t-\tau)\,d\tau$$

We want the output $y(t)$ to equal the cross-correlation $R_{xg}(t)$. Comparing the two expressions above requires:

$$h(t-\tau) = g^*(\tau - t)$$

Setting $t = 0$ in the relation above gives the necessary condition for the filter's impulse response:

$$h(\tau) = g^*(\tau) \Rightarrow h(t) = g^*(-t)$$

Since $g(t) = e^{at}u(-t)$ is real, the impulse response is obtained by simple time reversal:

$$h(t) = g(-t) = e^{-\alpha t}\,u(t), \qquad \alpha > 0$$

(b) The frequency response $H(\Omega)$ is calculated from:

$$H(\Omega) = \mathcal{F}\{h(t)\} = \frac{1}{a + j\Omega}, \qquad a > 0$$

Comments:

- Although the signal $g(t) = e^{\alpha t}u(-t)$ is noncausal, existing only at negative times and vanishing in the future, the reversal $g(-t)$ reverses time, making the impulse response $h(t) = e^{-\alpha t}u(t)$ causal. This means the resulting system can be physically implemented using real electronic components.
- The frequency response $H(\Omega) = 1/(\alpha + j\Omega)$ corresponds to the standard first-order low-pass filter. Therefore, the cross-correlator smooths the input signal, suppressing high frequencies, where noise usually dominates, and emphasizing low frequencies, where the energy of the pulse $g(t)$ lies.
- In modern digital communication systems, the cross-correlator, or matched filter, is the main receiver component. When the signal $g(t)$ travels through a channel and is distorted by noise, a filter with impulse response $h(t) = g(-t)$ ensures the maximum possible signal-to-noise ratio (SNR) at the sampling instant, allowing the receiver to decide correctly which symbol was sent.

📖 Example 8

Formulate mathematically the detection of a periodic signal buried in additive noise. Assume that the cross-correlation between the signal and noise is zero, and that the noise autocorrelation tends to zero as $|\tau| \to \infty$.

Answer: Let the signal under study be $x(t) = s(t) + n(t)$, where $s(t)$ is the periodic signal and $n(t)$ is the additive noise. The autocorrelation of $x(t)$ is:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t-\tau) dt$$

Substituting $x(t) = s(t) + n(t)$ and $x(t-\tau) = s(t-\tau) + n(t-\tau)$, we obtain:

$$\begin{aligned} R_{xx}(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [s(t) + n(t)]\, [s(t-\tau) + n(t-\tau)]\, dt \\ &= R_{ss}(\tau) + R_{nn}(\tau) + R_{ns}(\tau) + R_{sn}(\tau) \end{aligned}$$

Since $n(t)$ and $s(t)$ are uncorrelated, as $n(t)$ is noise, we have:

$$R_{ns}(\tau) = 0 \;\; and \;\; R_{sn}(\tau) = 0$$

Therefore:

$$R_{xx}(\tau) = R_{ss}(\tau) + R_{nn}(\tau) \tag{4.149}$$

The autocorrelation of a periodic signal is also periodic, with the same frequency. According to the problem statement, the noise autocorrelation tends to zero for large lags:

$$R_{nn}(\tau) \to 0, \qquad |\tau| \to \infty$$

Thus, for large lags:

$$R_{xx}(\tau) \approx R_{ss}(\tau)$$

Therefore, if the autocorrelation $R_{xx}(\tau)$ is periodic for large values of τ, we conclude that the signal $s(t)$ is present in the additive noise; otherwise, it is absent.

If $R_{xx}(\tau)$ exhibits such periodicity, we can distinguish the periodic and aperiodic components.

In practice, calculating the autocorrelation over an infinite interval is impossible. Instead, it is approximated over a large but finite interval T:

$$\hat{R}_{xx}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x(t-\tau)dt$$

This integration introduces an error, since the cross-correlation functions $R_{sn}(\tau)$ and $R_{ns}(\tau)$ are zero only when calculated over $(-\infty, +\infty)$. Therefore, for finite T, they are nonzero and contribute to the error in equation (4.149). To minimize this error, we should make the observation interval T as long as possible.

Autocorrelation-based detection is widely used in radar and sonar reception, detecting periodic components in brain waves, and identifying cyclic components in ocean-wave analysis.

Comments:

- Autocorrelation is useful for detecting periodic signals buried in noise, because periodicity is retained in $R_{ss}(\tau)$, while the noise term $R_{nn}(\tau)$ decreases for large lags.
- The assumption that the signal and noise are uncorrelated is crucial. It makes the signal–noise cross-correlations $R_{sn}(\tau)$ and $R_{ns}(\tau)$ zero in the ideal case. In practice, however, they do not become completely zero, so the final plot always contains small noise fluctuations (a noise floor).
- In practice, autocorrelation is calculated over a finite interval, so the terms $R_{ns}(\tau)$ *and* $R_{sn}(\tau)$ vanish only approximately. Increasing the observation duration improves detection reliability.
- Additive noise, including additive white Gaussian noise (AWGN), has a strong autocorrelation only at very small shifts ($\tau \approx 0$), since each noise sample is then compared with itself. At larger lags, the noise values are random and uncorrelated.

Note: Examples marked with an asterisk (*) also appear in the theory lecture slides.