UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Signals and Systems

Lecture 10: Inverse Fourier Transform, Fourier Transform of Power Signals

Michalis Paraskevas
Professor

Lecture Contents

1. Inverse Fourier transform
2. Fourier transform of power signals
3. Correlations (autocorrelation, cross-correlation) and spectral densities

1. Inverse Fourier Transform

Definition of the Inverse Fourier Transform

The inverse Fourier transform is defined as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\Omega)\, e^{j\Omega t} d\Omega$$

The properties of the forward Fourier transform also apply to the inverse Fourier transform.

2. Signal Windowing

Signal Windowing

Windowing multiplies a signal by a window function: $y(t) = x(t)\, w(t)$. It selects a segment from the entire, possibly infinite, signal, with duration equal to the window duration.

How does windowing affect the spectrum $Y(\Omega)$ of the truncated signal $y(t)$?

Let the window function be a rectangular pulse of duration $2T$:

$$w(t) = u(t+T) - u(t-T), \qquad T > 0$$

By the Fourier transform multiplication property, the truncated signal is:

$$y(t) = x(t)\, w(t) = w(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)\, e^{j\lambda t}\, d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)\, w(t)\, e^{j\lambda t}\, d\lambda$$

By frequency shifting, the Fourier transform $Y(\Omega)$ of $y(t)$ is:

$$Y(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)\, F\{w(t)\, e^{j\lambda t}\}\, d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)\, W(\Omega - \lambda)\, d\lambda$$

Signal Windowing

Thus, the Fourier transform $Y(\Omega)$ is:

$$Y(\Omega) = \frac{1}{2\pi}[X(\Omega) * W(\Omega)]$$

where the function $W(\Omega)$ is:

$$W(\Omega) = F\{w(t)\} = \frac{1}{s}[e^{Ts} - e^{-Ts}]_{s=j\Omega} = \frac{2\sin(\Omega T)}{\Omega}$$

- The signal $y(t)$ has finite duration in time, while convolution in frequency gives $Y(\Omega)$ and $W(\Omega)$ infinite bandwidth.
- The spectrum $Y(\Omega)$ of the truncated signal $y(t)$ differs from the spectrum $X(\Omega)$ of the original signal $x(t)$. It is the convolution $X(\Omega) * W(\Omega)$ of the spectra, divided by 2π.
- Thus, the spectrum of the truncated signal $y(t)$ depends on both the original signal spectrum $x(t)$ and the window spectrum $w(t)$.
- This spectral change is undesirable. Since windowing is useful in signal processing, we need window functions, such as Hamming, Hanning and Kaiser, that distort the truncated signal spectrum $Y(\Omega)$ as little as possible.

3. Fourier Transform of Power Signals

Fourier Transform of Power Signals (1/2)

- Some signals, including periodic, impulse and step functions, do not satisfy the Dirichlet conditions but still have Fourier transforms.
- Most have infinite energy, but their power $\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} |x(t)|^2 dt$ is often finite (they are power signals).
- These signals can have Fourier transforms if impulse functions are allowed in the transform.
- The Fourier transforms of periodic signals $e^{j\Omega t}, cos(\Omega_0 t), \ sin(\Omega_0 t)$ contain impulse functions.
- For a periodic signal $x(t)$, define the **truncated** signal $x_{_T}(t)$ from one period of $x(t)$, from the relation:

$$x_T(t) = \begin{cases} x(t), & -T_0/2 < t < T_0/2 \\ 0, & \text{otherwise} \end{cases}$$

Fourier Transform of Power Signals (2/2)

- The Fourier transform of a periodic signal $x(t)$ is an infinite sum. Each term is the Fourier transform of the truncated signal $x_T(t)$, multiplied by a frequency-shifted impulse $\delta(\Omega)$:

$$X(\Omega) = \Omega_0 \sum_{n=-\infty}^{\infty} X_T(n\Omega_0)\, \delta(\Omega - n\Omega_0)$$

  Here, $X_T(\Omega)$ denotes the Fourier transform of the truncated signal $x_T(t)$.

- Thus, the transform of a periodic signal is an infinite sum of impulses at multiples of its fundamental frequency (Ω_0). Each impulse is weighted by the product $\Omega_0\, X_T(k\Omega_0)$.

- These products equal the coefficients of the periodic signal's Fourier series expansion:

$$X_k = \frac{1}{T_0} X_T(k\Omega_0) = \frac{\Omega_0}{2\pi} X_T(k\Omega_0)$$

Observations

- The relation $X_k = \Omega_0 \, X_T(k\Omega_0)$ confirms the connection between Fourier series and the Fourier transform.
- It suggests a simpler method for calculating the exponential Fourier series coefficients of a periodic function $x(t)$,:
  - From the periodic signal $x(t)$, obtain the truncated signal $x_{_T}(t)$, over one period.
  - Calculate the Fourier transform of the truncated signal $x_{_T}(t)$, using the definition, transform properties or known transform pairs.
  - Evaluate the Fourier transform at frequencies $\Omega_0 = 2\pi k/T_0$ to find the k −th harmonic, then multiply by Ω_0.
  - The coefficients X_k are samples of the truncated signal's Fourier transform at the harmonic frequencies $k\Omega_0$, weighted by Ω_0.
- We can therefore find the Fourier series coefficients X_k simply by **sampling** the truncated signal's Fourier transform, avoiding difficult integrals.

4. Correlations
(Autocorrelation, Cross-Correlation)
and Spectral Densities

The Autocorrelation Function

- **Autocorrelation** measures the similarity between observations of a signal as a function of the time interval separating them.
- It is a mathematical tool for finding repeated patterns, such as a periodic signal hidden in strong noise.
- Its many applications include detecting repeated signal segments in signal processing, Markov chain Monte Carlo analysis and electromagnetism.

Autocorrelation of Energy Signals

For a **real** energy signal $x(t)$, the **autocorrelation** function $R_{xx}(\tau)$ is defined as:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt = \int_{-\infty}^{\infty} x(t+\tau)\, x(t)\, dt$$

The quantity τ is the **time lag.** Setting $\tau = 0$ in the definition gives the **energy** E_x of the signal $x(t)$:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt = R_{xx}(0)$$

Thus, the first coefficient (at zero lag, $R_{xx}(0)$) of the autocorrelation gives the signal energy.

For a complex energy signal $x(t)$, the autocorrelation $R_{xx}(\tau)$ is:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t-\tau)\, dt = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt$$

Fourier Transform of the Autocorrelation Function

The Fourier transform of the autocorrelation function is:

$$R_{xx}(\tau) \overset{F}{\longleftrightarrow} |X(\Omega)|^2$$

The Fourier transform of the autocorrelation $R_{xx}(\tau)$ equals the energy spectral density $|X(\Omega)|^2$ of the signal $x(t)$.

Using this result, or equivalently Parseval's theorem, gives:

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\Omega)|^2 \, d\Omega$$

This relation provides two equivalent methods of calculating signal energy: one uses the time representation $x(t)$, and the other the frequency representation $X(\Omega)$.

The quantity $S_x(\Omega) = |X(\Omega)|^2$ is the **energy spectral density**. It expresses signal energy per unit bandwidth (one radian per second) at each frequency.

Fourier Transform of the Autocorrelation Function

The energy spectral density $S_x(\Omega)$ gives the signal energy in the symmetric frequency band $[-\Omega_1, -\Omega_2]$:

$$E_x(\Omega_1, \Omega_2) = \frac{1}{2\pi}\left[\int_{-\Omega_2}^{-\Omega_1} S_x(\Omega)\, d\Omega + \int_{\Omega_1}^{\Omega_2} S_x(\Omega)\, d\Omega\right]$$

Since $S_x(\Omega) = |X(\Omega)|^2$ is even, we can restrict the calculation to positive frequencies ($\Omega > 0$):

$$E_x(\Omega_1, \Omega_2) = \frac{1}{\pi}\left[\int_{\Omega_1}^{\Omega_2} S_x(\Omega)\, d\Omega\right]$$

When an energy signal $x(t)$ enters an LTI system with impulse response $h(t)$, its output is the convolution $y(t) = x(t) * h(t)$. The output energy spectral density is:

$$S_y(\Omega) = S_x(\Omega)\, |H(\Omega)|^2$$

Autocorrelation of Power Signals

The **time-averaged autocorrelation** of a power signal $x(t)$, generally complex, over a period T_0 is:

$$R_{xx}(\tau) = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^*(t)\, x(t-\tau)\, dt$$

The power of a power signal is:

$$P_x = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, x^*(t)\, dt = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2\, dt = R_{xx}(0)$$

The **power spectral density** $S_x(\Omega)$, or **power spectrum**, of the signal $x(t)$ is the Fourier transform of its time-averaged autocorrelation:

$$R_{xx}(\tau) \overset{F}{\longleftrightarrow} S_x(\Omega)$$

The power of the signal $x(t)$ is:

$$P_x = R_{xx}(0) = \int_{-\infty}^{+\infty} S_x(\Omega) d\Omega$$

Autocorrelation of Periodic Signals

For a complex periodic signal $x(t)$, with period T_0 and fundamental frequency $\Omega_0 = 2\pi/T_0$, the autocorrelation is:

$$R_{xx}(\tau) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\, x^*(t-\tau)\, dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\, x^*(t+\tau)\, dt$$

Every periodic signal has an exponential Fourier series expansion:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{jk\Omega_0\tau}$$

Its **time-averaged autocorrelation** is:

$$R_{xx}(\tau) = \sum_{k=-\infty}^{\infty} |X_k|^2 e^{jk\Omega_0\tau}$$

where X_k are the Fourier series coefficients of the periodic signal $x(t)$.

The time-averaged autocorrelation of a periodic signal:

- is also periodic, with the same period as the original signal,
- can be expanded in a Fourier series whose coefficients are the squared magnitudes of the original signal's Fourier series coefficients.

Spectral Density of Power and Periodic Signals

The output of an LTI system is the convolution:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

The **time-averaged autocorrelation** of the output signal is:

$$R_{yy}(\tau) = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} y(t)\, y(t-\tau)\, dt$$

The following relation holds:

$$S_y(\Omega) = S_x(\Omega)\, |H(\Omega)|^2$$

The **power spectral density** of a periodic signal is the Fourier transform of its time-averaged autocorrelation:

$$S_{\mathrm{x}}(\Omega) \;=\; 2\pi \sum_{k=-\infty}^{\infty} |X_{\mathrm{k}}|^2\, \delta(\Omega - k\Omega_0)$$

Spectral Density of Power and Periodic Signals

The **power** P_x of a periodic signal $x(t)$ is:

$$P_x = \sum_{k=-\infty}^{+\infty} |X_k|^2$$

If a periodic signal $x(t)$ excites an LTI system with frequency response $H(\Omega)$, its output $y(t)$ is periodic, with **power spectral density**:

$$S_y(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} |X_\mathrm{k}|^2 \, |H(k\Omega_0)|^2 \, \delta(\Omega - k\Omega_0)$$

The **output signal power** is:

$$P_y = \sum_{k=-\infty}^{\infty} |X_k|^2 \, |H(k\Omega_0)|^2$$

Cross-Correlation of Finite-Energy Signals

For two real signals $x(t)$ and $y(t)$ of finite energy, the **cross-correlation** function $R_{xy}(\tau)$ is defined as:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y(t-\tau)\, dt = \int_{-\infty}^{\infty} x(t+\tau)\, y(t)\, dt$$

Similarly, $R_{yx}(\tau)$ is defined as:

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t)\, x(t-\tau)\, dt = \int_{-\infty}^{\infty} y(t+\tau)\, x(t)\, dt$$

Cross-Correlation of Infinite-Energy Signals

For infinite-energy signals, such as periodic signals or noise over $(-\infty, +\infty)$, we use the **average cross-correlation**, defined as:

$$\bar{R}_{xy}(\tau) = \lim_{T_0 \to \infty} \frac{1}{T} \int_{-T_0/2}^{T_0/2} x(t)\, y(t-\tau)\, dt$$

and

$$\bar{R}_{yx}(\tau) = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} y(t)\, x(t-\tau)\, dt$$

For periodic signals, these expressions become:

$$\bar{R}_{xy}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, y(t-\tau)\, dt$$

and

$$\bar{R}_{yx}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} y(t)\, x(t-\tau)\, dt$$

Exercise 1

Find the inverse Fourier transform of the function $X(\Omega) = 2\,cos(2\Omega + \pi/4)$.

Answer: Using Euler's formula, the function $X(\Omega)$ becomes:

$$X(\Omega) = 2cos(2\Omega + \pi/4) = e^{j\,[2\Omega+(\pi/4)]} + e^{-j\,[2\Omega+(\pi/4)]} = e^{2j\Omega}e^{j\pi/4} + e^{-2j\Omega}e^{-j\pi/4}$$

Substituting this into the definition of the inverse Fourier transform gives:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\Omega)\, e^{j\Omega t} d\Omega = \frac{e^{j\pi/4}}{2\pi}\int_{-\infty}^{+\infty} e^{j\Omega(t+2)}\, dt + \frac{e^{-j\pi/4}}{2\pi}\int_{-\infty}^{+\infty} e^{j\Omega(t-2)} dt$$
$$= e^{j\pi/4}\delta(t+2) + e^{-j\pi/4}\delta(t-2)$$

We used the fact that $(t) \overset{F}{\longleftrightarrow} 1$ and that the inverse Fourier transform of unity, $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\Omega t} d\Omega$, gives:

$$\delta(t - t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\Omega(t-t_0)} d\Omega$$

Exercise 1 (continued)

The same result follows from the time-shift property:

$$x(t-t_0) \overset{F}{\longleftrightarrow} e^{-j\Omega t_0}\, X(\Omega)$$

Setting $x(t) = \delta(t)$ and $X(\Omega) = 1$ gives:

$$\delta(t) \overset{F}{\longleftrightarrow} 1$$

$$\delta(t-t_0) \overset{F}{\longleftrightarrow} e^{-j\Omega t_0}\, 1 = \; e^{-j\Omega t_0}$$

$$\delta(t-2) \overset{F}{\longleftrightarrow} e^{-2j\Omega} \text{ and } \delta(t+2) \overset{F}{\longleftrightarrow} e^{2j\Omega}$$

Therefore:

$$X(\Omega) = e^{2j\Omega} e^{j\pi/4} + e^{-2j\Omega} e^{-j\pi/4} \Longrightarrow x(t) = \delta(t+2)\, e^{j\pi/4} + \delta(t+2) e^{-j\pi/4}$$

Exercise 2

Find the inverse Fourier transform of the function:

$$H(\Omega) = \frac{1}{(1 + 3j\Omega)(1 + j\Omega)}$$

Answer: Factor the denominator and split the expression into two partial fractions:

$$H(\Omega) = \frac{1}{(1 + 3j\Omega)(1 + j\Omega)} = \frac{A}{(1 + 3j\Omega)} + \frac{B}{(1 + j\Omega)} = \frac{A + jA\Omega + B + j3B\Omega}{(1 + 3j\Omega)(1 + j\Omega)}$$

The first and last fractions are equal. Since their denominators are identical, their numerators must also be equal. Thus:

$$A + jA\Omega + B + j3B\Omega = 1$$

Exercise 2 (continued)

This gives two equations in the two unknowns A and B:

$$A + B = 1 \quad \text{and} \quad A + 3B = 0$$

Solving the system gives $A = 3/2$ and $B = -1/2$. Therefore:

$$H(\Omega) = \frac{3}{2} \cdot \frac{1}{(1 + 3j\Omega)} - \frac{1}{2} \cdot \frac{1}{(1 + j\Omega)}$$

Observe that:

$$\frac{1}{1 + 3j\Omega} = \frac{1}{3} \cdot \frac{1}{\frac{1}{3} + j\Omega}$$

Thus:

$$H(\Omega) = \frac{3}{2} \cdot \frac{1}{3} \cdot \frac{1}{\frac{1}{3} + j\Omega} - \frac{1}{2} \cdot \frac{1}{1 + j\Omega} = \frac{1}{2} \cdot \frac{1}{\frac{1}{3} + j\Omega} - \frac{1}{2} \cdot \frac{1}{1 + j\Omega}$$

Exercise 2 (continued)

Using known transform pairs gives:

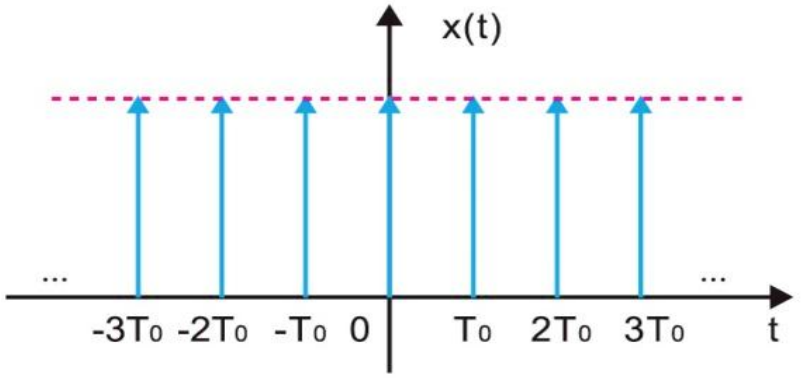
$$h(t) = \frac{3}{2}e^{-3t}u(t) - \frac{1}{2}e^{-t}u(t) = \frac{1}{2}[3e^{-3t} - e^{-t}]u(t)$$

Observations:

- The roots of the denominator of the spectrum $H(\Omega)$ are system poles at $j\Omega = -1/3$ and $j\Omega = -1$. Each frequency-domain pole produces an exponential term in time. Their negative real parts make $e^{-t/3}$ and e^{-t} decay, so $h(t)$ is BIBO stable and has finite energy.
- The term e^{-t} has time constant $\tau_1 = 1\ s$, while $e^{-t/3}$ has time constant $\tau_2 = 3\ s$. Thus, the second exponential decays three times more slowly. In a physical system, the pole nearest the imaginary axis dominates and determines the overall rate at which the signal decays.
- The unit step function $u(t)$ in the result makes the signal causal.

Exercise 3

Calculate the Fourier transform of the impulse train.



Answer: The signal is periodic and is expressed mathematically as:

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

Since $x(t)$ is periodic, its exponential Fourier series expansion is:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{jk\Omega_0 t}, \qquad \text{where } \Omega_0 = \frac{2\pi}{T_0}$$

The exponential Fourier series coefficients X_k are:

$$X_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \, e^{-jk\Omega_0 t} \, dt = \frac{1}{T_0} \int_{-\infty}^{\infty} \delta(t) \, e^{-jk\Omega_0 t} \, dt = \frac{1}{T_0}$$

Exercise 3 (continued)

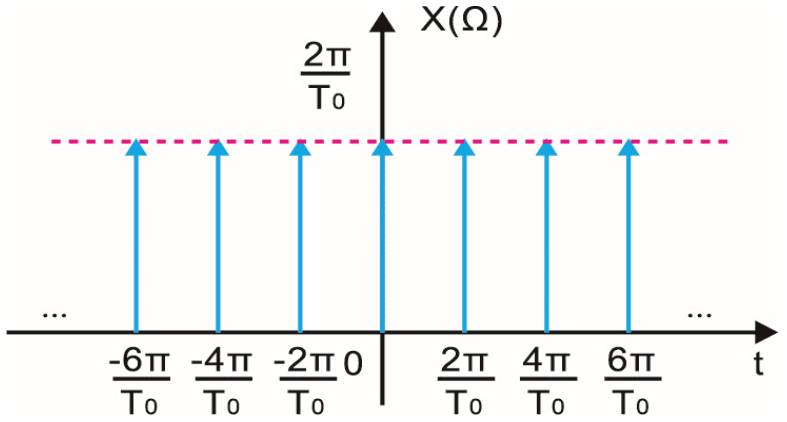
From the last two relations, $x(t)$ can be written as:

$$x(t) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{jk\Omega_0 t}$$

Taking the Fourier transform gives:

$$X(\Omega) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} F\{e^{jk\Omega_0 t}\} = \frac{2\pi}{T_0}\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0)$$

Thus, the Fourier transform of an impulse train is another impulse train.

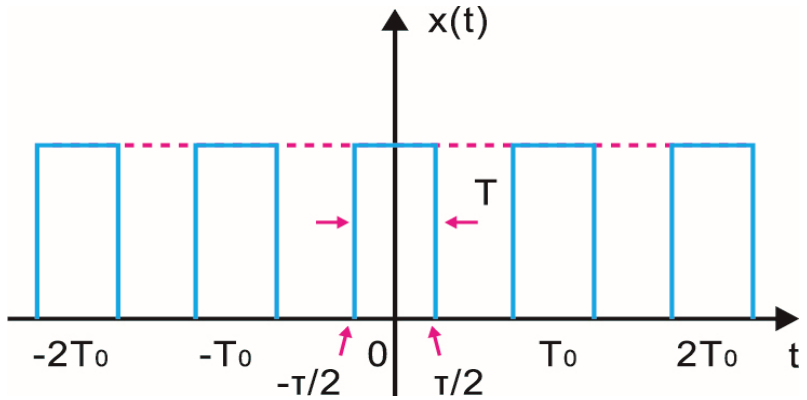


Exercise 3 (continued)

- The Fourier transform of a periodic impulse train in time, with period T_0, is a periodic impulse train in frequency, with period $\Omega_0 = 2\pi/T_0$. Thus, the spectrum is **discrete**.
- The periods T_0 and Ω_0 are inversely proportional. Closely spaced impulses in time produce widely spaced impulses in frequency, and vice versa.
- As the time-domain period tends to infinity $T_0 \to \infty$, only one impulse remains, $\delta(t)$. The frequency-domain impulse spacing tends to zero ($\Omega_0 \to 0$), so the spectrum becomes **continuous**.
- This transform pair underlies the Nyquist sampling theorem. Multiplying an analog signal by a time-domain impulse train (ideal sampling) corresponds, by the multiplication property, to convolution of its spectrum with a frequency-domain impulse train. This explains why **a sampled signal's spectrum repeats** every Ω_0.

Exercise 4

Calculate the FT of the periodic rectangular signal $x(t)$, with pulse duration τ and period T_0.



Answer: Follow the method used in the previous exercise.

The periodic rectangular signal has the Fourier series expansion:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{jk\Omega_0 t}$$

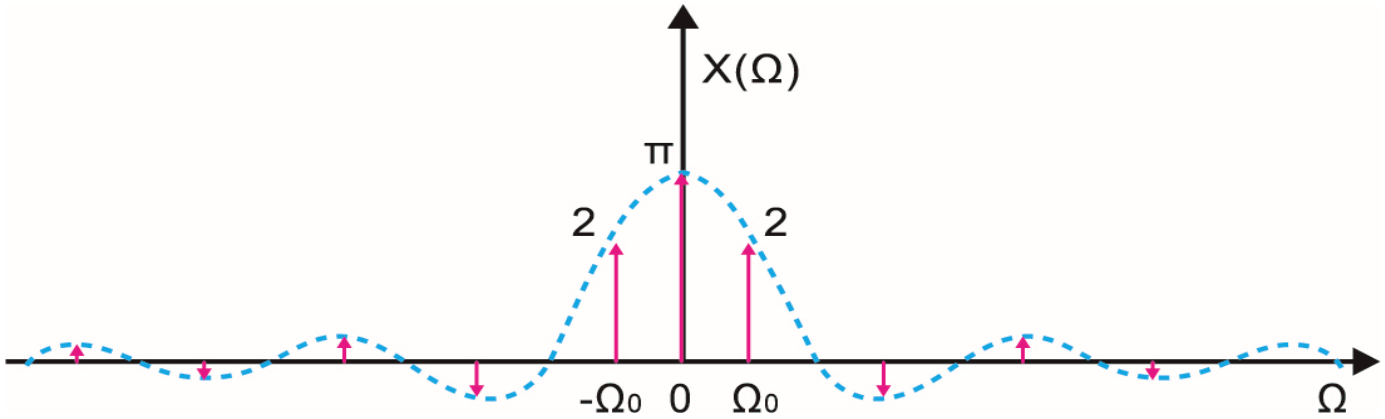
where:

$$X_k = \frac{\tau\Omega_0}{2\pi} \frac{sin\left(\frac{k\Omega_0 t}{2}\right)}{\left(\frac{k\Omega_0 t}{2}\right)} = \frac{\tau\Omega_0}{2\pi} sinc\left(\frac{k\Omega_0 t}{2}\right)$$

Exercise 4 (continued)

By linearity of the Fourier transform, and since $e^{j\Omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\delta(\Omega - \Omega_0)$,, the FT of the periodic signal is:

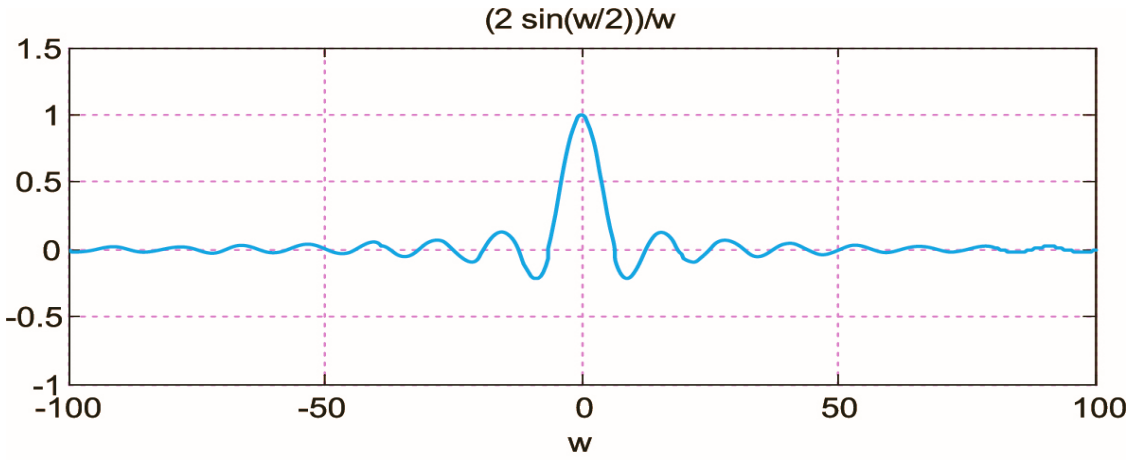
$$X(\Omega) = 2\pi \sum_{n=-\infty}^{+\infty} X_k \, \delta(\Omega - k\Omega_0)$$



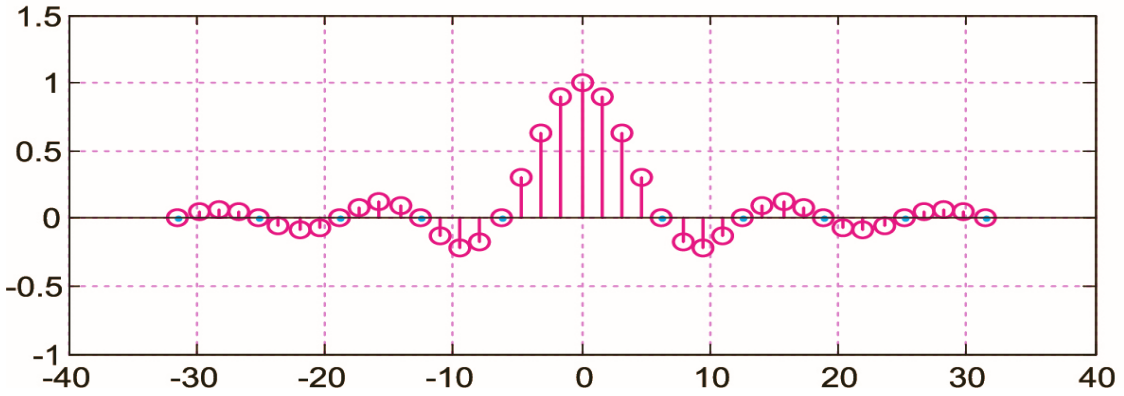
Spectrum of the periodic rectangular signal

Exercise 4 (continued)

(a) Fourier transform of the truncated function $x_T(t)$



(b) Fourier transform of the periodic signal $x(t)$



Exercise 4 (continued)

- The Fourier transform of a periodic rectangular pulse train is a train of impulses at harmonic frequencies $k\Omega_0$, rather than a continuous function.
- The impulse amplitudes in frequency are $2\pi X_k$, where X_k are the exponential Fourier series coefficients of the periodic signal.
- Equivalently, the line spectrum can be viewed as samples of the truncated signal's Fourier transform, with sampling interval $\Omega_0 = 2\pi/T_0$.
- The sinc envelope of the periodic signal's spectrum matches the spectrum of one truncated period, because each period contains a rectangular pulse of duration τ. Shorter pulses produce spectral lines that decay more slowly.
- The spectral line spacing is $\Omega_0 = 2\pi/T_0$. Thus, the period in time directly determines the density of lines in frequency.

Exercise 5

Calculate the autocorrelation of the signal $x(t) = A\cos(\Omega_0 t + \theta)$.

Answer: Since the signal is periodic, its autocorrelation is:

$$R_{xx}(\tau) = \frac{1}{T_0}\int_{T_0} x(t)\,x(t+\tau)\,dt = \frac{1}{T_0}\int_{T_0} A\cos(\Omega_0 t + \theta)\,A\cos(\Omega_0(t+\tau)+\theta)\,dt$$

$$= \frac{A^2}{T_0}\int_{T_0}\frac{1}{2}[\cos(\Omega_0 t + \theta + \Omega_0(t+\tau)+\theta) + \cos(\Omega_0 t + \theta - \Omega_0(t+\tau)-\theta)]\,dt$$

$$= \frac{A^2}{2T_0}\int_{T_0}[\cos(2\Omega_0 t + 2\theta + \Omega_0\tau) + \cos(\Omega_0\tau)]\,dt = \frac{A^2}{2T_0}\int_{T_0}\cos(2\Omega_0 t + \Omega_0\tau + 2\theta)\,dt + \frac{A^2}{2T_0}\cos(\Omega_0\tau)\int_{T_0} dt$$

$$= \frac{A^2}{2T_0}\int_{T_0}\cos(2\Omega_0 t + \Omega_0\tau + 2\theta)\,dt + \frac{A^2}{2}\cos(\Omega_0\tau)$$

The first integral is zero because it integrates a cosine over one period.

Therefore, the autocorrelation is:

$$R_{xx}(\tau) = \frac{A^2}{2}\cos(\Omega_0\tau)$$

Exercise 5 (continued)

- The initial phase θ is absent from the result. Autocorrelation measures similarity between a signal and itself as a function of the time shift τ. The relative shift between identical waves depends only on their frequency, rather than when they began.
- The autocorrelation of a periodic signal is also periodic, with the same frequency Ω_0. Here, the autocorrelation of a cosine is also a cosine at frequency Ω_0.
- The autocorrelation has:
  - maximum value $A^2/2$ at lag $\tau = kT_0,\ k \in Z$,
  - minimum value $-A^2/2$ at lag $\tau = (2k+1)T_0/2,\ k \in Z$.
- For $\tau = 0$, the autocorrelation gives the average signal power $R_{xx}(0) = A^2/2$. This is the familiar average power of a sinusoid of amplitude A, equal to the square of its RMS value, $V_{rms}^2 = \left(A/\sqrt{2}\right)^2 = A^2/2$.

Exercise 6

Calculate the autocorrelation function, energy spectral density and energy of the signal $x(t) = e^{-at}u(t)$, $a > 0$..

Answer: The Fourier transform is:

$$X(\Omega) = \frac{1}{\alpha + j\Omega}$$

The energy spectral density is:

$$S_x(\Omega) = |X(\Omega)|^2 = \frac{1}{\alpha^2 + \Omega^2}$$

The autocorrelation function $R_{xx}(\tau)$ is the inverse Fourier transform of the signal's energy spectral density $S_x(\Omega)$:

$$R_{xx}(\tau) = \mathcal{F}^{-1}\{S_x(\Omega)\} = \frac{1}{2a}e^{-a|t|}$$

The energy equals the autocorrelation at $\Omega = 0$:

$$E_x = R_{xx}(0) = \frac{1}{2a}$$

Exercise 7

Find the energy spectral density of the signal $x(t) = 10e^{-3|t|}$ using autocorrelation.

Answer: First calculate the autocorrelation $R_{xx}(\tau)$ of $x(\mathrm{t})$. By the Wiener–Khinchin theorem, the energy spectral density is $S(\Omega) = \mathcal{F}\{R_{xx}(\tau)\}$:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)dt = \int_{-\infty}^{\infty} 10e^{-3|t|}\, 10e^{-3|t-\tau|}dt$$

$$= \int_{-\infty}^{\infty} 10e^{3t}u(-t)\, 10e^{3(t-\tau)}u(\tau-t)dt + \int_{-\infty}^{\infty} 10e^{-3t}u(t)\, 10e^{3(t-\tau)}u(\tau-t)dt$$

$$+ \int_{-\infty}^{\infty} 10e^{-3t}u(t)\, 10e^{-3(t-\tau)}u(t-\tau)dt = 100e^{-3\tau}\left(\tau + \frac{1}{3}\right), \qquad \tau > 0$$

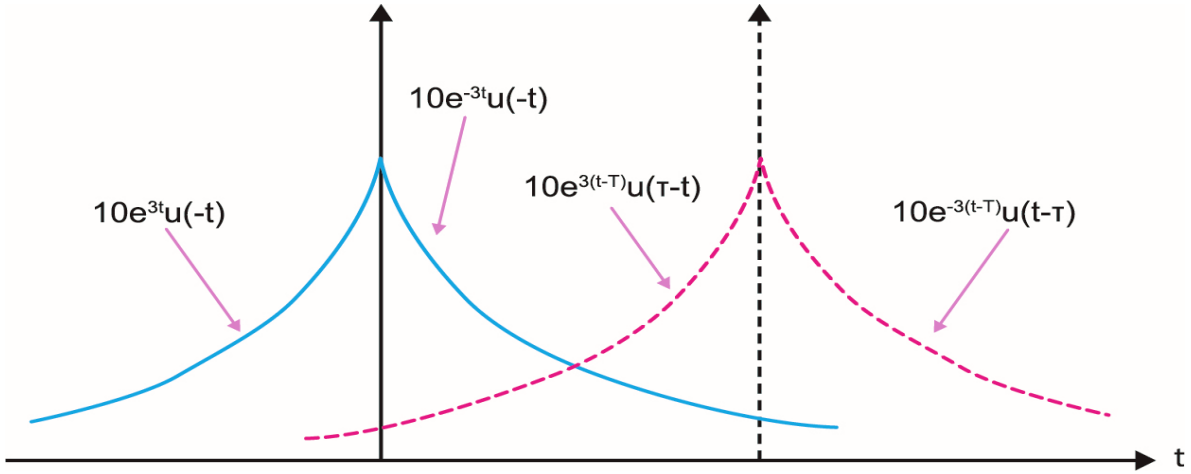
Exercise 7 (continued)

Similarly, calculate the autocorrelation for $\tau < 0$:

$$R_{xx}(\tau) = 100\, e^{3\tau}\left(-\tau + \frac{1}{3}\right), \tau < 0$$

Thus, for every t:

$$R_{xx}(\tau) = 100e^{-3|\tau|}\left(|\tau| + \frac{1}{3}\right), \qquad \forall \tau \in R \qquad (1)$$



Autocorrelation function of the signal $x(t) = 10e^{-3|t|}$

Exercise 7 (continued)

Calculate the energy spectral density from the autocorrelation. The Wiener–Khinchin theorem for energy signals gives:

$$S_x(\Omega) = F\{R_{xx}(\tau)\}$$

Thus:

$$Sx(\Omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\Omega\tau} d\tau$$

Since $R_{xx}(\tau)$ is even:

$$S_x(\Omega) = 2\int_{-\infty}^{\infty} R_{xx}(\tau)\cos(\Omega\tau)\, d\tau$$

For $\tau > 0$:

$$R_{xx}(\tau) = 100e^{-3\tau}\left(\tau + \frac{1}{3}\right)$$

Thus:

$$Sx(\Omega) = 2\int_{-\infty}^{\infty} 100e^{-3\tau}\left(\tau + \frac{1}{3}\right)\cos(\Omega\tau)\; d\tau = 200\left[\int_{0}^{\infty} \tau e^{-3\tau}\cos(\Omega\tau)\, d\tau + \frac{1}{3}\int_{0}^{\infty} e^{-3\tau}\cos(\Omega\tau)\, d\tau\right]$$

Exercise 7 (continued)

Use the known integrals:

$$\int_0^\infty e^{-a\tau}cos(\Omega\tau)d\tau = \frac{a}{a^2+\Omega^2}, \qquad a > 0$$

and:

$$\int_0^\infty \tau e^{-a\tau}cos(\Omega\tau)d\tau = \frac{a^2-\Omega^2}{(a^2+\Omega^2)^2}$$

For $a = 3$:

$$\int_0^\infty e^{-3\tau}cos(\Omega\tau)d\tau = \frac{3}{9+\Omega^2}$$

and:

$$\int_0^\infty \tau e^{-3\tau}cos(\Omega\tau)d\tau = \frac{9-\Omega^2}{(9+\Omega^2)^2}$$

Thus:

$$S_x(\Omega) = 200\left[\frac{9-\Omega^2}{(9+\Omega^2)^2} + \frac{1}{3}\cdot\frac{3}{9+\Omega^2}\right] = 200\left[\frac{9-\Omega^2}{(9+\Omega^2)^2} + \frac{1}{9+\Omega^2}\right] = 200\left[\frac{9-\Omega^2+9+\Omega^2}{(9+\Omega^2)^2}\right] = 200\frac{18}{(9+\Omega^2)^2}$$
$$= \frac{3600}{(9+\Omega^2)^2}$$

Exercise 7 (continued)

This agrees with direct calculation from the signal's Fourier transform. Indeed, since:

$$x(t) = 10e^{-3|t|}$$

we have:

$$X(\Omega) = \frac{60}{9 + \Omega^2}$$

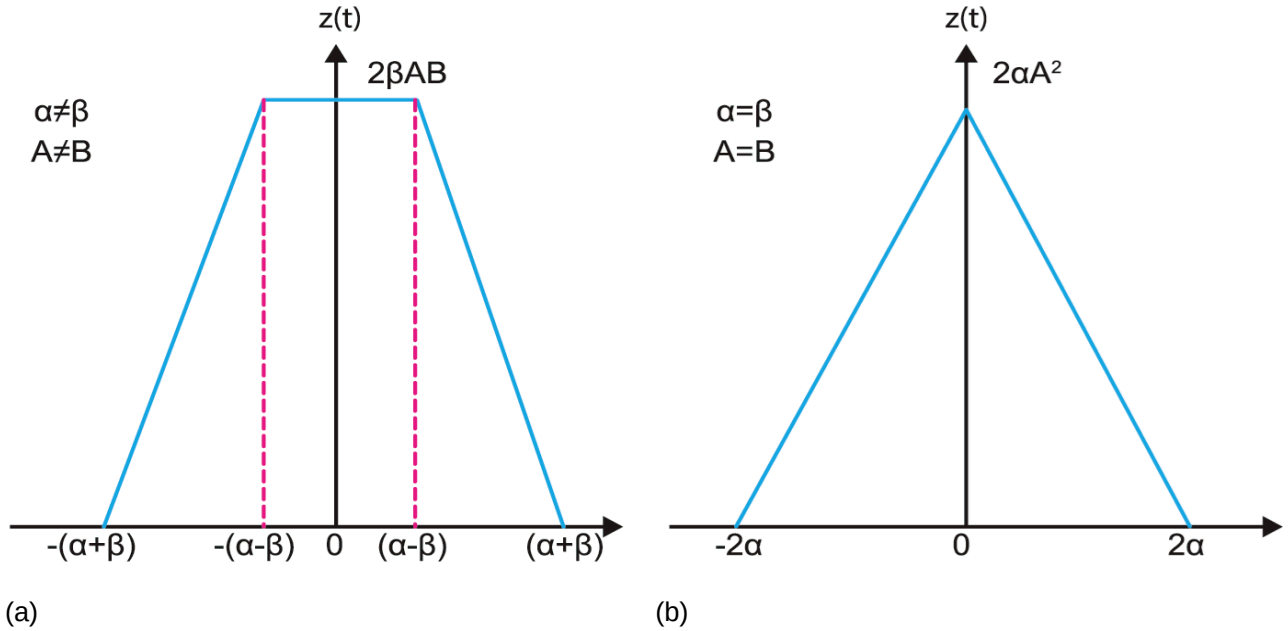
and therefore:

$$|X(\Omega)|^2 = \frac{3600}{(9 + \Omega^2)^2} = S_x(\Omega)$$

- The spectrum $X(\Omega) = 60/(9 + \Omega^2)$ is a simple Lorentzian curve, while the energy spectral density $S_x(\Omega) = 3600/(9 + \Omega^2)^2$ is a second-order Lorentzian. Signal energy therefore falls much faster at high frequencies (as $1/\Omega^4$) than the signal amplitude (which falls as $1/\Omega^2$).
- Agreement between the two methods, directly squaring $|X(\Omega)|^2$ and transforming $R_{xx}(\tau)$, confirms the Wiener–Khinchin theorem. In practice, for an unknown or stochastic signal, we cannot find $X(\Omega)$. Finding $R_{xx}(\tau)$ and taking its transform is then the **only** way to determine how energy is distributed over frequency.

Exercise 8

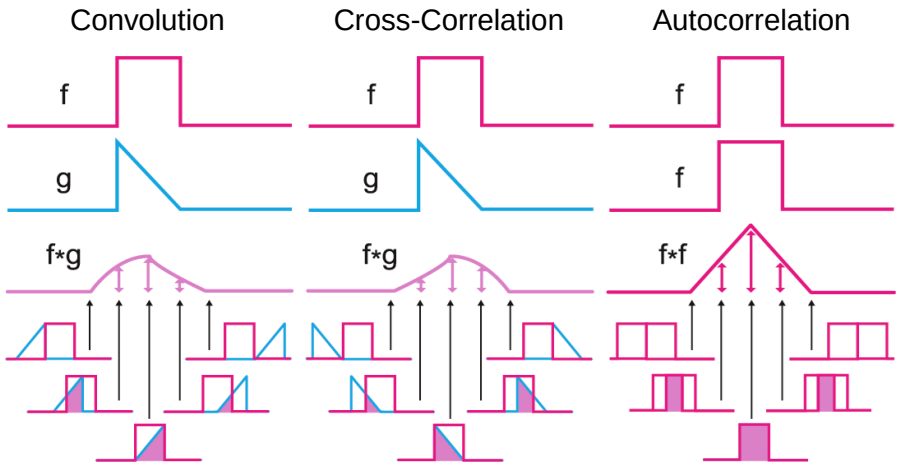
The cross-correlation of two signals $x(t)$ and $y(t)$ can be found **graphically**, using a method similar to convolution. The cross-correlation is defined only in the time regions where the two signals overlap.



Graphical cross-correlation of signals $x(t) = A\,rect(t/\alpha)$ and $y(t) = B\,rect(t/\beta)$ for (a) $\alpha \neq \beta, A \neq B$ and (b) $\alpha = \beta, A = B$.

Exercise 8 (continued)

For lag $\tau \geq 0$, the signals overlap over $(-\tau - \beta, \alpha)$. For lag $\tau < 0$, the overlap interval is $(-\alpha, -\tau - \beta)$. Thus, when calculating cross-correlation in these two cases, the integration limits follow those intervals.



<u>Observations</u>:

- Cross-correlation of two rectangular pulses measures their overlap length as one shifts relative to the other.
- For unequal durations, cross-correlation is trapezoidal. For equal durations, the trapezoid becomes a triangle.
- The maximum cross-correlation is $R_{xy,max} = AB\ min(\alpha, \beta)$, attained when the shorter pulse lies entirely within the longer pulse.

Exercise 9

Calculate the cross-correlation of the signals $x(t) = A\cos(\Omega_0 t)$ and $y(t) = B\sin(\Omega_0 t)$.

Answer: The signals $x(t)$ and $y(t)$ are real and periodic, and thus are power signals. We therefore seek the average power cross-correlation:

$$R_{xy}(t) = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x(\tau)\, y(\tau - t)d\tau = \frac{AB}{T_0}\int_{-T_0/2}^{+T_0/2} \cos(\Omega_0\tau)\sin[\Omega_0(\tau - t)]d\tau$$

Using the trigonometric identity:

$$\cos\alpha \sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

the integrand becomes:

$$\cos(\Omega_0\tau)\sin[\Omega_0(\tau - t)]d\tau = \frac{1}{2}\left[\sin\big(\Omega_0(2\tau - t)\big) - \sin(\Omega_0 t)\right]$$

Exercise 9 (continued)

The cross-correlation of the two signals is:

$$\bar{R}_{xy}(t) = \frac{AB}{2T_0}\left\{\int_{-T_0/2}^{+T_0/2} \sin(2\Omega_0 t + \Omega_0\tau)d\tau + \sin(\Omega_0 t)\int_{-T_0/2}^{+T_0/2} d\tau\right\}$$

The first integral is zero because it integrates a sine over one period.

Therefore:

$$\bar{R}_{xy}(t) = \frac{AB}{2T_0}\sin(\Omega_0 t)\int_{-T_0/2}^{+T_0/2} d\tau = \frac{AB}{T_0}\sin(\Omega_0\tau)$$

- The result is sinusoidal in the lag τ, since both signals have the same fundamental frequency Ω_0.
- The cross-correlation has:
  - maximum value $B/2$ when $\Omega_0\tau = \pi/2 + 2k\pi$, that is, at lag $\tau = (4k+1)T_0/4$,
  - minimum value $-AB/2$ when $\Omega_0\tau = -\pi/2 + 2k\pi$, that is, at lag $\tau = (4k-1)T_0/4$.
- For $\tau = 0$, the cross-correlation is zero: a sine and cosine at the same frequency are orthogonal over one period.