UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

SIGNALS AND SYSTEMS
Worked Examples – Tutorial Sheet 9

Instructor: Michalis Paraskevas, Professor

FOURIER TRANSFORM

- Calculating the forward and inverse Fourier transforms
- Relationship between Fourier and Laplace transforms
- Properties of the Fourier transform
- Parseval's theorem

1. Calculating the Forward and Inverse Fourier Transforms from the Definition

📖 Example 1 (*)
Calculate and plot the Fourier transform of the signal $x(t) = cos(\Omega_0 t)$.

Answer: Using Euler's identity, we express the cosine as a sum of complex exponential terms:

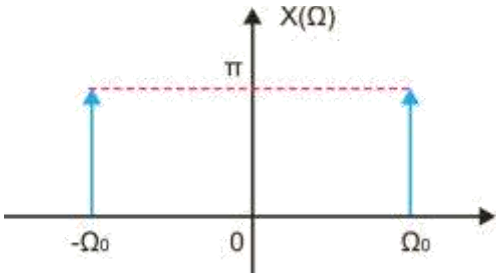
$$x(t) = \frac{1}{2}\, e^{j\Omega_0 t} + \frac{1}{2}\, e^{-j\Omega_0 t}$$

From Example 1, we know that:

$$e^{j\Omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\, \delta(\Omega - \Omega_0)$$

Therefore:

$$F\{x(t)\} = F\left\{\frac{1}{2}\, e^{j\Omega_0 t} + \frac{1}{2}\, e^{-j\Omega_0 t}\right\}$$
$$= \frac{1}{2} F\{e^{j\Omega_0 t}\} + \frac{1}{2}\, F\{e^{-j\Omega_0 t}\}$$
$$= \frac{1}{2} 2\pi\, \delta(\Omega - \Omega_0) + \frac{1}{2} 2\pi\, \delta(\Omega + \Omega_0)$$
$$= \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$$



Amplitude spectrum of $x(t) = cos(\Omega_0 t)$. The phase spectrum is zero.

📖 Example 2

Calculate the Fourier transform of the triangular pulse $\Lambda_T(t)$ with duration $2T$ and amplitude A:

$$x(t) \equiv \Lambda_T(t) \equiv A\, tri\left(\frac{t}{T}\right) = \begin{cases} A\left(1 - \dfrac{|t|}{T}\right), & |t| \le T \\ 0, & |t| > T \end{cases}$$

Answer: Since the signal is zero for $t < -T/2$ and $t > T/2$, its Fourier transform is:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-T}^{T} A\left(1 - \frac{t}{T}\right) e^{-j\Omega t} dt$$

$$= \int_{-T}^{0} \left(A + \frac{t}{T}\right) e^{-j\Omega t} dt + \int_{0}^{T} \left(A - \frac{t}{T}\right) e^{-j\Omega t} dt$$

$$= A\int_{-T}^{0} e^{-j\Omega t} dt + \frac{A}{T}\int_{-T}^{0} t e^{-j\Omega t} dt$$

$$+ A\int_{0}^{T} e^{-j\Omega t} dt - \frac{A}{T}\int_{0}^{T} t e^{-j\Omega t} dt$$

$$= \frac{A}{-j\Omega} e^{-j\Omega t}\Big|_{-T}^{0} + \frac{A}{-j\Omega T}\int_{-T}^{0} t\, d(e^{-j\Omega t})$$

$$+ \frac{A}{-j\Omega} e^{-j\Omega t}\Big|_{0}^{T} + \frac{A}{-j\Omega T}\int_{0}^{T} t\, d(e^{-j\Omega t})$$

$$= -\frac{A}{j\Omega}(1 - e^{-j\Omega T}) + \frac{-A}{j\Omega T}\left[t\, e^{-j\Omega t}\Big|_{-T}^{0} - \int_{-T}^{0} e^{-j\Omega t} dt\right]$$

$$+ \frac{-A}{j\Omega}(e^{-j\Omega T} - 1) + \frac{A}{j\Omega T}\left[t\, e^{-j\Omega t}\Big|_{0}^{T} - \int_{0}^{T} e^{-j\Omega t} dt\right]$$

$$= \frac{A}{j\Omega} e^{j\Omega T} - \frac{A}{j\Omega} - \frac{A}{j\Omega T}\left[0 - (-T)e^{j\Omega T} - \frac{A}{-j\Omega} e^{-j\Omega t}\Big|_{-T}^{0}\right]$$

$$- \frac{A}{j\Omega} e^{-j\Omega T} + \frac{A}{j\Omega} + \frac{A}{j\Omega T}\left[Te^{-j\Omega T} - 0 - \frac{A}{-j\Omega} e^{-j\Omega t}\Big|_{0}^{T}\right]$$

$$= \frac{A}{j\Omega}(e^{j\Omega T} - e^{-j\Omega T}) - \frac{A}{j\Omega T}\left[Te^{j\Omega T} + \frac{A}{j\Omega}(1 - e^{j\Omega T})\right]$$

$$+ \frac{A}{j\Omega T}\left[Te^{-j\Omega T} + \frac{A}{j\Omega}(e^{-j\Omega T} - 1)\right]$$

$$= \frac{A}{j\Omega}(e^{j\Omega T} - e^{-j\Omega T}) - \frac{A}{j\Omega}(e^{j\Omega T} - e^{-j\Omega T}) + \frac{A}{\Omega^2 T}(1 - e^{j\Omega T}) - \frac{A}{\Omega^2 T}(e^{-j\Omega T} - 1)$$

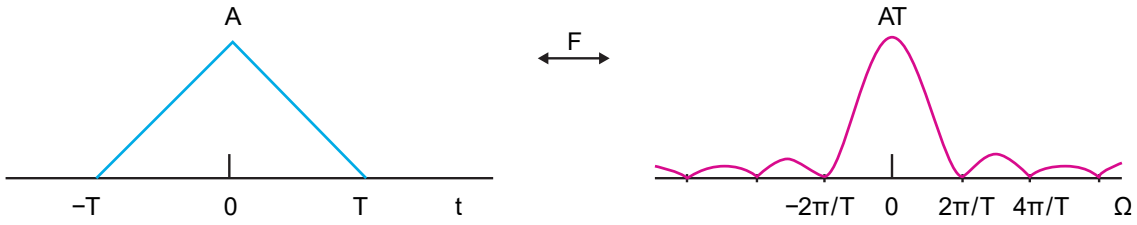
$$= \frac{A}{\Omega^2 T}(1 - e^{j\Omega T} - e^{-j\Omega T} + 1) = \frac{A}{\Omega^2 T}[2 - (e^{j\Omega T} + e^{-j\Omega T})] = \frac{A}{\Omega^2 T}[2 - 2\cos(\Omega T)]$$

$$= \frac{A}{\Omega^2 T}\left[2 - 2(1 - 2\sin^2\left(\frac{\Omega T}{2}\right)\right] = \frac{A}{\Omega^2 T}\left[2 - 2 + 4\sin^2\left(\frac{\Omega T}{2}\right)\right] = \frac{A}{\Omega^2 T}\left[4\sin^2\left(\frac{\Omega T}{2}\right)\right]$$

$$= AT\frac{\sin^2\left(\frac{\Omega T}{2}\right)}{\left(\frac{\Omega T}{2}\right)^2} = AT\, sinc^2\left(\frac{\Omega T}{2\pi}\right)$$

Therefore:

$$tri\left(\frac{\Omega}{B}\right) \overset{F}{\longleftrightarrow} AT sinc^2\left(\frac{\Omega T}{2\pi}\right)$$



(a) The triangular pulse of duration $2T$ and (b) its Fourier transform

Comments:

- The amplitude spectrum of the triangular pulse equals the square of the spectrum of the rectangular pulse. The spectrum is real, even, and nonnegative.
- The amplitude spectrum of the triangular pulse is always positive, and its main lobe has a sharper shape. The faster decay shows that the signal is smoother than the rectangular pulse, reducing the high-frequency content of its spectrum.
- The result can also be interpreted as the convolution of two rectangular pulses in time.
- The triangular window is used in smoothing filters to reduce high-frequency noise and as a window function, since it reduces spectral leakage more effectively than the rectangular window.

📖 Example 3

Calculate the Fourier transform of the rectangular pulse $\Pi_{T/2}(t)$ with duration T and amplitude A:

$$x(t) \equiv \Pi_{T/2}(t) \equiv A\, rect\left(\frac{t}{T}\right) = \begin{cases} A, & |t| \le T/2 \\ 0, & |t| > T/2 \end{cases}$$

Answer: Since the signal is zero for $t < -T/2$ and $t > T/2$, its Fourier transform is:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} dt = A \int_{-T/2}^{T/2} e^{-j\Omega t} dt$$

$$= \frac{A}{-j\Omega} e^{-j\Omega t} \Big|_{-T/2}^{T/2} = \frac{A}{-j\Omega} \left[e^{-j\Omega T/2} - e^{-j\Omega(-T/2)} \right]$$

$$= \frac{A}{j\Omega} \left[e^{j\Omega T/2} - e^{-j\Omega T/2} \right] = \frac{2A}{\Omega} sin(\Omega T/2)$$

$$= AT \frac{sin(\Omega T/2)}{\Omega T/2} = AT\ sinc\left(\frac{\Omega T}{2\pi}\right)$$

Therefore:

$$A\ rect\left(\frac{t}{T}\right) \overset{F}{\longleftrightarrow} AT\ sinc\left(\frac{\Omega T}{2\pi}\right)$$

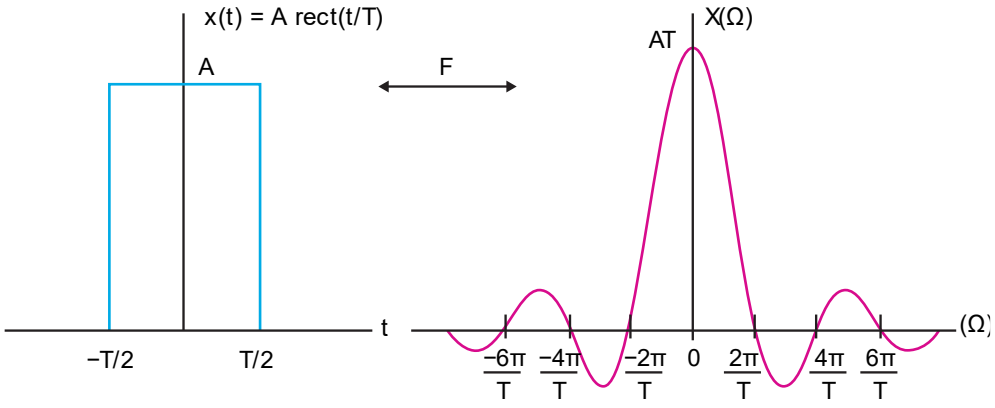
The solution uses Euler's identity: $sin(\varphi) = \left[e^{j\varphi} - e^{-j\varphi}\right]/2j$ with $\varphi = T/2$. Substituting $\Omega = 2\pi f$, we can express the result in terms of ordinary frequency f using:

$$X(f) = \frac{A}{\pi f} \sin(\pi f T)$$

A simpler form is:

- In terms of angular frequency Ω $\quad X(\Omega) = AT\ sinc(\Omega T/2\pi)$
- In terms of ordinary frequency f $\quad X(f) = AT\ sinc(fT)$

where $sinc(x)$is the normalized sinc function.1



(a) The rectangular pulse of duration T and (b) its Fourier transform.

The figure above shows the signal $x(t)$ and its Fourier transform. The function $X(\Omega)$ crosses zero

1 Recall that the sinc function is defined as $sinc(x) = sin(x)/x$, while the normalized sinc function is defined as $sinc(x) = sin(\pi x)/\pi x$.

periodically (spectral nulls), and the heights of its side lobes decrease asymptotically to zero.

Comments:

- Although $X(\Omega)$ is generally complex, it is real in the case of the rectangular pulse.
- The value of $X(\Omega)$ at zero frequency is given by the following limit, using l'Hôpital's rule:

$$X(0) = \lim_{\Omega \to 0} AT \frac{sin(\Omega T/2)}{\Omega T/2} = \lim_{\Omega \to 0} AT\, cos(\Omega T/2) = AT$$

  The maximum value of $X(\Omega)$ equals the total area of the rectangular pulse.
- The values at which $X(\Omega)$ is zero are the spectral nulls, given by:

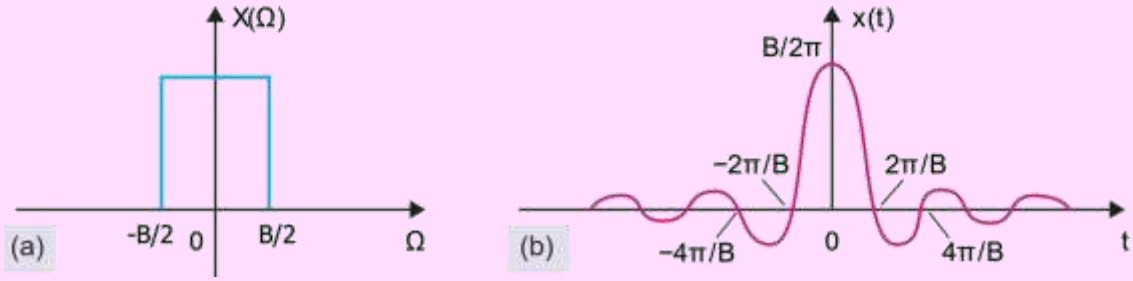
$$\Omega = \frac{2\pi n}{T}, \qquad n = \pm 1, \pm 2, \ldots.$$

- At high frequencies (i.e., $|\Omega| \to \infty$), the spectrum tends asymptotically to zero.
- The presence of side lobes shows that a time-limited signal is not band-limited.
- This analysis is used to determine signal bandwidth in telecommunications, since the pulse is a basic model for data transmission. It also applies to windowing, where the side lobes cause spectral leakage in the signal's spectrum.

Example 4

Find the signal whose Fourier transform is a rectangular frequency window of width B:

$$X(\Omega) = rect\left(\frac{\Omega}{B}\right) = \begin{cases} 1, & |\Omega| < B/2 \\ 0, & \text{elsewhere} \end{cases}$$



Representation of $x(t)$ in (a) the frequency domain and (b) the time domain

Answer: Since the Fourier transform is zero for frequencies $\Omega < -B/2$ and $\Omega > B/2$, the signal is calculated using the inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-B/2}^{B/2} e^{j\Omega t}\, d\Omega = \left[\frac{1}{2\pi jt} e^{j\Omega t}\right]_{-B/2}^{B/2}$$

$$= \frac{1}{2\pi jt}\left(e^{jBt/2} - e^{-jBt/2}\right) = \frac{1}{2\pi jt} 2j\, sin(Bt/2) =$$

$$\frac{1}{\pi t} sin(Bt/2) = \frac{B}{2\pi} sinc\left(\frac{Bt}{2\pi}\right)$$

Therefore:

$$rect\left(\frac{\Omega}{B}\right) \xleftrightarrow{F^{-1}} \frac{B}{2\pi} sinc\left(\frac{Bt}{2\pi}\right)$$

The figure shows $x(t)$ in the frequency and time domains, respectively.

Comments:

- As in the previous example, the solution is described by the normalized sinc function. Comparing the two examples reveals a form of symmetry: the Fourier transform of a rectangular pulse in time is a sinc function in frequency, and the inverse Fourier transform of a rectangular pulse in frequency is again a sinc function in time.
- We will make a similar observation about symmetry in the next example, when calculating the Fourier transform of the delta function. Using the duality property of the Fourier transform, we will confirm that this observation holds for all signals.
- The result also shows that information cannot be limited in both domains simultaneously.
- This property is fundamental to signal sampling, where reconstruction uses sinc interpolation. It is also used to design ideal filters, which cannot be implemented in practice because their impulse responses have infinite duration.

📖 Example 5

Calculate the inverse Fourier transforms of the functions:

(a) $X(\Omega) = \delta(\Omega)$ (b) $X(\Omega) = \delta(\Omega - \Omega_0)$

Answer: (a) From the definition of the inverse Fourier transform, we have:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\Omega)\, e^{j\Omega t}d\Omega = \frac{1}{2\pi}e^{j\Omega t}\Big|_{\Omega\,=\,0} = \frac{1}{2\pi}$$

Comparing the result with $\delta(t) \overset{F}{\longrightarrow} 1$ reveals a symmetric relationship between the time and frequency domains for this signal. This holds for all signals and is a fundamental property of the Fourier transform: symmetry, or duality.

(b) From the definition of the inverse Fourier transform, we have:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\, e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\Omega - \Omega_0)\, e^{j\Omega t}d\Omega = \frac{1}{2\pi}e^{j\Omega_0 t}$$

This is a complex vector of constant magnitude, whose phase rotates at a frequency of $\Omega_0 t$. Thus, we have the Fourier transform pair:

$$\frac{1}{2\pi}e^{j\Omega_0 t} \overset{F}{\longleftrightarrow} \delta(\Omega - \Omega_0) \qquad \text{or, equivalently} \qquad e^{j\Omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\,\delta(\Omega - \Omega_0)$$

2. Relationship between Fourier and Laplace Transforms

📖 Example 6
The periodic signal $x(t) = \sin(\Omega_0 t),\ -\infty < t < \infty$ is formed by adding a causal signal $y(t) = \sin(\Omega_0 t)\, u(t)$ and an anticausal signal $v(t) = \sin(\Omega_0 t)\, u(-t)$, with Laplace transforms $Y(s)$ and $V(s)$, respectively. Examine whether the Fourier transform of $x(t)$ can be calculated from the Laplace transforms $Y(s)$ and $V(s)$, by setting $s = j\Omega$.

Answer: The signals $y(t)$ and $v(t)$ satisfy $y(t) = -v(-t)$, and their Laplace transforms $Y(s)$ and $V(s)$ are:

$$V(s) = \frac{-\Omega_0}{(-s)^2 + \Omega_0^2}, \quad R_v = Re(s) < 0$$

$$Y(s) = \frac{\Omega_0}{s^2 + \Omega_0^2}, \qquad R_y = Re(s) > 0$$

The two transforms have the same poles $s = \pm j\Omega_0$, but different regions of convergence. For the signal $x(t)$, we have $x(t) = y(t) + v(t)$. Formally, the algebraic expressions would give:

$$X(s) = Y(s) + V(s) = \frac{\Omega_0}{s^2 + \Omega_0^2} + \frac{-\Omega_0}{(-s)^2 + \Omega_0^2} = 0$$

However, we cannot apply the linearity property of the Laplace transform, because the regions of convergence of $Y(s)$ and $V(s)$ do not intersect:

$$R_x = R_v \cap R_y = \emptyset$$

Therefore, the Laplace transform $X(s)$ is undefined, so we cannot calculate the required Fourier transform $X(\Omega)$.

Recall that, to calculate a Fourier transform from a Laplace transform, the axis $s = j\Omega$ must lie in the region of convergence. This is not the case here, since the region of convergence is empty. Therefore, even if we knew the Laplace transform $X(s)$, we could not calculate the Fourier transform $X(\Omega)$. For this periodic signal, the Fourier transform is calculated only from its Fourier series.

Comments:

- In the classical Fourier transform, we treat the signal as an ordinary function and require the defining integral to be meaningful in the classical sense. This works well when $x(t)$ is integrable, or at least square-integrable under suitable conditions.
- For singular signals such as the unit step function $u(t)$, the classical integral does not converge, because the signal does not decay as $t \to \infty$. In the generalized theory, these signals are treated as distributions (generalized functions), rather than ordinary functions that must be integrable. Their Fourier transform is defined in a different, generalized way.
- In this case, the Fourier transform is defined through its action on test functions. This allows us to give meaning to mathematical objects such as the functions $u(t), \delta(t)$, constants, and sinusoids of infinite duration. In particular, for $u(t)$, we obtain:

$$F\{u(t)\} = \pi\delta(\Omega) + PV\left(\frac{1}{j\Omega}\right)$$

where the term $\delta(\Omega)$ represents the zero-frequency component, and PV denotes the Cauchy principal value, a generalized way of giving meaning to $1/\Omega$.

3. Properties of the Fourier Transform

📖 Example 7 (*)
Solve Example 3 using the properties of the Fourier transform.

Answer: The unit triangular pulse can be written as a sum of unit ramp functions:

$$\Lambda_T(t) = \frac{1}{T}[r(t+T) - 2r(t) + r(t-T)]$$

We calculate the first and second derivatives of $\Lambda_T(t)$:

$$\Lambda'_T(t) = \frac{1}{T}[u(t+T) - 2u(t) + u(t-T)]$$

$$\Lambda''_T(t) = \frac{1}{T}[\delta(t+T) - 2\delta(t) + \delta(t-T)]$$

The Fourier transform of $\Lambda_T(t)$ can be calculated using the differentiation property:

$$F\{\Lambda_T(t)\} = F\left\{\frac{1}{(j\Omega)^2}\,\Lambda_T(t)\right\}$$

$$= F\left\{\frac{1}{(j\Omega)^2}\left[\frac{1}{T}[\delta(t+T) - 2\delta(t) + \delta(t-T)]\right]\right\}$$

$$= \frac{1}{\Omega^2 T}\left(2 - e^{j\Omega T} - e^{-j\Omega T}\right) = \cdots = 4\frac{sin^2(\Omega T/2)}{\Omega^2 T}$$

In compact form:

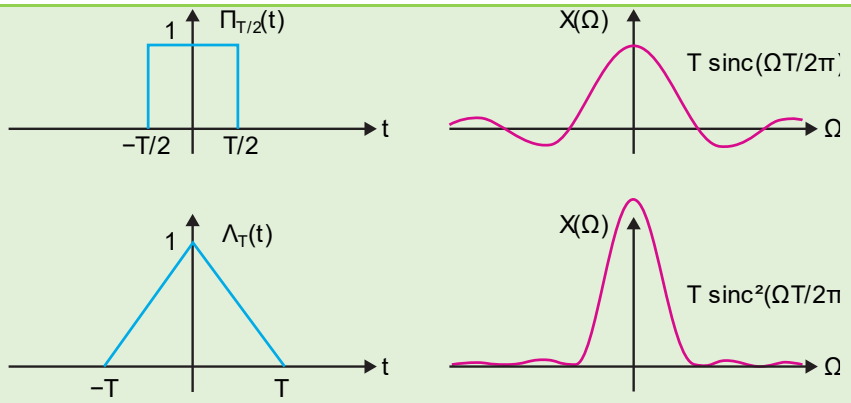
$$\Lambda_T(\Omega) = T\ sinc^2\left(\frac{\Omega T}{2\pi}\right)$$

Comments:

- The following relations hold, with frequency expressed as f:

$$rect(t) \overset{F}{\longleftrightarrow} sinc(f)$$
$$sinc(t) \overset{F}{\longleftrightarrow} rect(f)$$
$$tri(t) \overset{F}{\longleftrightarrow} sinc^2(f)$$
$$sinc^2(t) \overset{F}{\longleftrightarrow} tri(f)$$

- The figure shows the waveforms and Fourier transforms of rectangular and triangular pulses, which are common choices for signal window functions.



(a) Rectangular pulse and spectrum (b) Triangular pulse and spectrum

- For the triangular pulse, all lobes are positive. The main lobe is wider, while the side lobes have smaller amplitudes than those of the rectangular pulse.

📖 Example 8

Calculate the Fourier transform of $y(t) = \sin(\Omega_0 t)$ using the time-shifting property and the known Fourier transform of the cosine $x(t) = \cos(\Omega_0 t)$.

Answer: We know that the Fourier transform of $x(t) = \cos(\Omega_0 t)$ is:

$$X(\Omega) = \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$$

Sine and cosine have a phase difference of $\pi/2$; that is:

$$y(t) = \sin(\Omega_0 t) = \cos\left(\Omega_0 t - \frac{\pi}{2}\right) = \cos\left(\Omega_0\left[t - \frac{\pi}{2\Omega_0}\right]\right) = x\left(t - \frac{\pi}{2\Omega_0}\right)$$

From the time-shifting property of the Fourier transform:

$$y(t) = x\left(t - \frac{\pi}{2\Omega_0}\right) \overset{F}{\longleftrightarrow} e^{-j\Omega\pi/2\Omega_0}\, X(\Omega)$$

Thus:

$$\begin{aligned} Y(\Omega) &= \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]\, e^{-j\Omega\pi/2\Omega_0} \\ &= \pi\delta(\Omega - \Omega_0)e^{-j\pi/2} + \pi\delta(\Omega + \Omega_0)e^{j\pi/2} \\ &= -j\pi\,\delta(\Omega - \Omega_0) + j\pi\,\delta(\Omega + \Omega_0) \\ &= -j\pi\,[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)] \end{aligned}$$

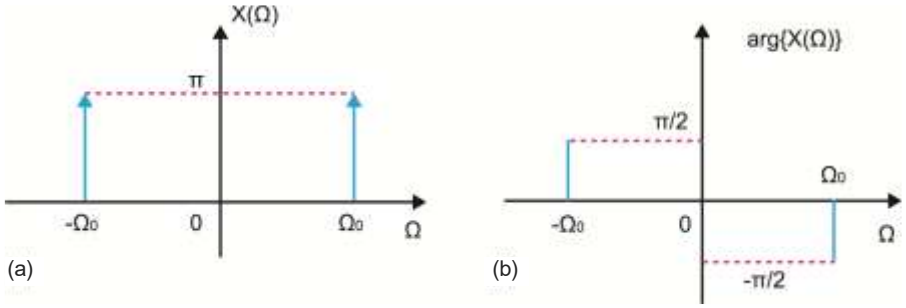
We also have:

$$Y(\Omega) = \frac{\pi}{j}\,[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)] = -\pi j\,[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$$

Therefore:

$$\sin(\Omega_0 t) \overset{F}{\longleftrightarrow} -\pi j\,[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$$

Comparing $Y(\Omega)$ and $X(\Omega)$shows that the Fourier transform of the sine $X(\Omega)$ differs from that of the cosine $Y(\Omega)$ only in phase, specifically in the phase of the term $\delta(\Omega - \Omega_0)$.



Spectral representation of the signal $x(t) = \sin(\Omega_0 t)$

📖 Example 9
Calculate the Fourier transform of the function $x(t) = 1$.

Answer: We know that $\delta(t) \overset{F}{\longleftrightarrow} 1$. Using the duality property, we find that the Fourier transform of the signal $x(t) = 1$ is:

$$X(\Omega) = 2\pi\delta(-\Omega) = 2\pi\delta(\Omega)$$

Therefore:

$$1 \overset{F}{\longleftrightarrow} 2\pi\delta(\Omega)$$

Comment: This pair illustrates the inverse relationship between time and frequency spread. An impulse $\delta(t)$ has zero duration in time but infinite spectral width. Conversely, a signal that is constant in time, such as $x(t) = 1$, has zero spectral width, with all its energy concentrated at the origin $\Omega = 0$.

📖 Example 10
Calculate the Fourier transform of the signal $x(t) = e^{j\alpha t}$.

Answer: From the defining equation of the Fourier transform, we obtain:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} dt = \int_{-\infty}^{\infty} e^{j\alpha t}\, e^{-j\Omega t} dt = \int_{-\infty}^{\infty} e^{j(\alpha-\Omega)t}\, dt = 2\pi\delta(\Omega - \alpha)$$

We obtain the same result using the frequency-shifting property $\mathcal{F}\{e^{j\Omega_0 t}x(t)\} = X(\Omega - \Omega_0)$ and the relation $\mathcal{F}\{1\} = 2\pi\delta(\Omega)$. Defining the function $y(t) = 1 \cdot x(t) = 1 \cdot e^{j\alpha t}$, we obtain:

$$Y(\Omega) = \mathcal{F}\{e^{j\alpha t} \cdot 1\} = \mathcal{F}\{1\}_{\Omega \to \Omega - \alpha} = 2\pi\delta(\Omega - \alpha)$$

Therefore:

$$e^{j\alpha t} \overset{F}{\longleftrightarrow} 2\pi\delta(\Omega - a)$$

which gives the same result as direct calculation. Following the same procedure, we can prove that:

$$e^{-j\alpha t} \overset{F}{\longleftrightarrow} 2\pi\delta(\Omega + a)$$

Comments:

- This pair illustrates the inverse relationship between time and frequency spread. An impulse $\delta(t)$ has zero duration in time but infinite spectral width. Conversely, a signal that is constant in time, such as $x(t) = 1$, has zero spectral width, with all its energy concentrated at the origin $\Omega = 0$.
- The duality property shows that the form of a function in the frequency domain can be used directly to find the transform of another function in the time domain.

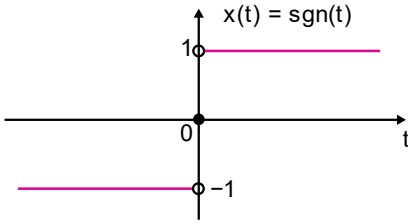
📖 Example 11

Calculate the Fourier transform of the signum function:

$$x(t) = sgn(t) \equiv \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$$

Answer: The signum function is plotted in the figure.

Signum function



The signum function can also be written as $sgn(t) = 2u(t) - 1$. Applying the Fourier transform to both sides, we use linearity and the fact that:

$$U(\Omega) = \frac{1}{j\Omega} + \pi\delta(\Omega)$$

to obtain:

$$\mathcal{F}\{\text{sgn}(t)\} = \frac{2}{j\Omega} + 2\pi\delta(\Omega) - 2\pi\delta(\Omega) = \frac{2}{j\Omega}$$

Comments:

- Unlike the step function $u(t)$, which has a mean value of $1/2$, the signum function $sgn(t)$ is balanced about the time axis. Since its mean value is zero, the final spectrum contains no impulses $\delta(\Omega)$.
- Since $sgn(t)$ is real and odd, its Fourier transform must be imaginary and odd. Indeed, $X(\Omega) = -j2/\Omega$ satisfies this property.

4. Parseval's Theorem

📖 Example 12

Calculate the energy of the causal exponential signal $x(t) = e^{-\alpha t}u(t),\ \alpha > 0$.

Answer: We know that the Fourier transform of the causal exponential signal $x(t) = e^{-\alpha t}u(t),\ \alpha > 0$ is:

$$X(\Omega) = \frac{1}{\alpha + j\Omega}$$

We calculate the signal energy in both the time and frequency domains. The time-domain energy is given by:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)^2 dt = \int_{0}^{\infty} e^{-2\alpha t} dt$$
$$= \frac{1}{2a}(e^{-2\alpha t})\Big|_0^{\infty} = -\frac{1}{2a}\left[\lim_{t\to\infty} e^{-2\alpha t} - e^0\right] = \frac{1}{2a}$$

The frequency-domain energy is given by:

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\Omega)|^2 d\Omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\frac{1}{\alpha + j\Omega}\right|^2 d\Omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\alpha^2 + \Omega^2} d\Omega = \frac{1}{\pi}\int_{0}^{+\infty} \frac{1}{\alpha^2 + \Omega^2} d\Omega$$

We know that:

$$\int_{\lambda_1}^{\lambda_2} \frac{1}{\alpha^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_{\lambda_1}^{\lambda_2}$$

Therefore, the frequency-domain energy is:

$$E_x = \frac{1}{a\pi}\tan^{-1}\left(\frac{\Omega}{a}\right)\Big|_0^{\infty} = \frac{1}{\alpha\pi}\left[\lim_{\Omega\to\infty} \tan^{-1}\left(\frac{\Omega}{a}\right) - \tan^{-1}(0)\right] = \frac{1}{\alpha\pi}\cdot\frac{\pi}{2} = \frac{1}{2\alpha}$$

Note: Examples marked with an asterisk (*) also appear in the theory lecture slides.