

UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Signals and Systems

Lecture 9: Fourier Transform

Michalis Paraskevas
Professor

Lecture Contents

1. Definition of the Fourier Transform
2. Physical Meaning of the Fourier Transform
3. Conditions for the Existence of the Fourier Transform
4. Relationship between Fourier and Laplace Transforms
5. Properties of the Fourier Transform
6. Parseval's Theorem
7. Summary of Fourier Transform Properties
8. Useful Fourier Transform Pairs

1. Definition of the Fourier Transform

Introduction to the Fourier Transform

- In practice, signals cannot be truly periodic because the requirement of infinite duration is not met.
- The Fourier transform extends Fourier series. It applies to both **periodic** and **nonperiodic signals**.
- Any signal, periodic or nonperiodic, can be expanded over the interval $(-\infty, +\infty)$ through the Fourier transform as a **linear combination of infinitely many harmonic exponential signals.**
- As in Fourier series, signals are expressed using complex exponentials at different frequencies. In the Fourier transform, however, frequencies are **continuous**, whereas in Fourier series they are discrete.

Definition of the Fourier Transform (1/2)

Forward Fourier Transform (FT)

- The **Fourier transform** of a function $x(t)$ is the complex function of a real variable $X(\Omega)$, defined by:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} dt$$

provided that the integral converges / exists, rather than diverging.

Inverse Fourier Transform

- This recovers the time function $x(t)$ when its Fourier transform $X(\Omega)$ **is known:**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)\, e^{j\Omega t} d\Omega$$

Definition of the Fourier Transform (2/2)

Using angular frequency Ω, or instead **ordinary frequency f** (where $f = \Omega/2\pi$), the forward and inverse Fourier transforms are defined as follows:

- Forward Fourier transform:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

- Inverse Fourier transform:

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

Fourier Transform Notation

- To indicate that the function $X(\Omega)$ is the Fourier transform of the signal $x(t)$, we use:

$$X(\Omega) = \mathcal{F}\{x(t)\}$$

- To indicate that the function $x(t)$ is the inverse Fourier transform of the function $X(\Omega)$, we use:

$$x(t) = \mathcal{F}^{-1}\{X(\Omega)\}$$

- The following shorthand notation is also used:

Forward Fourier transform	$x(t) \xrightarrow{F} X(\Omega)$
Inverse Fourier transform	$X(\Omega) \xrightarrow{F^{-1}} x(t)$

- or simply $x(t) \xleftrightarrow{F} X(\Omega)$

2. Physical Meaning of the Fourier Transform

Physical Meaning of the Fourier Transform (1/3)

The Fourier transform is generally complex and can therefore be written as:

$$\mathrm{X}(\Omega) = R(\Omega) + jI(\Omega)$$

where $R(\Omega)$ is the real part and $I(\Omega)$ is the imaginary part of $X(\Omega)$.

If the signal $x(t)$ is real, then $R(\Omega)$ is **even** and $I(\Omega)$ is **odd.** They are given by:

$$R(\Omega) = \int_{-\infty}^{\infty} x(t)\cos(\Omega t)\, d\Omega$$

$$I(\Omega) = -\int_{-\infty}^{\infty} x(t)\sin(\Omega t)\, d\Omega$$

Physical Meaning of the Fourier Transform (2/3)

As a complex function, the Fourier transform X(Ω) can be written as:

$$X(\Omega) = |X(\Omega)|\, e^{j\varphi_X(\Omega)}$$

- $|X(\Omega)|$: **amplitude or magnitude spectrum**
- $\varphi_X(\Omega)$: **phase spectrum**

The Fourier transform spectrum is **continuous**, unlike the discrete spectrum of Fourier series.

The magnitude $|X(\Omega)|$ and phase $\varphi_X(\Omega)$ are calculated as:

$$|X(\Omega)| = \sqrt{R^2(\Omega) + I^2(\Omega)}$$

$$\varphi_X(\Omega) = \tan^{-1}\left[\frac{I(\Omega)}{R(\Omega)}\right]$$

Physical Meaning of the Fourier Transform (3/3)

- For a real signal $x(t)$, the Fourier transform expansion uses the components $e^{j\Omega t}$..
- From Euler's formula $e^{j\Omega t} = \frac{1}{2}[\cos \Omega t + j \sin \Omega t]$, the term $e^{j\Omega t}$ separates into two mutually orthogonal components (vectors), $\cos \Omega t$ and $\sin \Omega t$.
- The signal $x(t)$ can be written as:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\cos\big(\Omega t + \varphi_X(\Omega)\big)\, d\Omega = \frac{1}{\pi}\int_{0}^{\infty} X(\Omega)\cos\big(\Omega t + \varphi_X(\Omega)\big)\, d\Omega$$

- Thus, the Fourier transform of a real signal is equivalent to an expansion into infinitely many sinusoidal signals. Each frequency contributes with amplitude $\frac{1}{\pi}X(\Omega)\, d\Omega$ and phase $\varphi_X(\Omega)$..
- In general, the functions $|X(\Omega)|$ and $\varphi_X(\Omega)$ fully describe the effect of a continuous-time linear time-invariant (LTI) system on its input signal.

The Fourier Transform Spectrum

- The Fourier transform decomposes a periodic or nonperiodic signal $x(t)$, over the time interval $(-\infty, +\infty)$, into a **continuous spectrum of periodic exponential signals**. This reveals and analyzes its **spectral content**.
- The spectral content in the infinitesimal frequency interval $[\Omega, \Omega + d\Omega]$ is $X(\Omega)$, and the energy contribution in that interval is:

$$dE = \frac{1}{2\pi} X(\Omega)\, d\Omega$$

- The Fourier transform is an **amplitude spectral density** rather than an amplitude spectrum. Its magnitude describes the frequency content of a signal and the relative sizes of the complex exponentials that compose the continuous signal $x(t)$.
- The **phase** describes the **relative phases** of the complex exponentials, rather than their magnitudes.
- Phase **strongly affects** the time-domain waveform. Signals with identical magnitudes but different phases can be **very different**. Calculating phase requires the correct quadrant, determined by the signs of $R(\Omega)$ and $I(\Omega)$.

3. Conditions for the Existence of the Fourier Transform

Conditions for the Existence of the Fourier Transform

The integrals defining the forward and inverse Fourier transforms (FT) do not always exist.

The Dirichlet conditions are sufficient conditions for the existence of a signal's Fourier transform.

- **Sufficient Condition 1**. The function $x(t)$ must be absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)|\, dt < +\infty$$

  Thus, the total area under the curve $|x(t)|$ must be finite. This is **sufficient but not necessary**. For example, a sinusoid is not absolutely integrable, but has a generalized Fourier transform.

- **Sufficient Condition 2**. The function $x(t)$ must be continuous or have finitely many discontinuities of finite size in each finite time interval.

- **Sufficient Condition 3**. The function $x(t)$ must have bounded variation. Equivalently, it must have finitely many maxima and minima in each finite time interval.

4. Relationship between Fourier and Laplace Transforms

Relationship between Fourier and Laplace Transforms

The definitions are almost identical except for the complex exponent: $e^{-j\Omega}$ in the Fourier transform and $e^{-(\sigma+j\Omega)}$ in the Laplace transform. Thus, Fourier lacks the damping factor present in Laplace.

This difference matters: the **damping factor** (σ) determines the **region of convergence** and ensures convergence of the Laplace transform. Fourier can therefore be viewed as a special case of Laplace for $\sigma = 0$:

$$X(\Omega) = X(s)\Big|_{s=j\Omega}$$

The transform $\sigma = 0$ is calculated only on the imaginary axis $j\Omega$. This relationship applies only when the Laplace region of convergence **includes** that axis $j\Omega$.

It applies to causal, anticausal and noncausal signals.

Relationship between Fourier and Laplace Transforms

The following rules apply:

- For a bounded, finite-duration signal $x(t)$, the Fourier transform converges.
- For an infinite-duration signal $x(t)$, if its Laplace region of convergence includes the imaginary axis $j\Omega$, substitute s into $j\Omega$, giving $X(s)|_{s=j\Omega}$.
- For a periodic signal $x(t)$, calculate the Fourier transform using Fourier series.
- In all other cases, $x(t)$, signals with discontinuities such as $x(t) = u(t)$, discontinuous signals of infinite energy $x(t) = u(t)\cos(\Omega_0 t)$, and signals with spectral discontinuities even at finite energy $x(t) = sinc(t)$, use Fourier series and their properties.
- For LTI systems, Laplace describes both transient and steady-state behavior, whereas Fourier describes only the steady state.

5. Properties of the Fourier Transform

Properties of the Fourier Transform

1. Linearity
2. Time Shift
3. Frequency Shift
4. Time Scaling
5. Frequency Scaling
6. Reflection
7. Conjugation
8. Symmetry (Duality)
9. Frequency Differentiation
10. Time Differentiation
11. Time Integration
12. Convolution
13. Time Multiplication
14. Even/Odd Signal Parts – Real/Imaginary Spectrum Parts
15. Spectral Symmetries
16. Areas under Curves

Properties of the Fourier Transform (1/12)

1. Linearity

If $x_i(t) \overset{F}{\longleftrightarrow} X_i(\Omega)$, for any real or complex constant c_i, and $i = 1,2, \dots n$, then:

$$c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) \overset{F}{\longleftrightarrow} c_1 X_1(\Omega) + c_2 X_2(\Omega) + \cdots + c_n X_n(\Omega)$$

The Fourier transform of a linear combination of signals is the same linear combination of their Fourier transforms.

2. Time Shift

If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$, for any real t_0:

$$x(t \pm t_0) \overset{F}{\longleftrightarrow} e^{\pm j\Omega t_0}\, X(\Omega) = |X(\Omega)|\, e^{j\varphi(\Omega) \pm \Omega t_0}$$

A time shift of $\pm t_0$ multiplies its Fourier transform by $e^{\pm j\Omega t_0}$. Its magnitude remains the same as $|e^{\pm j\Omega t_0}| = 1$, while its phase changes linearly.

Properties of the Fourier Transform (2/12)

3. Frequency Shift

If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$, for any real Ω_0:

$$x(t)\, e^{j\Omega_0 t} \overset{F}{\longleftrightarrow} X(\Omega - \Omega_0)$$

Multiplying the signal $x(t)$ by $e^{j\Omega_0 t}$ shifts its spectrum to Ω_0 without changing its shape.

This enables **modulation**: shifting a signal from baseband to higher frequencies for transmission over communication channels. It is also called the **modulation property**.

Properties of the Fourier Transform (3/12)

4. **Time Scaling**

If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$, for any real α $(\alpha \neq 0)$:

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\Omega}{a}\right)$$

5. Frequency Scaling

If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$, for any real α $(\alpha \neq 0)$:

$$\frac{1}{|a|} x\left(\frac{t}{a}\right) \overset{F}{\longleftrightarrow} X(a\Omega)$$

Scaling a signal in time changes the extent of its Fourier transform inversely.

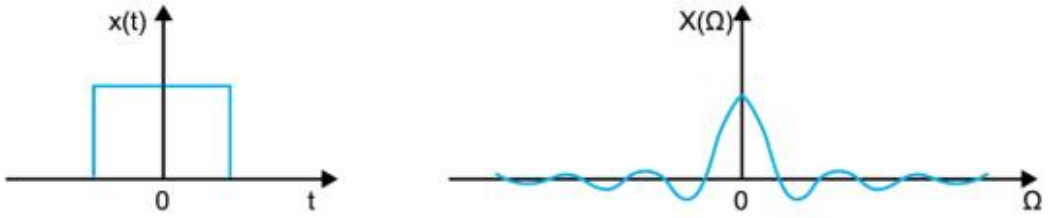
Faster changes over shorter times correspond to higher frequencies and a wider spectrum.

Thus, a spectrum can be narrowed or widened by expanding or compressing the signal in time.

Illustration of Time and Frequency Scaling

$a = 1/2$

$$X(\Omega) = \frac{2sin(\Omega/2)}{\Omega}$$



(a) Signal x(t) and its spectrum $X_1(\Omega)$

$a = 1$

$$X(\Omega) = \frac{2sin(\Omega)}{\Omega}$$



(b) Signal $x_1(t) = x(at)$, $a > 1$, and its spectrum $X_1(\Omega)$

$a = 2$

$$X(\Omega) = \frac{2sin(2\Omega)}{\Omega}$$



(c) Signal $x_2(t) = x(at)$, $a < 1$, and its spectrum $X_2(\Omega)$

Observations on Frequency Scaling

- When $\alpha > 1$ [figure (c)], the signal expands in time and changes more slowly per unit time. Since low-frequency signals change slowly, its spectrum is compressed.
- When $0 < \alpha < 1$ [figure (a)], the signal is compressed in time and changes faster. Rapid time-domain changes correspond to higher frequencies, so its spectrum expands.
- Transmitting very short pulses, as in digital communications, therefore requires channels with a wide frequency bandwidth.
- Key conclusion: the Fourier transform cannot simultaneously provide a narrow extent, and thus high resolution, in both time and frequency. High resolution in one domain is traded for lower resolution in the other.
- Other transforms, such as the wavelet transform, provide adaptive time-frequency resolution and localization, while respecting the fundamental trade-off between the two domains.

Properties of the Fourier Transform (4/12)

6. Reflection

Setting $\alpha = -1$ in the time-scaling property gives the reflection property:

$$x(-t) \overset{F}{\longleftrightarrow} X(-\Omega)$$

7. Conjugation

If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$, then:

$$x^*(t) \overset{F}{\longleftrightarrow} X^*(-\Omega) \quad \text{and} \quad x^*(-t) \overset{F}{\longleftrightarrow} X^*(\Omega)$$

Properties of the Fourier Transform (5/12)

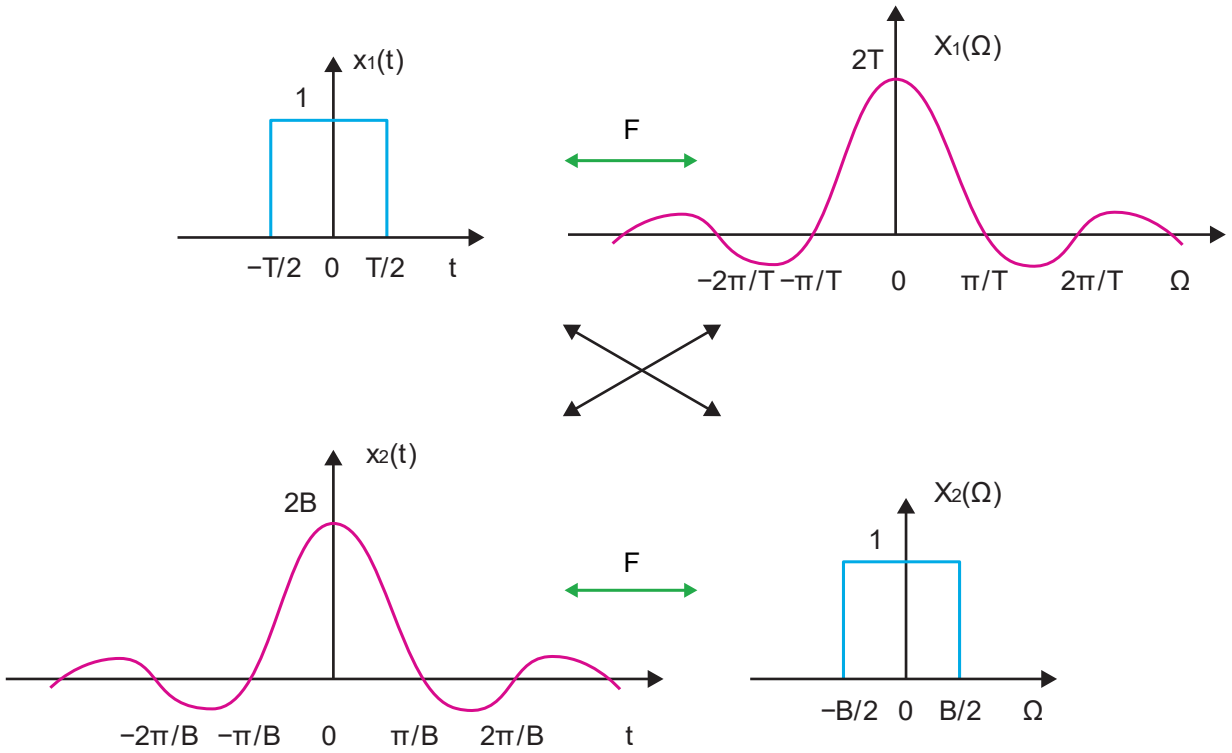
8. Symmetry (Duality)

If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$, , the signal $y(t) = X(t)$ has Fourier transform $Y(\Omega) = 2\pi x(-\Omega)$.

- Here, $X(t)$ denotes a function of the independent variable t, with mathematical expression $X(\Omega)$.
- Similarly, $2\pi x(-\Omega)$ denotes a function of the independent variable Ω, with expression $x(t)$.
- Duality is useful for finding a transform when direct evaluation is difficult but the reverse transform pair is known.
- It provides another method, alongside the definition and the relationship with the Laplace transform.
- In this way, frequency-domain characteristics become characteristics of $X(\Omega)$ in the time domain.

Graphical Illustration of Duality

The Fourier transform of a rectangular pulse $x(t) = \Pi_{T/2}\,(t) = rect(t/T)$ is a sampling (sinc) function $X(\Omega) = Tsinc(\Omega T/2\pi)$.



The Fourier transform of a sampling (sinc) function $x(t) = (B/2\pi)\,sinc(Bt/2\pi)$ is a rectangular frequency pulse $X(\Omega) = \Pi_{B/2}(t) = rect(\Omega/B)$.

Properties of the Fourier Transform (6/12)

9. Frequency Differentiation

If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$, then:

$$t\, x(t) \overset{F}{\longleftrightarrow} jX'(\Omega)$$

10. Time Differentiation

If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$, and the Fourier transform of the derivative $d^n x(t)/dt^n$ exists, then:

$$\frac{dx(t)}{dt} \overset{F}{\longleftrightarrow} j\Omega X(\Omega)$$

or, more generally:

$$\frac{d^n x(t)}{dt^n} \overset{F}{\longleftrightarrow} (j\Omega)^n X(\Omega)$$

and, in the frequency domain:

$$(-jt)^k x(t) \overset{F}{\longleftrightarrow} \frac{d^k X(\Omega)}{d\Omega^k}$$

Properties of the Fourier Transform (7/12)

11. Time Integration

If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$, then:

$$\int_{-\infty}^{t} x(\tau)\, d\tau \overset{F}{\longleftrightarrow} \frac{1}{j\Omega}\, \mathrm{X}(\Omega) + \pi X(0)\delta(\Omega)$$

12. Convolution

The Fourier transform of the convolution of two LTI signals is the product of their Fourier transforms. Specifically, if $x_1(t) \overset{F}{\longleftrightarrow} X_1(\Omega)$ and $x_1(t) \overset{F}{\longleftrightarrow} X_1(\Omega)$, then:

$$x(t) = x_1(t) * x_2(t) \overset{F}{\longleftrightarrow} X_1(\Omega)\, X_2(\Omega)$$

Properties of the Fourier Transform (8/12)

For an LTI system with impulse response $h(t)$ and input $x(t)$, the output $y(t)$ is given by convolution:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$ and $h(t) \overset{F}{\longleftrightarrow} H(\Omega)$, the convolution property gives:

$$y(t) = h(t) * x(t) \overset{F}{\longleftrightarrow} Y(\Omega) = H(\Omega)\,X(\Omega)$$

Thus, the output spectrum $Y(\Omega)$ can be calculated from the input spectrum $X(\Omega)$ and the Fourier transform $H(\Omega)$ of the system's impulse response $h(t)$.

The difficult task of evaluating a convolution becomes multiplication of two transforms.

This useful property underlies the design of LTI systems in the frequency domain (analog filters).

Properties of the Fourier Transform (9/12)

13. Time Multiplication

A corresponding reverse property applies to the convolution of the Fourier transforms $X(\Omega)$ and $Y(\Omega)$ of the signals $x(t)$ and $y(t)$.

Specifically, if $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$ and $y(t) \overset{F}{\longleftrightarrow} Y(\Omega)$, and $Z(\Omega) = X(\Omega) * Y(\Omega)$, then:

$$x(t)\, y(t) \overset{F}{\longleftrightarrow} \frac{1}{2\pi}[X(\Omega) * Y(\Omega)]$$

Properties of the Fourier Transform (10/12)

14. Even/Odd Signal Parts – Real/Imaginary Spectrum Parts

Any signal $x(t)$ can be expressed as the sum of its even part $x_e(t)$ and odd part $x_o(t)$. If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$, then:

$$x_e(t) \overset{F}{\longleftrightarrow} Re\{\, X(\Omega) \,\}$$

$$x_o(t) \overset{F}{\longleftrightarrow} j\, Im\{\, X(\Omega) \,\}$$

The function $Re\{\,\}$ returns the real part of $X(\Omega)$, while $Im\{\,\}$ returns the imaginary part.

Observations:

- If the signal is even, its Fourier transform is real. Thus, the real part of the transform comes from the even part of the signal.
- If the signal is odd, its Fourier transform is imaginary. Thus, the imaginary part of the transform comes from the odd part of the signal.

Properties of the Fourier Transform (11/12)

15. Spectral Symmetries

For a real signal $x(t)$, periodic or nonperiodic, its magnitude $|X(\Omega)|$ and **real part** $Re\{X(\Omega)\}$ are **even** functions of frequency Ω, while its **phase** $\measuredangle X(\Omega)$ and **imaginary part** $Im\{X(\Omega)\}$ are **odd**:

$$|X(\Omega)| = |X(-\Omega)| \quad \text{and} \quad Re\{X(\Omega)\} = Re\{X(-\Omega)\}$$

$$\measuredangle X(\Omega) = -\measuredangle X(-\Omega) \quad \text{and} \quad Im\{X(\Omega)\} = -Im\{X(-\Omega)\}$$

- For a real signal, the amplitude spectrum and the real part of its Fourier transform are even.
- For a real signal, the phase spectrum and the imaginary part of its Fourier transform are odd.
- These symmetries do not hold for complex signals.
- Negative frequencies do not exist in practice, but the concept is needed to construct real signals.

Properties of the Fourier Transform (12/12)

16. Areas under Curves

If $x(t) \overset{F}{\longleftrightarrow} X(\Omega)$, the definitions of the forward and inverse transforms give:

(a) The area under the signal $x(t)$ equals the value of its Fourier transform $X(\Omega)$ at frequency $\Omega = 0$:

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

(b) The area under the Fourier transform $X(\Omega)$ equals the value of the signal $x(t)$ at $t = 0$, multiplied by $1/2\pi$:

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)\, d\Omega$$

6. Parseval's Theorem

Parseval's Theorem (1/2)

- The theorem expresses **conservation of energy in the frequency domain**:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega$$

- The total energy of a signal can be calculated equally in the time domain or the frequency domain.
- In the time domain, energy per unit time is $|x(t)|^2$; integrating over the entire duration gives the total energy.
- In the frequency domain, energy per unit angular frequency is $|X(\Omega)|^2/2\pi$; integrating over all frequencies gives the total energy.

Parseval's Theorem (2/2)

- The function $S_x(\Omega) = |X(\Omega)|^2$ is the **energy spectral density** of the signal $x(t)$. It represents **energy per unit bandwidth** (one radian per second) at each frequency.
- To find the energy contributed by frequencies from f_1 to f_2, integrate $|X(\Omega)|^2$ between those frequencies:

$$E_{[\Omega_1,\Omega_2]} = \frac{1}{2\pi}\int_{\Omega_1}^{\Omega_2} |X(\Omega)|^2 d\Omega$$

7. Summary of Fourier Transform Properties

Summary of Fourier Transform Properties (1/3)

Property	Time Domain	Frequency Domain
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(\Omega) + a_2 X_2(\Omega)$
Time Shift	$x(t - t_0)$	$e^{-j\Omega t_0} X(\Omega)$
Frequency Shift	$e^{j\Omega t_0} x(t)$	$X(\Omega - \Omega_0)$
Symmetry: If $x(t) \overset{F}{\leftrightarrow} X(\Omega)$	$y(t) = X(t)$	$Y(\Omega) = 2\pi x(-\Omega)$
Multiplication (Modulation)	$x_1(t)\, x_2(t)$	$\frac{1}{2\pi}[X_1(\Omega) * X_2(\Omega)]$
Convolution	$x_1(t) * x_2(t)$	$X_1(\Omega)\, X_2(\Omega)$

Summary of Fourier Transform Properties (2/3)

Property	Time Domain	Frequency Domain
Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{\Omega}{a}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Time Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\Omega} X(\Omega) + \pi X(\Omega)\delta(\Omega)$
Time Differentiation	$\frac{dx(t)}{dt}$	$j\Omega\, X(\Omega)$
Frequency Differentiation	$tx(t)$	$j\frac{dX(\Omega)}{d\Omega}$
Even Signal	$x(t) = x(-t)$	$X(\Omega) \in R$
Odd Signal	$x(t) = -x(-t)$	$X(\Omega) \in I$

Summary of Fourier Transform Properties (3/3)

Property	Time Domain	Frequency Domain
Even Signal Part Real Spectrum Part	$x_e(t) = \frac{1}{2}[x(t) + x^*(-t)]$	$Re\{X(\Omega)\}$
Odd Signal Part Imaginary Spectrum Part	$x_o(t) = \frac{1}{2}[x(t) - x^*(-t)]$	$j\ Im\{X(\Omega)\}$
Time Conjugation	$x^*(t)$	$X^*(-\Omega)$
Frequency Conjugation	$x^*(-t)$	$X^*(\Omega)$
Conjugate Symmetry	$x^*(t)$	$X^*(-\Omega)$
Parseval's Theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$	

8. Useful Fourier Transform Pairs

Fourier Transforms of Basic Functions (1/3)

Time Domain	Frequency Domain
$\delta(t)$	1
1	$2\pi\ \delta(\Omega)$
A	$2\pi A\ \delta(\Omega)$
$u(t)$	$\frac{1}{j\Omega} + \pi\delta(\Omega)$
$\lvert t\rvert$	$-\frac{2}{\Omega^2}$
$sgn(t)$	$\frac{2}{j\Omega}$
$A\ \delta(t \pm t_0)$	$A\ e^{\pm j\Omega t_0}$
$e^{\pm j\Omega_0 t}$	$2\pi\delta(\Omega \mp \Omega_0)$
$\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0)$

Fourier Transforms of Basic Functions (2/3)

Time Domain	Frequency Domain
$cos(\Omega_0 t)$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$
$sin(\Omega_0 t)$	$\frac{\pi}{j}\,[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)]$
$cos(\Omega_0 t)\,u(t)$	$\frac{\pi}{2}[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]+j\frac{\Omega}{\Omega_0^2-\Omega^2}$
$sin(\Omega_0 t)\,u(t)$	$\frac{\pi}{2j}\,[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)]+\frac{\Omega}{\Omega_0^2-\Omega^2}$
$e^{-at}cos(\Omega_0 t)u(t), \qquad Re(a)>0$	$\frac{\alpha+j\Omega}{(\alpha+j\Omega)^2+\Omega_0^2}$
$e^{-at}sin(\Omega_0 t)u(t), \qquad Re(a)>0$	$\frac{\Omega_0}{(\alpha+j\Omega)^2+\Omega_0^2}$
$\Pi_{T/2}(t)\equiv rect\left(\frac{t}{T}\right)=\begin{cases}1, & \lvert t\rvert<T/2\\ 0, & \lvert t\rvert>T/2\end{cases}$	$T\,sinc\left(\frac{\Omega T}{2\pi}\right)$

Fourier Transforms of Basic Functions (3/3)

Time Domain	Frequency Domain
$\frac{B}{2\pi} sinc\left(\frac{Bt}{2\pi}\right)$	$\Pi_{B/2}(t) \equiv rect\left(\frac{\Omega}{B}\right)$
$\Lambda_T(t) = tri\left(\frac{t}{T}\right)$	$Tsinc^2\left(\frac{\Omega T}{2\pi}\right)$
$\frac{B}{2\pi} sinc^2\left(\frac{Bt}{2\pi}\right)$	$\Lambda_B(\Omega) = tri\left(\frac{\Omega}{B}\right)$
$e^{-at}u(t), \quad Re(a) > 0$	$\frac{1}{\alpha + j\Omega}$
$e^{-at}u(-t), \quad Re(a) > 0$	$\frac{1}{\alpha - j\Omega}$
$te^{-at}u(t), \quad Re(a) > 0$	$\frac{1}{(\alpha + j\Omega)^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t), \quad Re(a) > 0$	$\frac{1}{(\alpha + j\Omega)^n}$
$e^{-a\lvert t\rvert}u(t), \quad Re(a) > 0$	$\frac{2a}{\alpha^2 + \Omega^2}$

Exercises

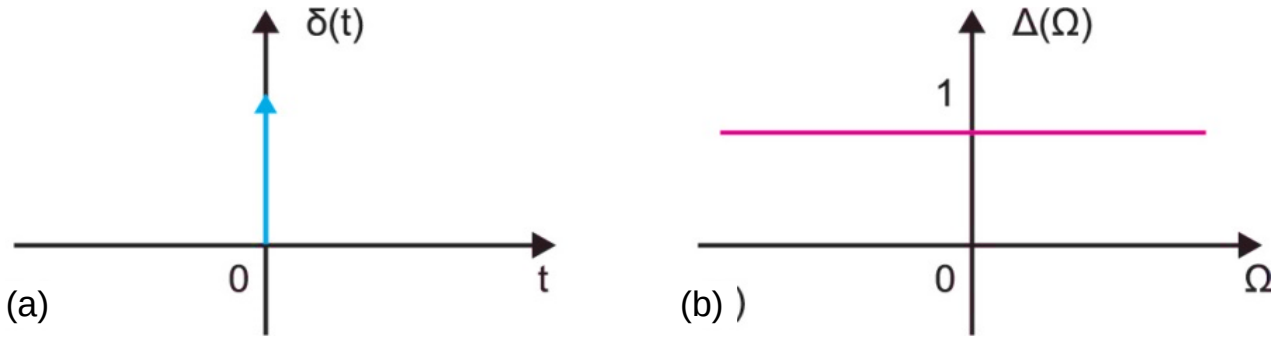
Exercise 1

Calculate the Fourier transform (FT) of the impulse function $\delta(t)$.

Answer: From the definition of the FT:

$$\Delta(\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = e^{0} = 1\measuredangle 0^{o}$$

The sifting property of $\delta(t)$ was used: $\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\, dt = x(t_0)$



The spectrum $\Delta(\Omega)$ contains infinitely many frequencies with unit amplitude and zero phase. The impulse function $\delta(t)$ is spectrally "white": it excites all frequencies equally, without preference. This explains the importance of the impulse response in LTI system analysis.

Exercise 2

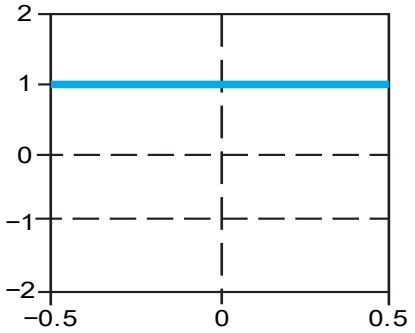
Calculate the Fourier transforms of the signals:

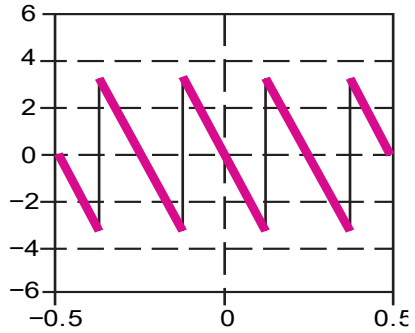
(a) $x_1(t) = \delta(t - t_0)$ (b) $x_2(t) = \delta(t + \alpha) + \delta(t - \alpha)$

Answer: (a) From the definition of the Fourier transform:

$$X_1(\Omega) = \int_{-\infty}^{\infty} x_1(t) e^{-j\Omega t} dt = \int_{-\infty}^{+\infty} \delta(t - t_0) e^{-j\Omega t} dt = e^{-j\Omega t}\Big|_{t = t_0} = e^{-j\Omega t_0} = 1\angle -\Omega t_0$$

The magnitude remains unity, as for $x(t) = \delta(t)$.
The phase is now **linear** in frequency, with
slope $-t_0$. Thus, a time shift of $\delta(t)$
leaves the transform magnitude unchanged,
but changes its phase linearly.





This holds for all signals and is a basic Fourier transform property: the **time-shift** property.

Exercise 2 (continued)

(b) From the definition of the Fourier transform:

$$X_2(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} dt = \int_{-\infty}^{\infty} \delta(t+\alpha)\, e^{-j\Omega t} dt + \int_{-\infty}^{\infty} \delta(t-\alpha)\, e^{-j\Omega t} dt$$

$$= e^{-j\Omega t}\Big|_{t=-\alpha} + e^{-j\Omega t}\Big|_{t=\alpha} = e^{j\alpha\Omega} + e^{-j\alpha\Omega} = 2\cos(\alpha\Omega)$$

- The Fourier transform magnitude of two impulses symmetric in time is the real, even cosine function $2cos(\alpha\Omega)$. Its frequency is determined by the time position α of the two impulses.
- The Fourier transform is real, so its phase is zero for $X(\Omega) > 0$ and π for $X(\Omega) < 0$ (undefined at spectral zeros).

Exercise 3

Calculate the Fourier transform of $x(t) = A\, e^{-\alpha t} u(t), \alpha > 0$ and plot its magnitude and phase spectra.

Answer: (a) From the definition of the Fourier transform:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt = A \int_{0}^{\infty} e^{-at}\, e^{-j\Omega t}\, dt = A \int_{0}^{\infty} e^{-(a+j\Omega)t}\, dt =$$

$$= -\frac{A}{a+j\Omega} e^{-(a+j\Omega)t} \Big|_{0}^{\infty} = -\frac{A}{a+j\Omega} \left[\lim_{t\to\infty} e^{-(a+j\Omega)t} - e^{0} \right] = -\frac{A}{a+j\Omega} \left[\lim_{t\to\infty} e^{-\alpha t} e^{-j\Omega t} - 1 \right]$$

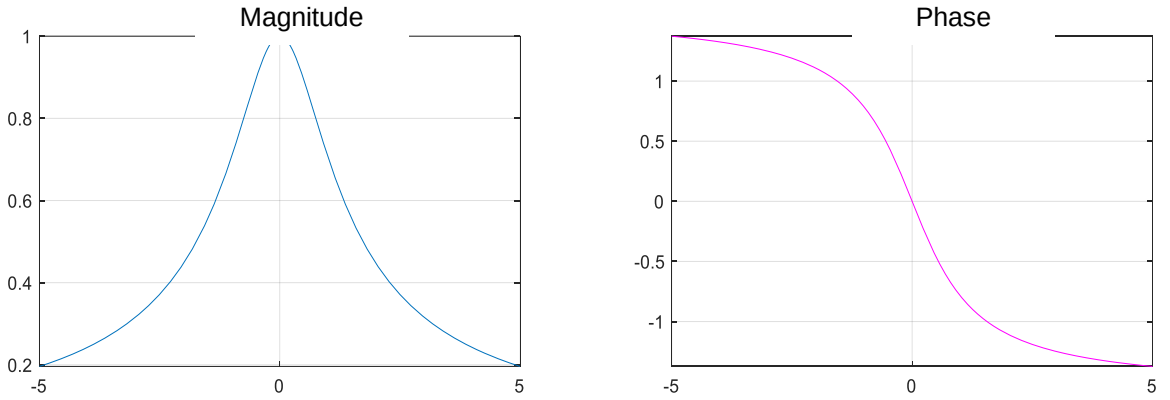
$$= -\frac{A}{a+j\Omega} [0 - 1] = \frac{A}{a+j\Omega}$$

It can be shown that $\lim_{t\to\infty} e^{(c+jd)t} = \lim_{t\to\infty} e^{ct} e^{jdt} = \begin{cases} 0, & c < 0 \\ \infty, & d > 0 \end{cases}$

Exercise 3 (continued)

(b) The magnitude and phase of the complex function $X(\Omega) = \mathrm{A}/(a + j\Omega)$ are :

$$|X(\Omega)| = \frac{|A|}{|a + j\Omega|} = \frac{A}{\sqrt{\alpha^2 + \Omega^2}} \quad \text{and} \quad \measuredangle X(\Omega) = \measuredangle X(A) - \measuredangle X(a + j\Omega) = -\tan^{-1}\left(\frac{\Omega}{\alpha}\right)$$



Magnitude spectrum $|X(\Omega)|$ and phase spectrum $\measuredangle X(\Omega)$

The magnitude indicates low-pass behavior, while the nonlinear phase affects the time delay of different frequencies. Exponential decay in time produces a smooth spectral distribution.

Exercise 4

Calculate the Fourier transform of the unit rectangular pulse $\Pi_{T/2}(t)$, of duration T:

$$x(t) \equiv \Pi_{T/2}(t) \equiv A\, rect\left(\frac{t}{T}\right) = \begin{cases} A, & |t| \le T/2 \\ 0, & |t| > T/2 \end{cases}$$

Answer: Since the signal is zero for $t < -T/2$ and $t > T/2$, its Fourier transform is:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = A \int_{-T/2}^{T/2} e^{-j\Omega t} dt = \frac{A}{-j\Omega} e^{-j\Omega t} \Big|_{-T/2}^{T/2} = \frac{A}{-j\Omega}\left[e^{-j\Omega T/2} - e^{-j\Omega(-T/2)}\right]$$

$$= \frac{A}{j\Omega}\left[e^{j\Omega T/2} - e^{-j\Omega T/2}\right] = \frac{2A}{\Omega} sin(\Omega T/2) = AT\, \frac{sin(\Omega T/2)}{\Omega T/2} = AT\, sinc\left(\frac{\Omega \mathrm{T}}{2\pi}\right)$$

Euler's relation $sin(\varphi) = [ej\varphi - e{-}j\varphi]/2j$ with $\varphi = T/2$ was used. Substituting $\Omega = 2\pi f$, we can express the result in ordinary frequency f:

$$X(f) = \frac{A}{\pi f} \sin(\pi f T)$$

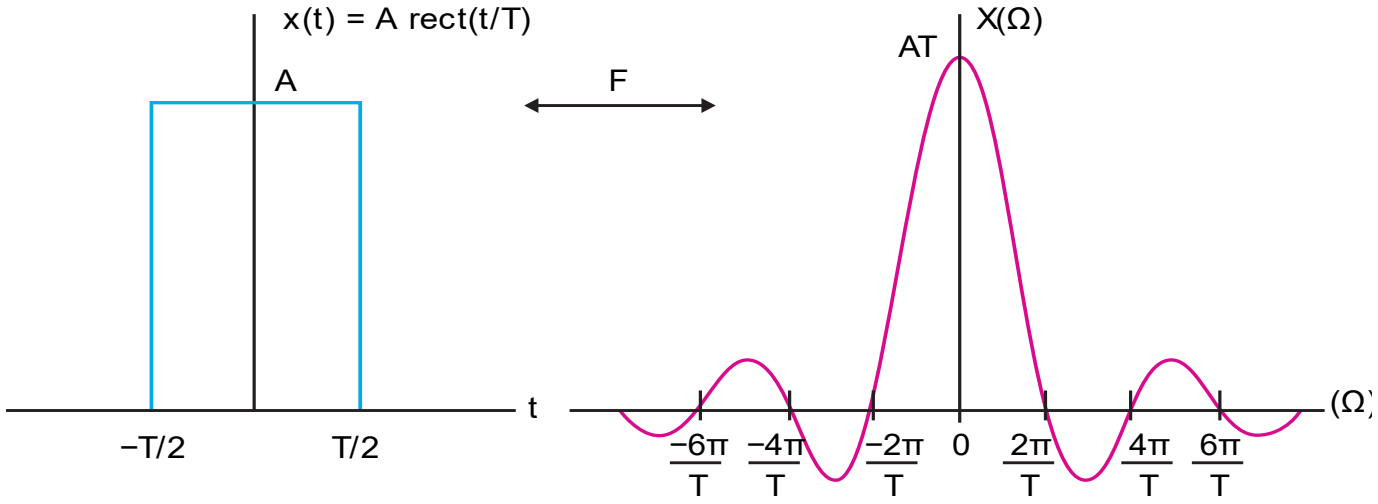
Exercise 4 (continued)

A simpler form is:

$$X(\Omega) = AT\, sinc\left(\frac{\Omega T}{2\pi}\right)$$

$$X(f) = AT\, sinc(fT)$$

where $sinc(x)$ is the normalized sampling (sinc) function.



The rectangular pulse of duration T and its Fourier transform.

Exercise 4 (continued)

Observations:

1. Although $X(\Omega)$ is generally complex, it is real for a rectangular pulse.
2. The value of $X(\Omega)$ at zero frequency is found from the limit:

$$X(0) = \lim_{\Omega \to 0} \frac{\sin(\Omega T/2)}{\Omega \mathrm{T}/2} = \lim_{\Omega \to 0} AT \cos(\Omega T/2) = 2T$$

3. The maximum value of $X(\Omega)$ equals the total area of the rectangular pulse.
4. The values where $X(\Omega)$ vanishes are the **spectral zeros**, given by $\Omega = 2\pi n/T$, where $n = \pm 1, \pm 2, \ldots$.
5. The spectrum tends to zero at high frequencies, that is, as $|\Omega| \to \infty$.
6. The Fourier transform crosses zero periodically, and the heights of the sidelobes tend asymptotically to zero.
7. The presence of sidelobes shows that a time-limited signal is not band-limited.

Exercise 5

Find the signal whose Fourier transform is a rectangular frequency window of width B:

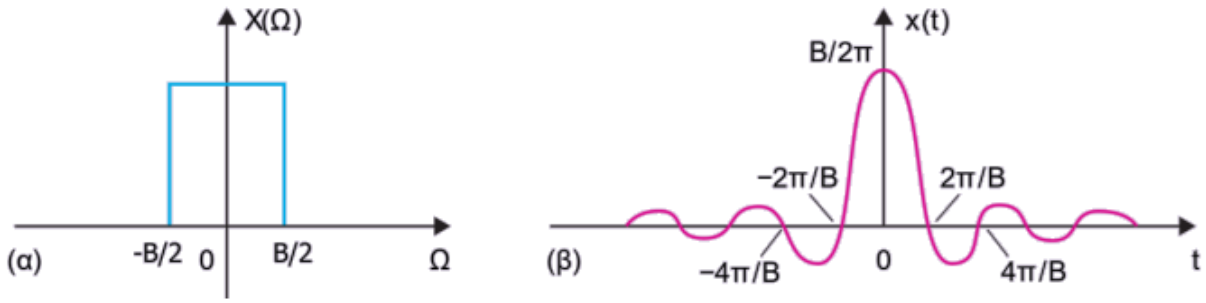
$$X(\Omega) = rect\left(\frac{\Omega}{\mathrm{B}}\right) = \begin{cases} 1, & |\Omega| < \mathrm{B}/2 \\ 0, & \text{otherwise} \end{cases}$$

Answer: Since the Fourier transform is zero for $\Omega < -\mathrm{B}/2$ and $\Omega > \mathrm{B}/2$, the signal is:

$$x(t) = \frac{1}{2\pi}\int_{-B/2}^{B/2} e^{j\Omega t}\, d\Omega = \frac{1}{2\pi jt} e^{j\Omega t}\Big|_{-B/2}^{B/2} =$$

$$\frac{1}{2\pi jt}\left(e^{jBt/2} - e^{-jBt/2}\right) = \frac{1}{2\pi jt} 2j\, sin(Bt/2) =$$

$$= \frac{1}{\pi t} sin(Bt/2) = \frac{B}{2\pi} sinc\left(\frac{Bt}{2\pi}\right)$$



Representation of $x(t)$ in (a) frequency and (b) time, respectively.

Exercise 5 (continued)

Observations:

- As in the previous exercise, the solution is described by the normalized sampling (sinc) function.
- Comparing the two exercises reveals a symmetry: the Fourier transform of a rectangular pulse in time is a sampling function in frequency, while the inverse Fourier transform of a rectangular pulse in frequency is again a sampling function in time.
- This observation will be confirmed below by the Fourier transform duality property, which applies to all signals.
- The result also shows that information cannot be confined simultaneously in both time and frequency.

Exercise 6

Calculate the Fourier transform of the signal $x(t) = cos(\Omega_0 t)$..

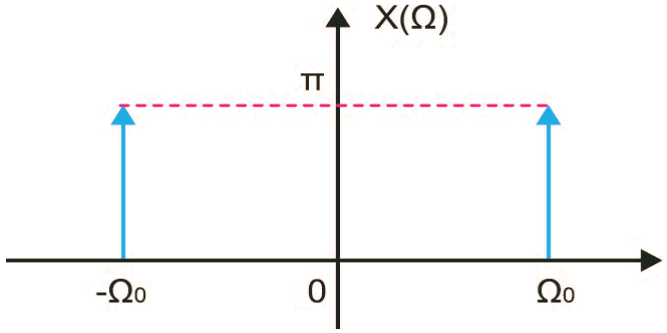
Answer: Euler's relation gives $x(t) = (1/2)[e^{j\Omega_0 t} + e^{-j\Omega_0 t}]$.. From Exercise 2, we know:

$$e^{j\Omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\,\delta(\Omega - \Omega_0)$$

Therefore:

$$F\{x(t)\} = F\left\{\frac{1}{2}\,e^{j\Omega_0 t} + \frac{1}{2}\,e^{-j\Omega_0 t}\right\} = \frac{1}{2}F\{e^{j\Omega_0 t}\} + \frac{1}{2}\,F\{e^{-j\Omega_0 t}\}$$

$$= \frac{1}{2}2\pi\,\delta(\Omega - \Omega_0) + \frac{1}{2}2\pi\,\delta(\Omega + \Omega_0) = \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$$



Magnitude spectrum of $x(t) = cos(\Omega_0 t)$. The phase spectrum is zero.

Exercise 7

Find the Fourier transform of the signal $x(t) = e^{-\alpha t}\, cos(\Omega_0 t)\, u(t),\ \alpha > 0$.

Answer: Using the definition of the Fourier transform and replacing the cosine with Euler's formula gives:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \frac{1}{2}\int_{0}^{\infty} e^{-(\alpha+j\Omega)t}\left[e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right] dt$$

$$= \frac{1}{2}\int_{0}^{\infty} e^{-(\alpha+j\Omega)t} e^{j\Omega_0 t} dt + \frac{1}{2}\int_{0}^{\infty} e^{-(\alpha+j\Omega)t} e^{-j\Omega_0 t} dt$$

$$= \frac{1}{2}\left[\int_{0}^{\infty} e^{-(\alpha+j(\Omega-\Omega_0))t} dt + \int_{0}^{\infty} e^{-(\alpha+j(\Omega+\Omega_0))t} dt\right] = \frac{1}{2}\left[\frac{1}{\alpha + j(\Omega-\Omega_0)} + \frac{1}{\alpha + j(\Omega+\Omega_0)}\right]$$

$$= \frac{\alpha + j\Omega}{\alpha^2 + 2j\Omega\alpha + (\Omega_0^2 - \Omega^2)}$$

Exercise 8

Determine whether the Fourier transforms of the following signals can be obtained from their Laplace transforms:

(a) $x_1(t) = \delta(t)$ (b) $x_2(t) = e^{-3t}u(t)$ (c) $x_3(t) = u(t)$

Answer: (a) The Laplace transform is $X_1(s) = 1$, with the entire s-plane as its region of convergence. Since this includes the imaginary axis $j\Omega$, the Fourier transform can be found from Laplace:

$$X(\Omega) = X(s)\Big|_{s=j\Omega} = 1$$

(b) The Laplace transform is $X_2(s) = 1/(s+3)$, with region of convergence $Re\{s\} > -3$. It includes the imaginary axis $j\Omega$, so the Fourier transform can be found from Laplace:

$$X(\Omega) = X(s)\Big|_{s=j\Omega} = \frac{1}{3 + j\Omega}$$

(c) The Laplace transform is $X_3(s) = 1/s$, with region of convergence $Re\{s\} > 0$. The imaginary axis $j\Omega$ lies outside this region, so the Fourier transform cannot be calculated using $X(\Omega) = X(s)|_{s=j\Omega}$.

Exercise 9

Use the time-shift property to calculate the Fourier transform of the signal $x(t) = \delta(t - t_0)$.

Answer: The Fourier transform of the delta function is $\Delta(\Omega) = 1$.

The spectrum of $\delta(t)$ is constant at all frequencies: all frequency components have equal weight. The time-shift property gives:

$$X(\Omega) = e^{-j\Omega t_0}\,\Delta(\Omega) = e^{-j\Omega t_0}$$

Therefore:

$$\delta(t - t_0) \overset{F}{\longleftrightarrow} e^{-j\Omega t_0}$$

- To synthesize $\delta(t - t_0)$, we need sinusoids at all possible frequencies ($-\infty < \Omega < \infty$), with constant unit amplitude and **linear phase**.
- The time position of the impulse is represented entirely in the spectral **phase**. For $\delta(t - t_0)$, the phase is linear and equals $-\Omega t_0$.
- This shows the close relationship between time and phase: a time delay corresponds to a linear phase shift in the frequency domain.

Exercise 10

Calculate the Fourier transform of the signal $y(t) = x(t)\cos(\Omega_0 t)$.

Answer: Using Euler's relation, the signal $y(t)$ can be written as:

$$y(t) = x(t)\cos(\Omega_0 t) = x(t)\frac{1}{2}\left[e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right] = \frac{1}{2}\,x(t)\,e^{j\Omega_0 t} + \frac{1}{2}x(t)e^{-j\Omega_0 t}$$

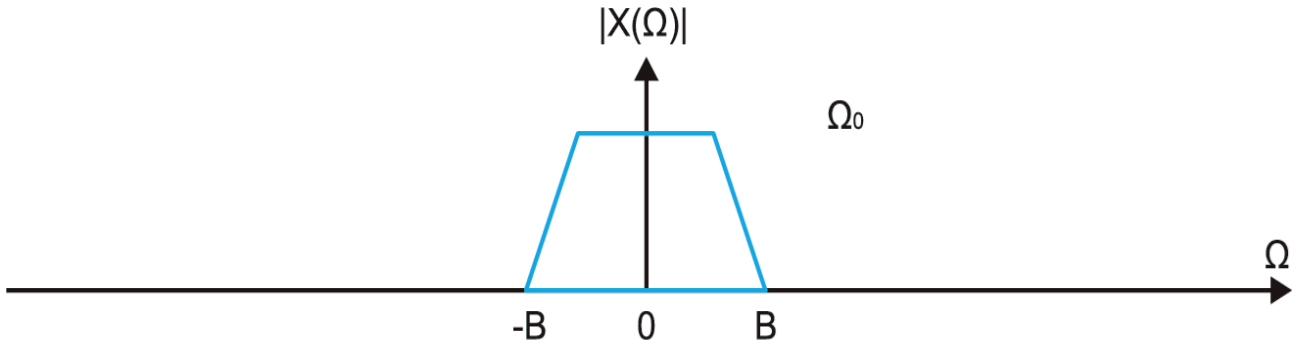
By linearity and frequency shifting, the FT of $y(t)$ is:

$$Y(\Omega) = \mathcal{F}\left\{\frac{1}{2}\,x(t)\,e^{j\Omega_0 t}\right\} + \mathcal{F}\left\{\frac{1}{2}x(t)e^{-j\Omega_0 t}\right\} = \frac{1}{2}\left[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)\right]$$

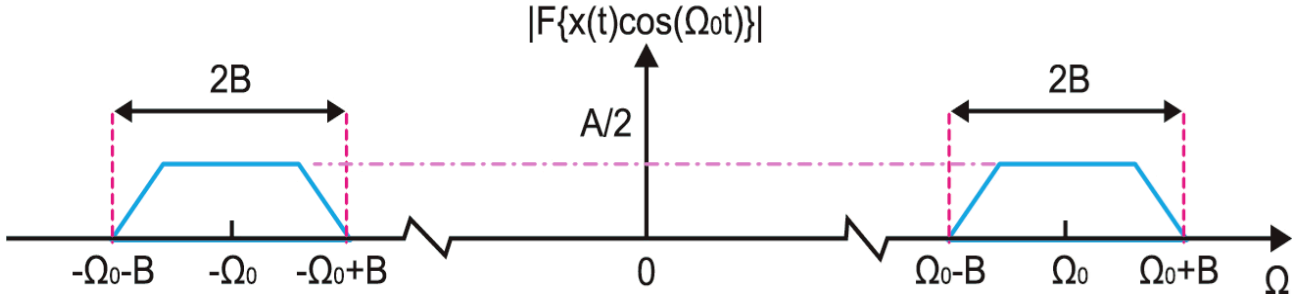
Multiplying the signal $x(t)$ by $\cos(\Omega_0 t)$ leaves the shape of $X(\Omega)$ unchanged, while moving its spectrum to the frequency regions $\pm\Omega_0$. This process is called **modulation.**

Exercise 10 (continued)

Original signal spectrum



Modulated signal spectrum



In modulation, an information signal $x(t)$ multiplies a sinusoidal carrier $\cos(\Omega_0 t)$, shifting the information to the **carrier** frequency for transmission over a channel.

Exercise 10 (continued)

- Multiplying the signal $x(t)$ by the cosine $\cos(\Omega_0 t)$ leaves the shape of its spectrum $X(\Omega)$ unchanged, while moving it to the frequency regions $\pm\Omega_0$. Thus, the output $y(t)$ carries the same information at frequencies $\pm\Omega_0$. suitable for transmission.
- This property is central to telecommunications and **modulation.** To transmit an information signal, the input $x(t)$ is multiplied by a carrier $\cos(\Omega_0 t)$, usually at a much higher frequency.
- In this case, $y(t) = x(t)\cos(\Omega_0 t)$ gives **double-sideband amplitude modulation** (AM/DSB), one of the simplest modulation schemes.
- The carrier frequency is usually much higher than the maximum signal frequency: $\Omega_0 \gg B$.
- Modulation systems are neither linear nor time-invariant. Thus, they are not LTI systems.

Exercise 11

Calculate the Fourier transform of the signal $x(t) = cos(\Omega_0 t)\, \Pi_T(t)$.

Answer: The cosine $cos(\Omega_0 t)$ does not have infinite duration: it is confined to the duration $2T$ of a rectangular pulse. Multiplication by the rectangular pulse selects the corresponding portion $cos(\Omega_0 t)$ (windowing). We wish to determine the effect on the spectrum.

From Euler's relation $cos(\Omega_0 t) = \frac{1}{2} e^{j\Omega_0 t} + \frac{1}{2} e^{-j\Omega_0 t}$, the signal $x(t)$ is:

$$x(t) = \Pi_T(t) \left[\frac{1}{2} e^{j\Omega_0 t} + \frac{1}{2} e^{-j\Omega_0 t} \right] = \frac{1}{2} e^{j\Omega_0 t} \Pi_T(t) + \frac{1}{2} e^{-j\Omega_0 t} \Pi_T(t)$$

We also know that:

$$\Pi_T(t) \overset{F}{\longleftrightarrow} \frac{2\, sin(\Omega T)}{\Omega}$$

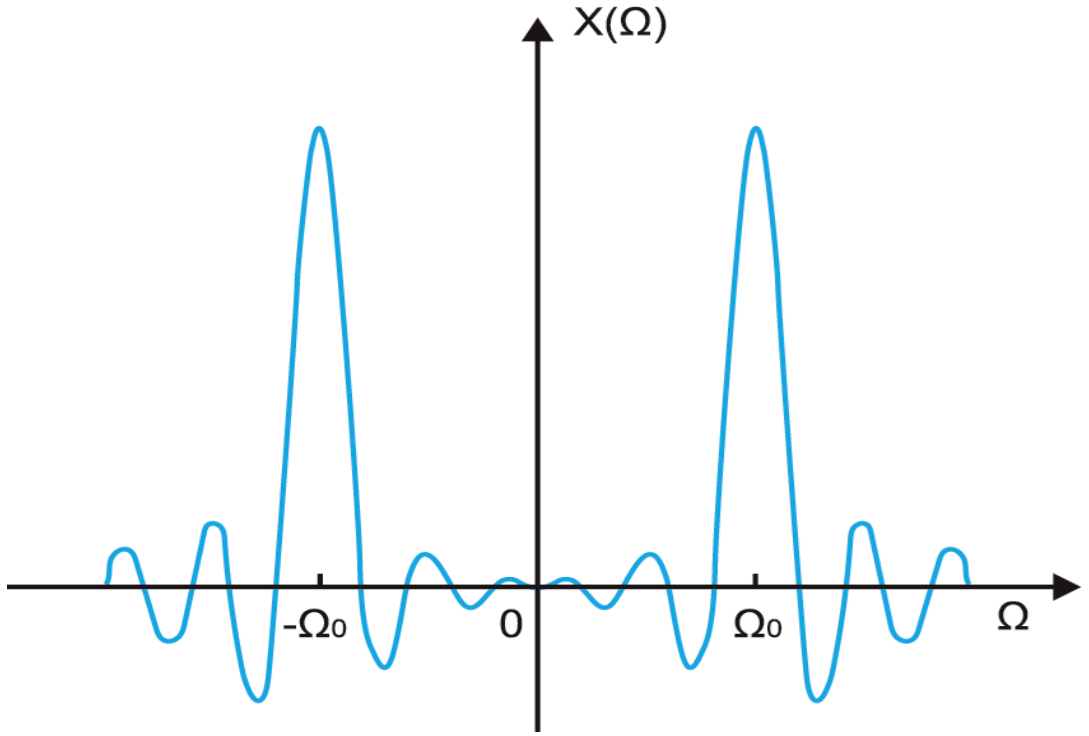
Using the frequency-shift property gives:

$$x(t) \overset{F}{\longleftrightarrow} \frac{sin(\Omega - \Omega_0)T}{(\Omega - \Omega_0)} + \frac{sin(\Omega + \Omega_0)T}{(\Omega + \Omega_0)}$$

Exercise 11 (continued)

Observations:

- The FT consists of two **sampling (sinc) functions**, shifted to the frequencies $-\Omega_0$ and Ω_0..
- Spectral **leakage** occurs from the signal into neighboring frequencies around $\pm\Omega_0$.
- This is undesirable, especially when locating multiple cosines at different frequencies.



- Windowing distorts the spectrum. This matters because it is a common and essential signal-processing operation.

Exercise 11 (continued)

- To reduce the effect of rectangular windowing on the spectrum, increase the window duration T. This narrows the width of the main lobe of the sampling function.
- A longer window causes less distortion. An infinitely long window has no effect on the spectrum, but no longer performs windowing.
- For a signal with rapid time variations, short windows are required. A suitable window can improve the spectrum, as discussed in detail in Section 12.14 of the book.
- Choosing a window trades time resolution against frequency resolution. A narrow window localizes the signal better in time, while a wide window gives better frequency resolution.

Exercise 12

Find the Fourier transform of the signal $x(t) = 1$.

Answer: We know that $\delta(t) \overset{F}{\longleftrightarrow} 1$. By duality, the Fourier transform of $x(t) = 1$ is:

$$X(\Omega) = 2\pi\delta(-\Omega) = 2\pi\delta(\Omega)$$

- The time and frequency domains are inversely related: an impulse $\delta(t)$ of zero duration has an infinitely wide spectrum, while the constant signal 1 has zero bandwidth, with all energy concentrated at frequency $\Omega = 0$.
- Duality shows that the shape of a function in frequency can be used directly in time to calculate the transform of another function, a useful technique.

Exercise 13

Find the output $y(t)$ of an LTI system initially at rest, described by the following differential equation, when the input is $x(t) = \delta(t)$:

$$6y(t) + 5\frac{dy(t)}{dt} + \frac{d^2y(t)}{dt^2} = x(t)$$

Answer: Apply the differentiation property to both sides of the differential equation:

$$6Y(\Omega) + 5j\Omega\, Y(\Omega) + (j\Omega)^2 Y(\Omega) = \Delta(\Omega) \Rightarrow Y(\Omega)[6 + 5j\Omega + (j\Omega)^2] = 1 \Rightarrow$$

$$Y(\Omega) = \frac{1}{6 + 5j\Omega + (j\Omega)^2} \Rightarrow Y(\Omega) = \frac{1}{(3 + j\Omega)} \cdot \frac{1}{(2 + j\Omega)} = \cdots = -\frac{1}{3 + j\Omega} + \frac{1}{2 + j\Omega}$$

Taking the inverse Fourier transform gives: $y(t) = [-e^{-3t} + e^{-2t}]\, u(t)$

- The differentiation property multiplies by $j\Omega$ in the frequency domain. Thus, the differential equation becomes an algebraic equation for $Y(\Omega)$.
- This method requires the system to be initially at rest. Otherwise, the unilateral Laplace transform is needed.

Exercise 14

Find the Fourier transform of a unit triangular pulse of duration 2T, given by:

$$\Lambda_{\mathrm{T}}(t) = \begin{cases} 1 - \dfrac{|t|}{\mathrm{T}}, & t < |\mathrm{T}| \\ 0, & t > |\mathrm{T}| \end{cases}$$

Answer: The unit triangular pulse can be written as the sum of three shifted unit ramps:

$$\Lambda_{\mathrm{T}}(t) = \frac{1}{T}\,[r(t+T) - 2r(t) + r(t-T)]$$

Calculate the first and second time derivatives:

$$\Lambda'_{\mathrm{T}}(t) = \frac{1}{T}[u(t+T) - 2u(t) + u(t-T)]$$

$$\Lambda''_{\mathrm{T}}(t) = \frac{1}{T}[\delta(t+T) - 2\delta(t) + \delta(t-T)]$$

Exercise 14 (continued)

Find the Fourier transform of the triangular pulse $\Lambda_{\mathrm{T}}(t)$ using the differentiation property:

$$F\{\Lambda_{\mathrm{T}}(t)\} = F\left\{\frac{1}{(j\Omega)^2}\Lambda_{\mathrm{T}}(t)\right\} = F\left\{\frac{1}{(j\Omega)^2}\left[\frac{1}{T}[\delta(t+T) - 2\delta(t) + \delta(t-T)]\right]\right\}$$

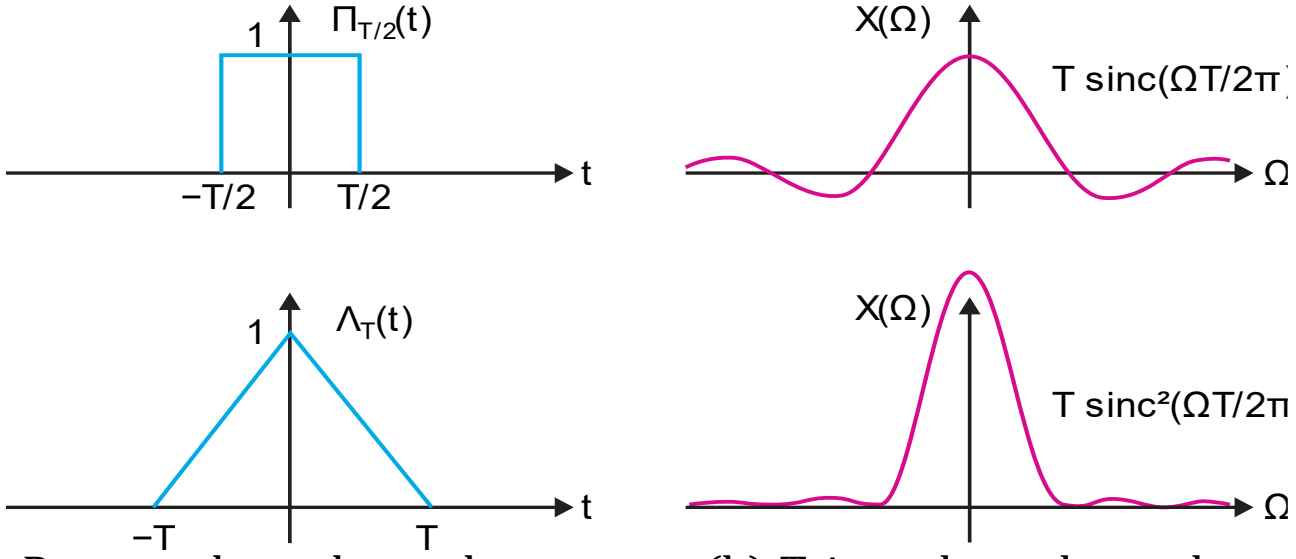
$$= \frac{1}{\Omega^2 \mathrm{T}}\left(2 - e^{j\Omega \mathrm{T}} - e^{-j\Omega \mathrm{T}}\right) = \cdots = 4\frac{\sin^2\left(\frac{\Omega T}{2}\right)}{\Omega^2 \mathrm{T}}$$

Recall the Fourier transform of a unit rectangular pulse of duration T:

$$\Pi_{\mathrm{T}/2}(t) = Arect(t/\mathrm{T}) \overset{F}{\longleftrightarrow} \mathrm{AT}\ \mathrm{sinc}\left(\frac{\Omega \mathrm{T}}{2\pi}\right)$$

Exercise 14 (continued)

The waveforms and Fourier transforms of rectangular and triangular pulses are shown below. These are common window functions.



(a) Rectangular pulse and spectrum. (b) Triangular pulse and spectrum.

For the triangular pulse, all sidelobes are positive. Its main lobe is wider than that of the rectangular pulse, while its sidelobes are smaller.

Exercise 15

Calculate the Fourier transform of the unit step function $u(t)$.

Answer: The function $u(t)$ can be written as:

$$u(t) = \int_{-\infty}^{t} \delta(t)\, dt$$

We know that $\delta(t) \overset{F}{\longleftrightarrow} 1$. Using the integration property, the Fourier transform of the unit step is:

$$u(t) \overset{F}{\longleftrightarrow} U(\Omega) = \frac{1}{j\Omega} + \pi\delta(\Omega)$$

- The unit step $u(t)$ is not absolutely integrable, so its transform must be interpreted in terms of distributions.
- Since there is a discontinuity at $\Omega = 0$, the term $1/j\Omega$ is interpreted as a Cauchy principal value.

Exercise 15 (continued)

- In the result $U(\Omega) = 1/j\Omega + \pi\delta(\Omega)$, the term $\pi\delta(\Omega)$ represents the DC component of the step function, with value $1/2$.
- The complex term $1/j\Omega$ is a continuous spectrum expressing the abrupt change of $u(t)$ at $t = 0$, which excites high frequencies that decay as $1/\Omega$.
- Unlike the impulse, which is instantaneous and spectrally flat, the step maintains the constant value 1 for all positive time. This temporal extent concentrates most of its energy near low frequencies, becoming stronger as $\Omega \to 0$.

Exercise 16

Prove the convolution property of the Fourier transform.

Answer: The convolution of two LTI signals is:

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

Using the definition of the Fourier transform in the above relation gives:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{\infty} [x_1(t) * x_2\,(t)]\; e^{-j\Omega t}\, dt =$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right) e^{-j\Omega t}\, dt =$$

$$= \int_{-\infty}^{\infty} x_1(t) \left(\int_{-\infty}^{\infty} x_2(t-\tau)\; e^{-j\Omega t}\, dt \right) d\tau =$$

$$= \int_{-\infty}^{\infty} x_1(t)\, \mathrm{X}_2(\Omega)\; e^{-j\Omega\tau}\, d\tau = X_2(\Omega) \int_{-\infty}^{\infty} x_1(\tau)\; e^{-j\Omega\tau}\, d\tau = X_2(\Omega)\, X_1(\Omega)$$

Exercise 17

Prove the time-multiplication property of the Fourier transform.

Answer: From the definition of the inverse Fourier transform:

$$X(\Omega) = 1/2\pi \int_{-\infty}^{\infty} [\mathrm{X}_1(\Omega) * X_2(\Omega)]\, e^{j\Omega t}\, d\Omega = 1/2\pi \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mathrm{X}_1(\lambda)\, X_2(\Omega - \lambda) d\lambda \right) e^{j\Omega t}\, d\Omega$$

Set $\Omega - \lambda = \rho$ and reverse the order of integration:

$$X(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{X}_1(\lambda) \left(\int_{-\infty}^{\infty} X_2(\rho)\, e^{j(\lambda+\rho)t}\, d\rho \right) d\lambda =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{X}_1(\lambda)\, e^{j\lambda t} \left(\int_{-\infty}^{\infty} X_2(\rho)\, e^{j\rho t}\, d\rho \right) d\lambda =$$

$$= 2\pi \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\Omega)\, e^{j\Omega t}\, d\Omega \right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\Omega)\, e^{j\Omega t}\, d\Omega \right) = 2\pi\, x_1(t)\, x_2(t)$$

Exercise 18

Find the Fourier transform of the signal $x(t) = e^{-a|t|}$, where $a > 0$.

Answer: Consider the signal $g(t) = e^{-at}u(t)$. Its FT is:

$$G(\Omega) = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-j\Omega t}\, dt = \int_{0}^{\infty} e^{-at}\, e^{-j\Omega t}\, dt = \frac{1}{a + j\Omega}$$

The signal $g(t)$ can be decomposed into even and odd components: $g(t) = g_e(t) + g_0(t)$.

Its even part is $g_e(t) = \frac{g(t)+g(-t)}{2} = \frac{e^{-a|t|}}{2}$, and $x(t) = 2\, g_e(t)$..

By the time-multiplication property, the Fourier transform of $g_e(t)$ is:

$F\{g_e(t)\} = R_e[G(\Omega)]$. Thus:

$$G_e(\Omega) = R_e[G(\Omega)] = \frac{a}{a^2 + \Omega^2}$$

Finally, using linearity of the Fourier transform:

$$X(\Omega) = 2\, G_e(\Omega) = \frac{2\, a}{a^2 + \Omega^2}$$

Exercise 19

Calculate the energy of the causal exponential signal $x(t) = e^{-\alpha t}u(t),\ \alpha > 0$.

Answer: The Fourier transform of $x(t) = e^{-\alpha t}u(t),\ \alpha > 0$ is known:

$$X(\Omega) = \frac{1}{\alpha + j\Omega}$$

Its energy in the time domain is:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)^2 dt = \int_0^{\infty} e^{-2\alpha t} dt = -\frac{1}{2a}(e^{-2\alpha t})\Big|_0^{\infty} = -\frac{1}{2a}\left[\lim_{t\to\infty} e^{-2\alpha t} - e^0\right] = \frac{1}{2a}$$

Its energy in the frequency domain is:

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1}{\alpha + j\Omega}\right|^2 d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha^2 + \Omega^2} d\Omega = \frac{1}{\pi}\int_0^{\infty} \frac{1}{\alpha^2 + \Omega^2} d\Omega$$

We know that $\int_{\lambda_1}^{\lambda_2} \frac{1}{\alpha^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_{\lambda_1}^{\lambda_2}$. Therefore, the frequency-domain energy is:

$$E_x = \frac{1}{a\pi}\tan^{-1}\left(\frac{\Omega}{a}\right)\Big|_0^{\infty} = \frac{1}{\alpha\pi}\left[\lim_{\Omega\to\infty} \tan^{-1}\left(\frac{\Omega}{a}\right) - \tan^{-1}(0)\right] = \frac{1}{\alpha\pi}\frac{\pi}{2} = \frac{1}{2\alpha}$$

This confirms Parseval's theorem.

Exercise 19 (continued)

Observations:

- This example shows that signal energy is a physical quantity that does not change between representations. Whether the signal is studied in time or frequency, its individual frequency components have the same total energy: $E_x = 1/2\alpha$.
- The quantity $|X(\Omega)|^2$ integrated in the frequency domain is the energy spectral density: the energy distributed per unit frequency. All the energy lies at low frequencies (near $\Omega = 0$).
- Energy is inversely proportional to the parameter α. For small α, the signal $e^{-\alpha t}$ decays slowly (long duration), accumulating more energy. For large α, it decays faster and has less total energy.