

UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Signals and Systems

Lecture 8: Fourier Series

Michalis Paraskevas
Professor

Signal Analysis Using Fourier Series

1. Review of eigenmodes
2. Conditions for the existence of a Fourier series
3. Fourier series expansion of a signal:
   - Exponential form
   - Trigonometric form I
   - Trigonometric form II
4. Properties of Fourier series
5. Normalized power of a Fourier expansion
6. Parseval's theorem for periodic functions
7. Gibbs phenomenon
8. Relationship between Fourier series and the Fourier transform

Introduction (1/2)

- Analysis of signals and linear time-invariant (LTI) systems using Fourier series and Fourier transforms represents signals in the **frequency domain** rather than the time domain.
- Fourier analysis represents a broad class of signals as sums or integrals of sinusoidal components at different frequencies.
- Complex exponential and sinusoidal functions are eigenfunctions of LTI systems. Thus, the output changes only the amplitude and phase of each frequency component of the input, while leaving its frequency unchanged.

Introduction (2/2)

- **Fourier series**: Used to analyze periodic signals, giving a discrete spectral representation of their frequency content through amplitude and phase spectra.
- **Fourier transform**: Used mainly to analyze aperiodic signals, giving a continuous spectral representation.
- The Fourier transform is the limit of the Fourier series as the signal period tends to infinity. In this limit, the discrete spectrum becomes continuous.

1. Review of Eigenmodes

Review of Eigenmodes

When a causal, stable LTI system receives a complex exponential input at a given frequency:

$$x(t) = e^{j\Omega_0 t} \text{ and } -\infty < t < \infty$$

the system output is:

$$y(t) = x(t)\, H(\Omega_0) = e^{j\Omega_0 t}\, H(\Omega_0)$$

where:

$$H(\Omega_0) = \int_0^{+\infty} e^{-j\Omega_0 \tau}\, h(\tau)\, d\tau$$

- For a complex exponential input at frequency Ω_0, the output of a stable LTI system is the input multiplied by a complex constant $H(\Omega_0)$..
- The constant $H(\Omega_0)$ is the value of the system **frequency response** at frequency Ω_0.
- The input appears at the output **modified** by the frequency response $H(\Omega_0)$, which depends solely on the system.

Review of Eigenmodes

By linearity, the response to a sum of complex exponentials is the sum of their individual responses. For an input that is a linear combination of complex exponentials with different amplitudes, frequencies and phases:

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k \, e^{j\Omega_k t}$$

where X_k are complex values, the system output is:

$$y(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{j\Omega_k t} \, H(\Omega_k) = \sum_{k=-\infty}^{\infty} X_k \, |H(\Omega_k)| \, e^{j\left(\Omega_k t + \varphi(\Omega_\mathrm{k})\right)}$$

Here, $\phi(\Omega_k)$ is the phase of the frequency response at frequency Ω_k. This also applies to sinusoidal signals, since they can be expressed as sums of complex conjugate exponentials.

This discrete sum of complex exponentials corresponds to periodic signals (Fourier series). Nonperiodic signals are described by the Fourier transform, using an integral in place of the sum.

2. Conditions for the Existence of a Fourier Series

Conditions for the Existence of a Fourier Series

If a periodic signal $x(t)$ with period T_0 satisfies the following three Dirichlet conditions:

1. The signal is absolutely integrable over one period:

$$\int_0^{T_0} |x(t)|\, dt < +\infty$$

2. The signal has finitely many discontinuities of finite size in each period.

3. The signal has finitely many extrema in each period.

then $x(t)$ can be represented by a Fourier series in the three equivalent forms presented below.

The Dirichlet conditions ensure **convergence** of the Fourier series.

At continuity points, the series converges to the function value. At a discontinuity, it converges to the average of the one-sided limits $[x(t^-) + x(t^+)]/2$.

Most physical signals encountered in practice satisfy these conditions. However, some mathematical functions do not.

3. Fourier Series Expansion of a Signal

Exponential Form of the Fourier Series

For a periodic signal $x(t)$ with frequency Ω_0, its exponential Fourier series expansion is:

$$x(t) = X_0 + \sum_{k=1}^{\infty} (X_k\, e^{jk\Omega_0 t} + X_{-k}\, e^{-jk\Omega_0 t})$$

or, equivalently, in compact form:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{jk\Omega_0 t}$$

The exponential Fourier coefficients X_k are calculated as:

$$X_k = \frac{1}{T_0} \int_0^{T_0} x(t)\, e^{-jk\Omega_0 t}\, dt$$

For $k = 0$, in particular:

$$X_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt$$

Here, T_0 is the **fundamental period** of the periodic signal ($T_0 = 2\pi/\Omega_0$).

Exponential Form of the Fourier Series

- The exponential expansion defines the **double-sided amplitude spectrum** $|X_k| = |X_k(\Omega_k)|$ and the double-sided **phase spectrum** $\varphi_k = \varphi_k(\Omega_k)$..
- For real signals, the coefficient **magnitude** has **even symmetry**, while the **phase** has **odd symmetry**.
- Since the independent variable Ω takes the discrete values Ω_k, the magnitude and phase plots of the complex coefficients form the **discrete spectrum** of the periodic signal $x(t)$.
- The quantity $f_0 = 1/T_0$ is the **fundamental frequency** of the signal $x(t)$.
- The frequencies of the complex terms are multiples of the fundamental frequency.
  The k −th multiple of the fundamental frequency is the $\boldsymbol{k}$ −th harmonic.

Observations

- The exponential Fourier series expands a signal into the components $e^{jk\Omega_0 t}$ for $k \in Z$.
- The functions $e^{jk\Omega_0 t}$ are Fourier **basis vectors**, **periodic** with period T_0.
- They are also orthonormal over one period, satisfying:

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} e^{jn\Omega_0 t}\left[e^{jm\Omega_0 t}\right]^* dt = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

- Since $\cos(k\Omega_0 t) = \frac{1}{2}\left[e^{jk\Omega_0 t} + e^{-jk\Omega_0 t}\right]$,, each term $\cos(k\Omega_0 t)$ consists of the terms $e^{jk\Omega_0 t}$ and $e^{-jk\Omega_0 t}$, whose half-sum gives the harmonic component.
- A broad class of signals can be represented as linear combinations of suitable orthonormal functions, decomposing each signal into elementary components.
- The coefficients X_k minimize the mean squared error between the signal and its finite Fourier expansion. For signals with a convergent Fourier series, this error tends to zero as the number of terms increases.

Trigonometric Form I of the Fourier Series

Every periodic signal $x(t)$ satisfying the Dirichlet conditions can be represented by a convergent trigonometric series of infinitely many sine and cosine terms:

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k \, cos(k\Omega_0 t) + b_k \, sin(k\Omega_0 t)]$$

Here, Ω_0 is the fundamental angular frequency ($\Omega_0 = 2\pi/T_0$) of $x(t)$, and the coefficients α_0, α_k and b_k are:

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) \, dt$$

$$a_k = \frac{2}{T_0} \int_0^{T_0} x(t) \, cos(k\Omega_0 t) \, dt$$

$$b_k = \frac{2}{T_0} \int_0^{T_0} x(t) \, sin(k\Omega_0 t) \, dt$$

Trigonometric Form I — Observations

- For a real periodic signal, the expansion in trigonometric form I is:

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\Omega_0 t + \theta_k)$$

  where $A_0 = X_0$, $A_k = 2|X_k|$ and $\theta_k = \measuredangle X_k = \varphi_k$.

- The basis vectors of trigonometric form I are $\cos(k\Omega_0 t)$ and $\sin(k\Omega_0 t)$.
- The basis vectors are **periodic** and **orthogonal** over one period, satisfying:

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} \cos(k\Omega_0 t)\cos(m\Omega_0 t)\,dt = \begin{cases} 1/2 & m = k \\ 0 & m \neq k \end{cases}$$

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} \sin(k\Omega_0 t)\sin(m\Omega_0 t)\,dt = \begin{cases} 1/2 & m = k \\ 0 & m \neq k \end{cases}$$

- These basis vectors do not have the same normalization as the basis vectors of the exponential form.

Trigonometric Form I — Observations

- If the periodic signal $x(t)$ is even, the product $x(t)\cos(k\Omega_0 t)$ is also even, while $x(t)\,sin(k\Omega_0 t)$ is odd. Thus, α_0 and α_k can be calculated over half a period by doubling the corresponding integral, while all b_k coefficients are zero. The expansion of an **even signal** therefore contains only **cosine** terms.
- If the periodic signal $x(t)$ is odd, the product $x(t)\cos(k\Omega_0 t)$ is odd, while $x(t)\,sin(k\Omega_0 t)$ is even. Thus, α_0 and α_k are zero, while b_k can be calculated over half a period by doubling the corresponding integral. The expansion of an **odd signal** therefore contains only sine terms.

Trigonometric Form II

Every periodic signal $x(t)$ satisfying the Dirichlet conditions can be represented by a convergent trigonometric series of infinitely many cosine terms:

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \, cos(k\Omega_0 t + \theta_k)$$

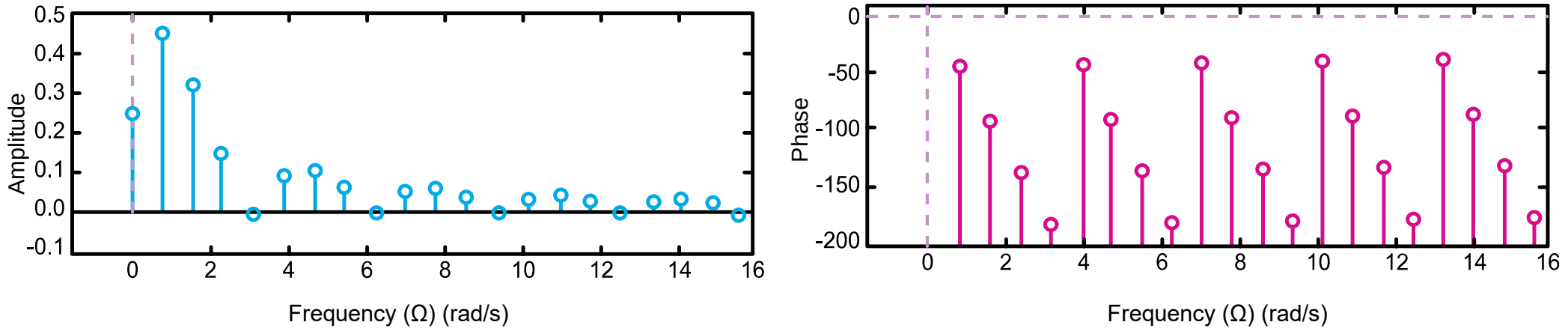
where:

$$c_0 = a_0, \; c_k = \mathrm{A_k} = 2|X_k| = \sqrt{a_k^2 + b_k^2} \quad \text{and} \quad \theta_k = \measuredangle X_k = -\tan^{-1}\left(\frac{b_k}{a_k}\right), \quad \theta_k \in (-\pi, \pi]$$

- The coefficient α_0 is the **DC component**, while α_k and b_k are the Fourier coefficients for the kth harmonic of the signal.
- As with the other two forms, an expansion with infinitely many terms describes the signal exactly where the series converges to its value. At discontinuities, the series converges to the average of the one-sided limits.
- A signal $x(t)$ can be fully specified by the coefficients $\{c_k, \theta_k\}$ and the fundamental frequency Ω_0. This allows us to define the amplitude and phase spectra of a periodic signal.

The Concept of a Spectrum

Plotting the amplitude coefficients c_k and phase coefficients φ_k against the harmonic frequencies $\Omega_\mathrm{k} = \mathrm{k}\Omega_0$ gives the **single-sided amplitude spectrum** $c_k = c_k(\Omega_k)$ and **phase spectrum** $\varphi_k = \varphi_k(\Omega_k)$, respectively. These display only positive frequencies.



Example of single-sided amplitude and phase spectra.

Trigonometric form I does not directly give a unified amplitude and phase representation as form II does. Instead, it describes the signal through the two coefficient sets a_k and b_k.

Summary

- The Fourier basis vectors in all Fourier series forms (exponential and trigonometric) are **periodic** with fundamental period T_0.
- A periodic signal is fully described by its period T_0 and its Fourier coefficients:
  - $\{a_k, b_k\}$ for trigonometric form I
  - $\{c_k, \theta_k\}$ for trigonometric form II
  - $\{X_k\}$ for the exponential form
- For signals satisfying the Dirichlet conditions, the magnitudes of $\{a_k, b_k\}$, $\{c_k\}$ and $\{|X_k|\}$ tend to zero as k tends to infinity.
- A finite number of terms therefore **approximates** $x(t)$. The approximation improves as the number of terms increases. The number required depends on the characteristics of the signal.

Summary

- Fourier series represent a periodic signal, defined by an **uncountable set of values in time**, through a **countable set of Fourier coefficients**, with a small, controllable error. This gives a more compact and efficient description of the signal.
- In all Fourier series forms, an expansion with infinitely many terms represents the signal exactly at continuity points. At discontinuities, it converges to the average of the one-sided limits.

4. Properties of Fourier Series

Properties of Fourier Series (1/5)

1. Linearity: If $x(t) \overset{FS}{\longleftrightarrow} X_k, \quad y(t) \overset{FS}{\longleftrightarrow} Y_k$

then:

$$Ax(t) + By(t) \overset{FS}{\longleftrightarrow} AX_k + BY_k$$

2. Time Shifting: A time shift of a periodic signal changes only the phase of its Fourier coefficients, leaving their magnitudes unchanged.

$$x(t \pm t_0) \overset{FS}{\longleftrightarrow} \sum_{k=1}^{\infty} X_k \, e^{jk\Omega_0(t \pm t_0)} = \sum_{k=1}^{\infty} \left[X_k \, e^{\pm jk\Omega_0 t_0}\right] e^{jk\Omega_0 t}$$

$$x(t \pm t_0) \overset{FS}{\longleftrightarrow} c_0 + \sum_{k=1}^{\infty} c_k \cos(k\Omega_0 t + \theta_k \pm k\Omega_0 t_0)$$

Properties of Fourier Series (2/5)

3. Frequency Shifting:

$$\text{If } x(t) \overset{FS}{\longleftrightarrow} X_k \quad \text{then} \quad e^{jM\Omega_0 t}x(t) \overset{FS}{\longleftrightarrow} X_{k-M}$$

A frequency shift of a periodic signal shifts its coefficients by the integer M corresponding to the frequency shift:

4. Differentiation:

$$\text{If } x(t) \overset{FS}{\longleftrightarrow} X_k \quad \text{then} \quad \frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} jk\Omega_0 X_k$$

5. Integration:

$$\text{If } x(t) \overset{FS}{\longleftrightarrow} X_k \text{ and } y(t) = \int x(t)dt \text{ and provided } X_0 = 0, \qquad \text{then } Y_k = \frac{1}{jk\Omega_0} X_k, \qquad k \neq 0$$

Periodic integration preserves periodicity only when the signal has zero mean ($X_0 = 0$).

Properties of Fourier Series (3/5)

6. Symmetry and Conjugation: Every periodic signal is the sum of an even component $x_e(t)$ and an odd component $x_o(t)$. Its coefficients X_n satisfy:

$$X_k = X_{e,k} + X_{o,k}, \text{ where } X_{e,k} = \frac{1}{2}(X_k + X_{-k}) \text{ and } X_{o,k} = \frac{1}{2}(X_k - X_{-k}).$$

If the periodic signal is **real** and **even**, its coefficients X_k are real. Thus, $X_k = X_{-k}$, and the expansion becomes:

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} X_k \cos(k\Omega_0 t)$$

If the periodic signal is **real** and **odd**, its coefficients X_k are purely imaginary and $X_0 = 0$. The expansion becomes:

$$x(t) = 2j\sum_{k=1}^{\infty} X_k \sin(k\Omega_0 t)$$

Properties of Fourier Series (4/5)

For a real periodic signal $x(t)$, such as a sinusoidal function:

$$X_k = X_{-k}^*$$

This relation reduces the calculations and shows that the **amplitude is even** and the **phase is odd**:

$$|X_k| = |X_{-k}| \quad \text{and} \quad \measuredangle X_k = -\measuredangle X_{-k}$$

Because of the symmetry of X_k for real signals, the **amplitude and phase spectra** need only be plotted for $k \geq 0$.

These relations apply only to real signals, not to complex signals.

Negative frequencies in double-sided amplitude and phase spectra arise from the complex exponent in the definition of X_k.

Properties of Fourier Series (5/5)

For real signals, only positive frequencies need to be displayed, since negative frequencies follow from conjugate symmetry.

Frequency counts repetitions of a phenomenon per unit time, so **negative** frequency has no direct intuitive physical meaning. However, it can be viewed as **clockwise** rotation of the basis vector $e^{-jk\Omega_0 t}$ at rate $k\Omega_0$.

Conversely, $e^{jk\Omega_0 t}$ describes a vector rotating **counterclockwise** at rate $k\Omega_0$, representing positive frequencies.

The **real part** of X_k **is even**, while the imaginary part **is odd**:

$$Re\{X_k\} = Re\{X_{-k}\} \quad \text{and} \quad Im\{X_k\} = -Im\{X_{-k}\}$$

5. Normalized Power of a Fourier Expansion

Normalized Power of a Fourier Expansion

The normalized power of a generally complex signal $x(t)$ in the time domain is:

$$S_x = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt$$

The normalized power for each Fourier series form is:

Trigonometric form I:

$$S_x = a_0^2 + \sum_{k=1}^{+\infty}\left(\frac{a_k^2}{2} + \frac{b_k^2}{2}\right)$$

Trigonometric form II:

$$S_x = c_0^2 + \sum_{k=1}^{+\infty}\left(\frac{c_k^2}{2}\right)$$

Exponential form:

$$S_x = \sum_{k=-\infty}^{+\infty} |X_k|^2$$

The last relation is Parseval's theorem for the exponential Fourier series.

Normalized Power of a Fourier Expansion

- The sequence $\{|X_k|^2\}$ is the **power spectrum** and describes the distribution of signal power over the harmonic frequencies $k\Omega_0, k = 0, \pm 1, \pm 2, \ldots$.
- Because harmonic frequencies are discrete, the plot has one line at each frequency and is called a **power line spectrum**.
- A large value of $|X_k|^2$ means that the corresponding harmonic contributes significantly to the total signal power.
- A periodic signal has all its power concentrated at harmonic frequencies. At nonharmonic frequencies, the power is zero.
- Since X_k are complex, we can define two spectra. The first plots the magnitude $|X_k|$ and is the amplitude line spectrum. The second plots the angle (phase) $\measuredangle X_k$ and is the **phase line spectrum**.
- The power spectrum $|X_k|^2,\ k = 0, \pm 1, \pm 2, \ldots$ is obtained by squaring the amplitude spectrum $|X_k|$,, since $|X_k|^2 = (|X_k|)^2$. It therefore shows directly which harmonics dominate the power.

6. Parseval's Theorem for Periodic Functions

Parseval's Theorem for Periodic Functions

Fourier series describe the distribution of signal power across frequencies through Parseval's theorem:

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$

- The left-hand side calculates power in the time domain over one period. The right-hand side calculates power in the frequency domain.
- A Fourier series expansion preserves the mean signal power. The total mean power equals the sum of the powers of the individual harmonic components.
- For a real periodic signal $x(t)$:

$$X_{-k} = X_k^*$$

- The amplitude is therefore even and the phase is odd:

$$|X_{-k}| = |X_k| \quad \text{and} \quad \measuredangle X_{-k} = -\measuredangle X_k$$

- Because the Fourier coefficients have conjugate symmetry, only the spectrum for $k \geq 0$ need be shown. The part for $k < 0$ follows directly by symmetry.

7. Gibbs Phenomenon

Gibbs Phenomenon

- If $x(t)$ has discontinuities of finite size, its Fourier series or Fourier transform expansion develops **ripples** whose amplitude peaks near the discontinuities.
- This unwanted effect, called the Gibbs phenomenon, arises because a finite sum of continuous sinusoids cannot fully represent an abrupt discontinuity.
- As the number of terms increases, the maximum overshoot does not tend to zero. Instead, it approaches about **9% of the jump height.**
- At a discontinuity, the Fourier expansion converges to the average of the values of $x(t)$ on either side:

$$\frac{x(t_0^-) + x(t_0^+)}{2}$$

- As the number of terms N increases, the oscillations become confined to an increasingly narrow region around the discontinuity.

8. Relationship between Fourier Series and the Fourier Transform

Fourier Series and the Fourier Transform

Let $x(t)$ be a finite-duration signal defined from $-T_0/2$ to $T_0/2$. Take the segment of $x(t)$ over $-T_0/2$ and $T_0/2$ and expand it in an exponential Fourier series of period $T_0 = \Omega_0/2\pi$:

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \, e^{jn\Omega_0 t} \quad -T_0/2 \leq t \leq T_0/2$$

where:

$$X_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \, e^{-jn\Omega_0 t} \, dt$$

The fundamental frequency $\Omega_0 = 2\pi/T_0$ is the spacing between adjacent harmonics in the amplitude spectrum, which consists of discrete spectral components. Substituting the second relation into the first gives:

$$x(t) = \sum_{n=-\infty}^{+\infty} e^{jn\Omega_0 t} \left[\frac{\Omega_0}{2\pi} \int_{-T_0/2}^{T_0/2} x(t) \, e^{-jn\Omega_0 t} \, dt \right]$$

Fourier Series and the Fourier Transform

If $T_0 \to \infty$, then $\Omega_0 = 2\pi/T_0$ tends to $d\Omega$. The spacing between harmonics becomes infinitesimal $d\Omega$, and their number becomes infinite ($n \to \infty$). Each harmonic has frequency Ω, given by $\Omega = n\Omega_0$. Thus, as the period tends to infinity, the previous sum becomes an integral:

$$x(t) = \int_{-\infty}^{+\infty} e^{j\omega t} \left[\frac{d\Omega}{2\pi} \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \right] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\Omega t} \left[\int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \right] d\Omega$$

This is the Fourier integral. Its inner integral is a complex function of Ω, namely the Fourier transform. Thus, we obtain the familiar forward and inverse Fourier transform pair:

$$X(\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\Omega)\, e^{j\Omega t}\, d\Omega$$

The Fourier transform is therefore the limit of the Fourier series as the period tends to infinity ($T_0 \to \infty$).

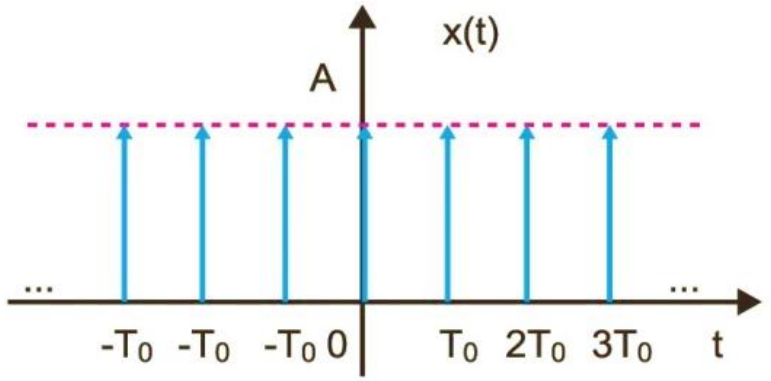
Exercises

Exercise 1

Find the Fourier expansion of an impulse train.

Answer: The impulse train is expressed mathematically as:

$$x(t) = A \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$



- **Coefficients of trigonometric form I:**

$$\alpha_0 = \frac{A}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt = \frac{A}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\,dt = \frac{A}{T_0}$$

$$\alpha_k = \frac{A}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,cos(k\Omega_0 t)\,dt = \frac{2A}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\,cos(k\Omega_0 t)\,dt = \frac{2A}{T_0}$$

$$b_k = \frac{A}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,sin(k\Omega_0 t)\,dt = \frac{2A}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\,sin(k\Omega_0 t)\,dt = 0$$

Exercise 1 (continued)

- **Coefficients of trigonometric form II:**

$$c_0 = a_0 = \frac{A}{T_0}, \qquad c_k = \sqrt{a_k^2 + b_k^2} = \frac{2A}{T_0}, \qquad \theta_k = -\,tan^{-1}\left(\frac{b_k}{a_k}\right) = 0$$

- **Coefficients of the exponential form:**

$$X_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\Omega_0 t}\, dt = \frac{1}{T_0}\int_{T_0/2}^{T_0/2} A\sum_{k=-\infty}^{\infty} \delta(t - kT_0)\, e^{-jk\Omega_0 t}\, dt$$

Only the term $n = 0$ remains within one period, so:

$$X_k = \frac{A}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-jk\Omega_0 t}\, dt = \frac{A}{T_0}$$

Exercise 1 (continued)

- **Signal expansion in trigonometric forms I and II:**

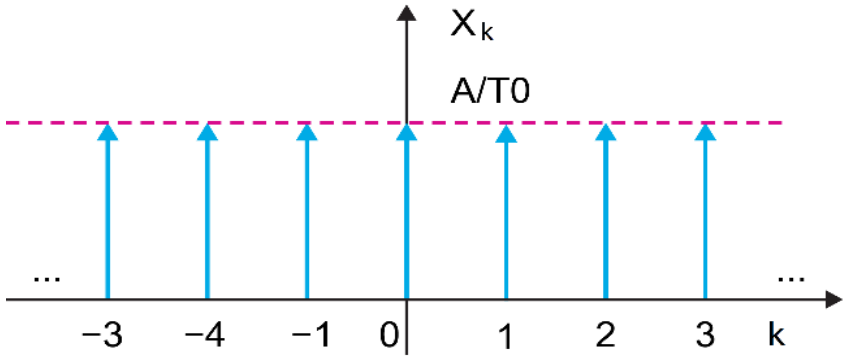
$$x(t) = \frac{A}{T_0} + \frac{2A}{T_0}\sum_{k=1}^{\infty} cos(k\Omega_0 t)$$

- **Signal expansion in exponential form:**

$$x(t) = \frac{A}{T_0}\sum_{k=-\infty}^{\infty} e^{jk\Omega_0 t}$$

The coefficients X_k are real and equal for every k, consistent with the even symmetry of the signal.
The amplitude spectrum is constant and equals:

$$|X_k| = \frac{A}{T_0}$$



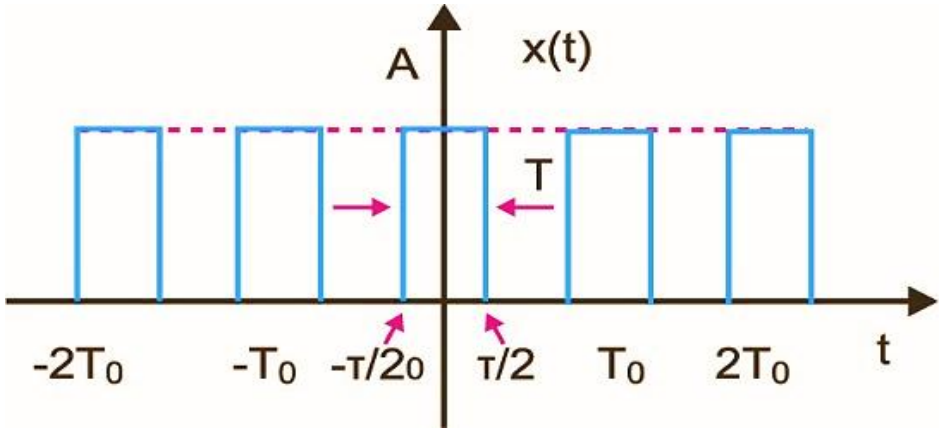
Amplitude spectrum

Exercise 1 (continued)

- The coefficients in all forms are real, as expected from the even symmetry of the signal.
- This is a generalized function rather than an ordinary periodic signal. Thus, power and Parseval's theorem do not apply here in the usual way.
- The delta train has Fourier coefficients $C_k = 1/T_0$, giving a constant amplitude spectrum with all harmonics of equal strength.
- Its even symmetry implies real coefficients and zero phase, a characteristic of even periodic signals.

Exercise 2

Find all Fourier series forms of the pulse train below, where T_0 is the period and τ the pulse duration. Evaluate for $\tau = T_0/2, T_0 = 2\pi$ and $A = 1$.



Answer: Choose the interval $[-T_0/2, T_0/2]$ as one period and calculate the Fourier coefficients using their defining relations.

Exercise 2 (continued)

Exponential form:

$$X_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)dt = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} A\,dt = \frac{A}{T_0}t\Big|_{-\tau/2}^{\tau/2} = \frac{A\tau}{T_0}$$

$$X_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,e^{-jk\Omega_0 t}dt = \frac{A}{T_0}\int_{-\tau/2}^{\tau/2} e^{-jk\Omega_0 t}dt = -\frac{\mathrm{A}}{\mathrm{T_0}}\frac{e^{-jk\Omega_0 t}}{jk\Omega_0}\Bigg|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{\mathrm{A}}{\mathrm{T_0}}\frac{1}{jk\Omega_0}\left[e^{\frac{jk\Omega_0\tau}{2}} - e^{-\frac{jk\Omega_0\tau}{2}}\right]$$

$$= \frac{2A}{T_0}\frac{1}{k\Omega_0}\sin(k\Omega_0\tau)$$

Since $\Omega_0 = 2\pi/T_0$, we have:

$$X_k = \frac{A}{k\pi}sin\left(\frac{k\pi\tau}{T_0}\right), \qquad k \neq 0$$

The exponential Fourier coefficients are therefore:

$$X_0 = \frac{A\tau}{T_0}, X_k = \frac{A}{k\pi}sin\left(\frac{k\pi\tau}{T_0}\right), k \neq 0$$

The Fourier coefficients are real, as expected, because the signal is even.

Exercise 2 (continued)

The ratio of pulse duration (τ) to period (T_0) is the **duty cycle**: the fraction of time for which the signal is active.

We study how the duty cycle affects the frequency spectrum for different values of τ, keeping the period T_0 fixed:

(a) For $\tau = T_0/2,\ T_0 = 2\pi, A = 1$, we have:

$$X_0 = \frac{1}{2} \qquad \text{and} \qquad X_k = \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right),\ k \neq 0$$

For even k, the sine and hence the Fourier coefficients are zero, so $X_k = 0$ for k even. For odd k, the sine takes nonzero values with alternating signs. Thus:

$$X_1 = X_{-1} = \frac{1}{\pi}, \qquad X_2 = X_{-2} = 0, \qquad X_3 = X_{-3} = -\frac{1}{3\pi},$$

$$X_4 = X_{-4} = 0, \qquad X_5 = X_{-5} = \frac{1}{5\pi}, \qquad X_6 = X_{-6} = 0, \ldots$$

Exercise 2 (continued)

(b) For $\tau = T_0/4,\ T_0 = 2\pi, A = 1$, we have:

$$X_0 = \frac{1}{4} \quad \text{and} \quad X_k = \frac{1}{k\pi}\sin\left(\frac{k\pi}{4}\right),\ k \neq 0$$

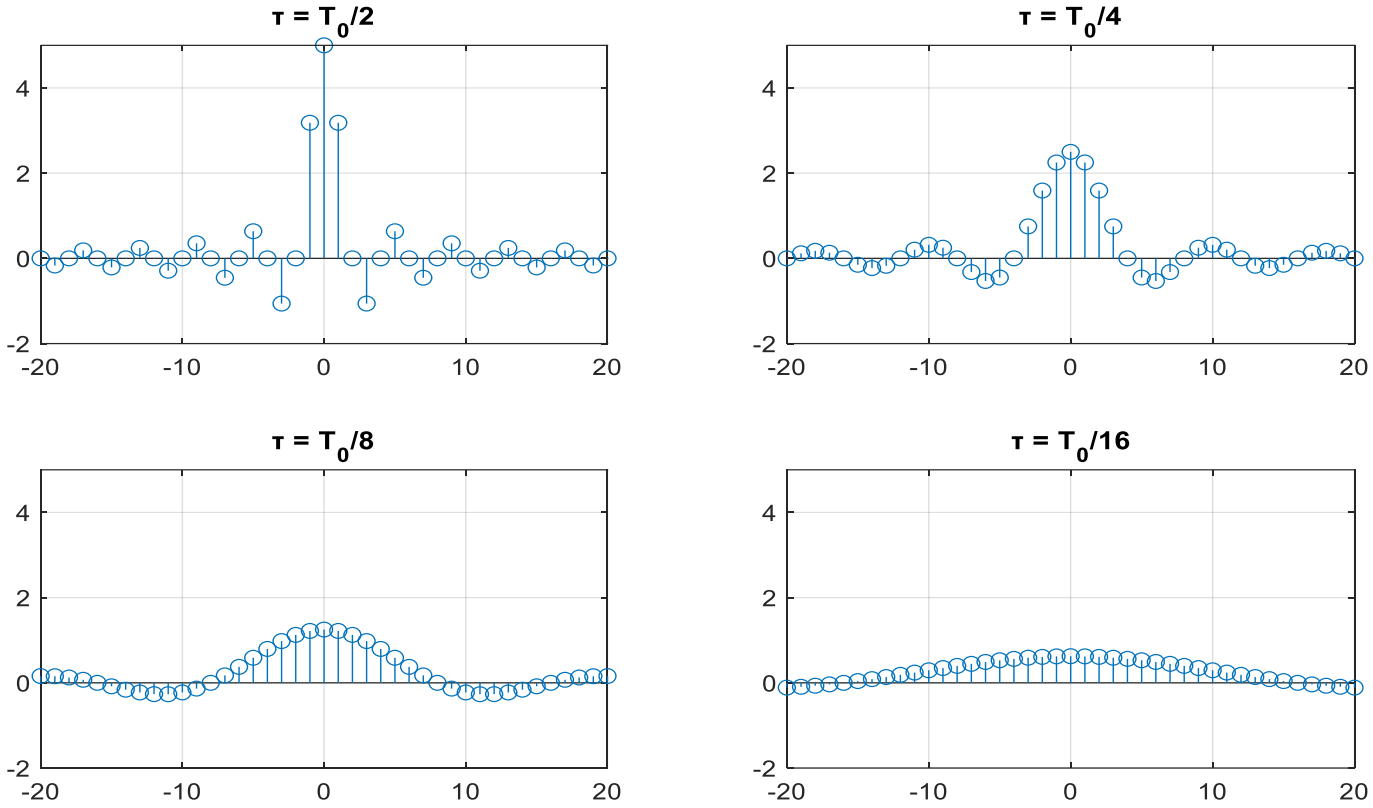
When k is an integer multiple of 4, the sine is zero, so $X_k = 0$ for $k = 4r, r = 1,2,3, \ldots$. In general, X_k are:

$$X_1 = X_{-1} = \frac{\sqrt{2}}{2\pi}, \quad X_2 = X_{-2} = \frac{1}{2\pi}, \quad X_3 = X_{-3} = \frac{\sqrt{2}}{6\pi}, \quad X_4 = X_{-4} = 0,$$

$$X_5 = X_{-5} = -\frac{\sqrt{2}}{10\pi}, \quad X_6 = X_{-6} = -\frac{1}{6\pi}, \quad X_7 = X_{-7} = -\frac{\sqrt{2}}{14\pi}, \quad X_8 = X_{-8} = 0, \ldots$$

The exponential Fourier coefficients can be calculated similarly for other values of τ.

Exercise 2 (continued)



Amplitude spectrum of the pulse train for values of τ:

(a) $\tau = T_0/2$, (b) $\tau = T_0/4$, (c) $\tau = T_0/8$, (d) $\tau = T_0/16$

Exercise 2 (continued)

- For $\tau = T_0/\alpha,\ \alpha \in \mathrm{N}$, spectral coefficients vanish at values of k satisfying $sin(k\pi/\alpha) = 0$, namely $k = \lambda\alpha, \lambda = \pm 1, \pm 2, \ldots$: integer multiples of α.
- The corresponding zero frequencies are $\Omega = k\Omega_0$.. For $k = \lambda\alpha$, this gives $\Omega = 2\pi\lambda/\tau$. Thus, the zero locations are determined by the duration τ.
- Shorter pulses extend the spectrum to higher frequencies.
- In all cases, the Fourier coefficients are equally spaced samples (at spacing $\Omega_0 = 2\pi/T_0$) of the envelope:

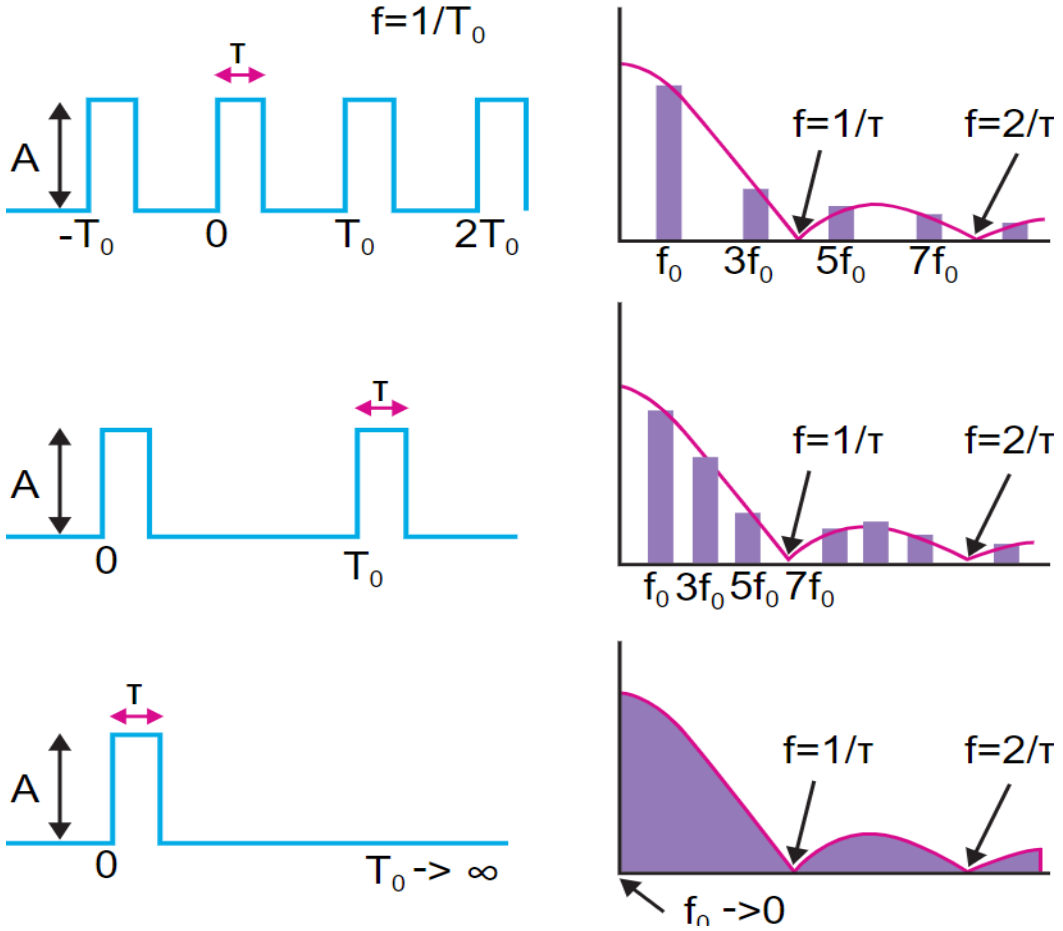
$$\frac{2\mathrm{A}}{\mathrm{T}_0} \cdot \frac{sin(\Omega\tau/2)}{\Omega}$$

- Increasing the period T_0 while keeping the pulse duration τ fixed relative to its period T_0 leaves the envelope unchanged but brings the spectral coefficients closer together, since the fundamental frequency decreases. The harmonics move closer in frequency, making the spectrum denser.

Exercise 2 (continued)

As the time between pulses, or the signal period, tends to infinity ($T_0 \rightarrow \infty$), the spacing between successive spectral coefficients (harmonics) tends to zero.

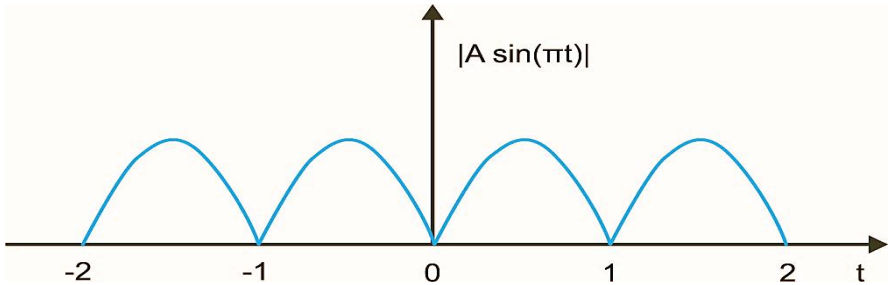
The discrete Fourier series spectrum then becomes continuous, described by the Fourier transform of the isolated pulse.



Amplitude spectrum of a pulse train for different values of the period T_0

Exercise 3

Find the Fourier series of the fully rectified sinusoidal signal and plot its single-sided and double-sided amplitude spectra and its phase spectrum.



Answer: The fully rectified sinusoid is $x(t) = |Asin(\pi t)|$. Its fundamental period is $T_0 = 1\ sec$, its fundamental frequency is $f_0 = 1\ (Hz)$, and its fundamental angular frequency is $\Omega_0 = 2\pi\ (rad/sec)$. Calculate the coefficients of the Fourier series forms:

- **Trigonometric form I:**

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \int_0^1 A\sin(\pi t)\,dt = \left[-\frac{A}{\pi}\cos(\pi t)\right]_0^1 =$$

$$= -\frac{A}{\pi}[\cos(\pi) - \cos(0)] = \frac{2A}{\pi}$$

$$a_k = \frac{2}{T_0}\int_0^{T_0} x(t)\,cos(k\Omega_0 t)\,dt = 2A\int_0^1 \sin(\pi t)\,cos(2k\pi t)\,dt = \ldots$$

Exercise 3 (continued)

$$= 2A\left[-\frac{\cos(\pi - 2k\pi)}{2(\pi - 2k\pi)}t - \frac{\cos(\pi + 2k\pi)}{2(\pi + 2k\pi)}t\right]_0^1 = 2A\left[\frac{\cos(2k\pi - \pi)}{2(2k\pi - \pi)} - \frac{\cos(2k\pi + \pi)}{2(2k\pi + \pi)}\right] = \cdots$$

$$= 2A\left[-\frac{1}{4k\pi - 2\pi} + \frac{1}{4k\pi + 2\pi}\right] = 2A\left[\frac{-4\pi}{(4k\pi)^2 - (2\pi)^2}\right] = -\frac{4A}{\pi(4k^2 - 1)} = \frac{4A}{\pi(1 - 4k^2)}$$

We used $\cos(2k\pi - \pi) = -1$ and $\cos(2k\pi + \pi) = -1,\ \forall t$, and the integral of the sine–cosine product:

$$\int sinPx\ cosQx\ dx = -\frac{\cos(P-Q)x}{2(P-Q)} - \frac{\cos(P+Q)x}{2(P+Q)}, \text{ where } P = \pi \text{ and } Q = 2k\pi.$$

The coefficients b_k are calculated below. They are zero because the function is even:

$$b_k = \frac{2}{T_0}\int_0^{T_0} x(t)sin(k\Omega_0 t)\ dt = \cdots = 0$$

Exercise 3 (continued)

- **Trigonometric form II:**

$$c_0 = a_0 = \frac{2A}{\pi}$$

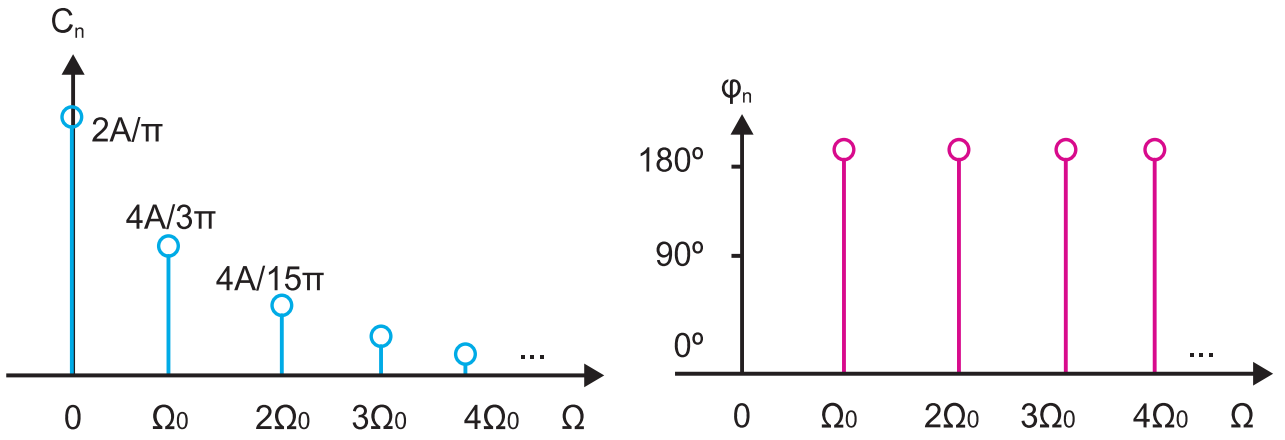
$$c_k = \sqrt{a_k^2 + b_k^2} = |\alpha_k| = \frac{4A}{\pi(4k^2 - 1)}$$

$$\theta_k = \tan^{-1}\left(\frac{b_k}{a_k}\right) = 0^0 = 180^0$$

From trigonometric form II, we can plot the single-sided amplitude spectrum $c_k = c_k(\Omega_k)$ and the phase spectrum $\theta_k = \theta_k(\Omega_k)$:

Thus, the expansion of $x(t)$ in trigonometric form II is:

$$x(t) = \frac{2A}{\pi} + \frac{4A}{\pi}\sum_{k=1}^{\infty}\frac{\cos(k\Omega_0 t)}{(1-4k^2)}, k \neq 0$$



Exercise 3 (continued)

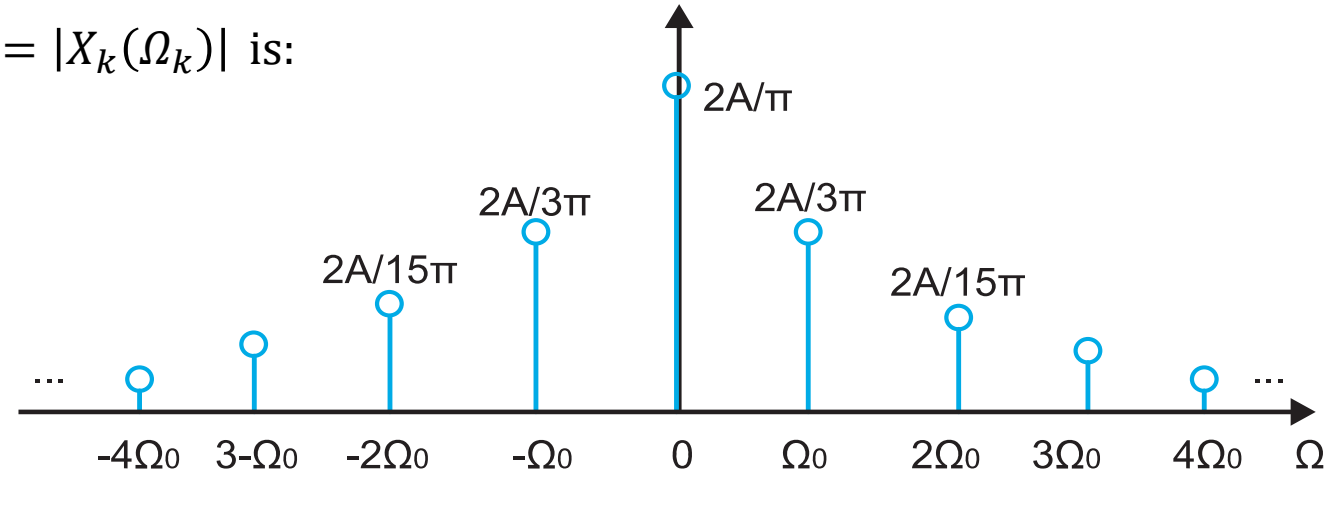
- **Exponential form:**

$$|X_0| = \alpha_0 = \frac{2A}{\pi} \quad \text{and} \quad |X_k| = \frac{1}{2}(\alpha_k - jb_k) = \frac{\alpha_k}{2} = \frac{2A}{\pi(4k^2-1)}$$

$$\theta_k = \measuredangle X_k = -\tan^{-1}\left(\frac{b_k}{a_k}\right) = 0^0 = 180^0, \qquad k \neq 0$$

The phase spectrum $\theta_k = \theta_k(\Omega_k)$ is the same as before.

The double-sided amplitude spectrum $|X_k| = |X_k(\Omega_k)|$ is:



The expansion of $x(t)$ in exponential form is:

$$x(t) = \frac{2A}{\pi} + \sum_{k=-\infty}^{\infty} \frac{2A}{\pi(1-4k^2)} e^{j2\pi kt}, k \neq 0$$

Exercise 3 (continued)

- Full rectification halves the sinusoid period, doubling its fundamental frequency. The spectrum contains a DC component and only even harmonics of the original frequency.
- The fully rectified sinusoid is even, so its Fourier series contains only cosine terms ($b_k = 0$).
- The positive signal mean produces a strong DC component.
- The cosine coefficients are negative, so the harmonics in trigonometric form II have phase π.
- The spectral coefficient magnitude decreases with harmonic order. Thus, a few harmonics give a satisfactory approximation.
- The signal is real and even, so its Fourier coefficients are real and symmetric ($X_{-k} = X_k$). The amplitude spectrum is therefore even, and the phase is either 0 or π.
- Conclusion: Rectification significantly changes the spectral structure, introducing a DC component and new harmonics.

Exercise 4

Using the double-sided amplitude spectrum of the previous example, calculate each harmonic's absolute power and its percentage of the total power.

Answer: Calculate the power spectrum coefficients $|X_k|^2,\ k = 0, \pm1, \pm2, \ldots$ to find the absolute power of each harmonic. The power spectrum is the square of the amplitude spectrum:

- For $k = 0 : P_0 = |X_0|^2 = \left(\frac{2A}{\pi}\right)^2 = \frac{4A^2}{\pi^2}$
- For $k = 1 : P_1 = |X_1|^2 + |X_{-1}|^2 = 2\left(\frac{2A}{3\pi}\right)^2 = \frac{8A^2}{9\pi^2}$
- For $k = 2 : P_2 = |X_2|^2 + |X_{-2}|^2 = 2\left(\frac{2A}{15\pi}\right)^2 = \frac{8A^2}{225\pi^2}$

Exercise 4 (continued)

Calculate the remaining harmonic terms similarly. To express each harmonic's power as a percentage of the total, first calculate the total power using Parseval's theorem:

$$P_{total} = \frac{1}{T}\int_0^T x^2(t)\,dt = \frac{1}{1}\int_0^1 A^2 \sin^2(2\pi t)\,dt = \frac{A^2}{2}$$

The percentage of power in the DC term is therefore:

$$P'_0 = \frac{P_0}{P_{total}}(\%) = \frac{\frac{4A^2}{\pi^2}}{\frac{A^2}{2}}(\%) = \frac{8}{\pi^2} \cong 81.06\%$$

Similarly, the percentages for the remaining terms are:

$$P'_k = \frac{P_k}{P_{total}} = \frac{2|X_k|^2}{P_{total}}, \qquad n = 1,2 \ldots$$

The first harmonic contains approximately: $P'_1 = \frac{16}{9\pi^2} \approx 18.01\%$

The second harmonic: $P'_2 = \frac{16}{225\,\pi^2} \approx 0.72\%$

The third harmonic: $P'_3 = \frac{16}{1225\,\pi^2} \approx 0.13\%$

Exercise 4 (continued)

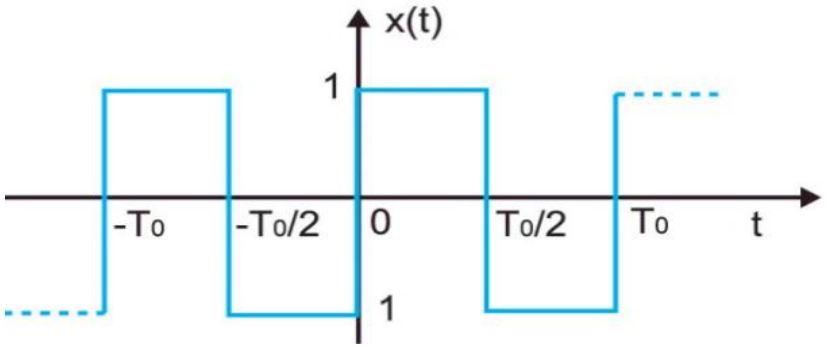
- The rectified sinusoid's power is not distributed uniformly across the harmonics. The DC component (81.06%) and first harmonic (18.01%) contain 99.07% of the total power. Higher harmonics contribute progressively less.
- The power spectrum is strongly concentrated at low frequencies. Thus, a small number of Fourier terms can describe the signal satisfactorily in terms of power.
- Since the signal is real and even, its exponential Fourier coefficients are symmetric in $k = 0$. The components at $+ k\Omega_0$ and $-k\Omega_0$ carry equal power. The total power of the kth harmonic, for $k \geq 1$, is therefore the sum of $|X_k|^2$ and $|X_{-k}|^2$.
- The total power calculated in the time domain agrees with the sum of the spectral component powers, as Parseval's theorem states. This demonstrates the equivalence of time-domain and frequency-domain descriptions of signal power.
- The rapid decrease in higher-harmonic power shows that high frequencies contribute little to the overall waveform. This explains why the first few harmonics usually approximate the rectified sinusoid well.

Exercise 5

Calculate trigonometric form I of the Fourier series of the periodic pulse train:

$$x(t) = \begin{cases} -1, & -T_0/2 < t < 0 \\ 1, & 0 < t < T_0/2 \end{cases} \quad \text{and} \quad x(t) = x(t + T_0)$$

Answer: The periodic function $x(t)$ is plotted below:



The coefficients a_0, a_k, b_k are calculated as:

Exercise 5 (continued)

$$\boldsymbol{a_0} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt = \frac{1}{T_0}\left[\int_{-T_0/2}^{0}(-1)dt + \int_{0}^{T_0/2}(1)\,dt\right] = \frac{1}{T_0}\left[-\frac{T_0}{2} + \frac{T_0}{2}\right] = 0$$

$$\boldsymbol{a_k} = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(k\Omega_0 t)\,dt = \frac{2}{T_0}\left[\int_{-T_0/2}^{0}(-1)\cos(k\Omega_0 t)\,dt + \int_{0}^{T_0/2}(1)\cos(k\Omega_0 t)\,dt\right]$$

$$= \frac{2}{T_0}\left[-\frac{1}{k\Omega_0}\sin(k\Omega_0 t)\Big|_{-T_0/2}^{0} + \frac{1}{k\Omega_0}\sin(k\Omega_0 t)\Big|_{0}^{T_0/2}\right] = 0$$

$$\boldsymbol{b_k} = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(k\Omega_0 t)\,dt = \frac{2}{T_0}\left[\int_{-T_0/2}^{0}(-1)\sin(k\Omega_0 t)\,dt + \int_{0}^{T_0/2}(1)\sin(k\Omega_0 t)\,dt\right]$$

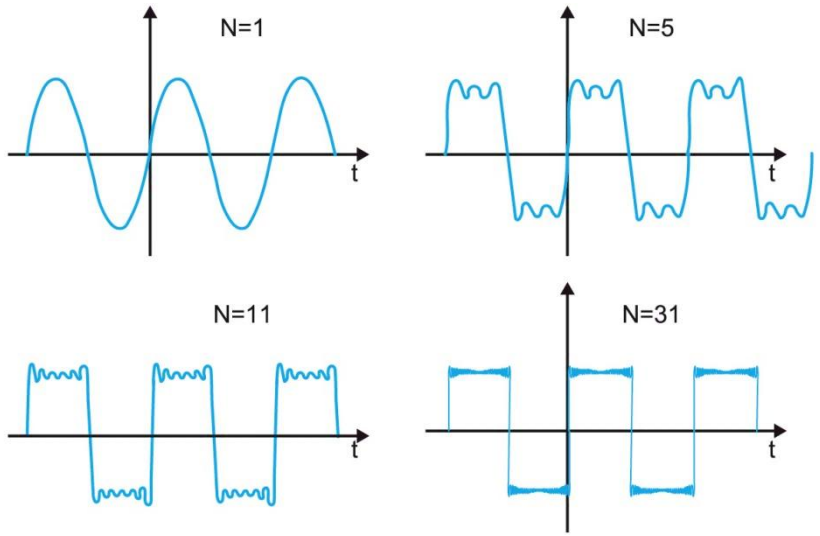
$$= \frac{2}{k\Omega_0 T_0}\left[\cos(k\Omega_0 t)\Big|_{-T_0/2}^{0} - \cos(k\Omega_0 t)\Big|_{0}^{T_0/2}\right] = \frac{2}{k\pi}(1-\cos(k\pi)) = \begin{cases} 0, & k \text{ even} \\ 4/k\pi, & k \text{ odd} \end{cases}$$

The Fourier series expansion of $x(t)$ is:

$$x(t) = \frac{4}{\pi}\left(\sin k\Omega_0 t + \frac{1}{3}\sin 3k\Omega_0 t + \frac{1}{5}\sin 5k\Omega_0 t + \cdots\right)$$

Exercise 5 (continued)

The figure shows successive approximations to $x(t)$ using the first $N = 1, 5, 11, 31$ harmonics.



- The approximation improves as N increases.
- However, oscillations remain at the discontinuities $k(T_0/2),\ k = \pm 1, \pm 2, \ldots$ because of the Gibbs phenomenon. Adding more terms does not eliminate them, but confines them to a narrower region.
- The approximation quality depends on the waveform: sharper transitions produce slower convergence.