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SIGNALS AND SYSTEMS
Solved Examples – Tutorial Sheet 7

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LAPLACE TRANSFORM

- Properties of the Laplace transform
- Inverse Laplace transform

1. Properties of the Laplace Transform

📖 Example 1

The signal $x(t) = e^{-2t}u(t)$ has the bilateral Laplace transform:

$$X(s) = \frac{1}{s+2}, \qquad Re(s) > -2$$

Find the bilateral Laplace transform of the signal $y(t) = x(-3t)$ and determine its region of convergence.

Answer: We use the scaling property of the bilateral Laplace transform:

$$x(at) \overset{L}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

Here $a = -3$, so:

$$Y(s) = \frac{1}{|-3|} X\left(\frac{s}{-3}\right) = \frac{1}{3} X\left(\frac{s}{-3}\right) = \frac{1}{3} \frac{1}{-\frac{s}{3}+2} = \frac{1}{3} \frac{3}{(-s+6)} = \frac{1}{6-s}$$

For the region of convergence:

$$Re\left(\frac{s}{-3}\right) > -2 \Rightarrow Re(s) < 6$$

Direct interpretation in the time domain:

$$y(t) = e^{6t}u(-t)$$

The signal is left-sided, which is consistent with a left half-plane ROC.

Comments:

- Time scaling by a negative factor combines two operations: scaling and time reversal. This directly affects the ROC as well.
- Scaling changes both the algebraic expression of the transform and the geometry of the region of convergence. For $a < 0$, the ROC is also reversed.
- This example helps avoid the common mistake of mechanically applying the formula $X(s/a)$ without checking the new ROC.

📖 Example 2

Find the Laplace transform of the signal $x(t) = t,\ 0 \le t \le 1$.

Answer: We write the signal $x(t) = t,\ 0 \le t \le 1$ as follows:

$$x(t) = t[u(t) - u(t-1)] = t\,u(t) - t\,u(t-1)$$

We know that the Laplace transform of the unit step function is:

$$u(t) \xrightarrow{L} \frac{1}{s}, \qquad R_u = Re(s) > 0$$

The time-shifting property of the Laplace transform gives:

$$u(t-t_0) \xrightarrow{L} e^{-st_0}\frac{1}{s}, \qquad R'_u = R_u = Re(s) > 0$$

For $t_0 = 1$, we have:

$$u(t-1) \xrightarrow{L} e^{-s}\frac{1}{s}, \qquad R'_u = R_u = Re(s) > 0$$

Therefore, the Laplace transform of the signal $y(t) = u(t) - u(t-1)$ is:

$$Y(s) = \frac{1}{s} - e^{-s}\frac{1}{s} = \frac{1}{s}(1 - e^{-s}), \qquad R_y = Re(s) > -\infty$$

From the differentiation property in the complex-frequency domain of the Laplace transform, we know that:

$$x(t) = t\,y(t) \xrightarrow{L} -\frac{dY(s)}{ds} = X(s), \qquad R_x = R_y = Re(s) > -\infty$$

Calculating the first derivative of the function:

$$Y(s) = \frac{1}{s}(1 - e^{-s})$$

we obtain:

$$X(s) = -\frac{dY(s)}{ds} = \cdots = \frac{e^{-s}(e^{s} - s - 1)}{s^2}, \qquad R_x = Re(s) > -\infty$$

📖 Example 3
Find the unilateral Laplace transform of the signal $x(t) = t\,cos(\Omega t)\,u(t)$.

Answer: Applying the differentiation property in the complex-frequency domain of the Laplace transform to the given signal, namely:

$$t\,x(t) \xrightarrow{L} -\frac{dX(s)}{ds}$$

we obtain:

$$L\{t\,cos(\Omega t)\} = -\frac{d}{ds}\{L\{cos(\Omega t)\}\}$$

Since $cos(\Omega t) \xrightarrow{L} s/(s^2 + \Omega^2)$, we obtain:

$$t\,cos(\Omega t) \xrightarrow{L} -\frac{d}{ds}\left(\frac{s}{s^2+\Omega^2}\right) = -\frac{(s^2+\Omega^2) - 2s^2}{(s^2+\Omega^2)^2} = \frac{s^2-\Omega^2}{(s^2+\Omega^2)^2}, Re(s) > 0$$

Comments:

- Multiplying a signal by t in the time domain corresponds to differentiating its Laplace transform with respect to s, with a negative sign.
- The differentiation property in the complex-frequency domain saves us the laborious calculation of the integral $\int t\,cos(\Omega t)\;e^{-st}dt$ using integration by parts.
- The region of convergence remains $Re(s) > 0$, because the factor t does not change the exponential order of the signal; it only introduces polynomial growth.

📖 Example 4
Find the unilateral Laplace transform of the signal $x(t) = t^2 cos(\Omega t)\,u(t)$.

Answer: From the previous example, we know that:

$$t\,cos(\Omega t)u(t) \xrightarrow{L} \frac{s^2-\Omega^2}{(s^2+\Omega^2)^2}, \quad Re(s) > 0$$

Using the property:

$$L\{t^n x(t)\} = (-1)^n \frac{d^n X(s)}{ds^n}$$

for $n = 2$, we obtain:

$$L\{t^2 cos(\Omega t)\,u(t)\} = \frac{d^2}{ds^2}\left(\frac{s}{s^2+\Omega^2}\right)$$

We calculate the derivatives. First derivative:

$$\frac{d}{ds}\left(\frac{s}{s^2+\Omega^2}\right) = \frac{\Omega^2 - s^2}{(s^2+\Omega^2)^2}$$

Second derivative:

$$\frac{d}{ds}\left(\frac{\Omega^2 - s^2}{(s^2+\Omega^2)^2}\right) = \frac{2s(s^2-3\Omega^2)}{(s^2+\Omega^2)^3}$$

Alternative solution:

$$L\{t^2 cos(\Omega t)\, u(t)\} = -\frac{d}{ds}\{L\{t\, cos(\Omega t)\}\} = -\frac{d}{ds}\left[\frac{s^2-\Omega^2}{(s^2+\Omega^2)^2}\right] = \cdots$$

$$= \frac{2s(s^2-3\Omega^2)}{(s^2+\Omega^2)^3}, \qquad Re(s) > 0$$

Comments:

- Multiplying a signal by t^n in the time domain corresponds to the n −th derivative of its Laplace transform with respect to s, with a factor of $(-1)^n$.
- For $n = 2$, the sign becomes positive. Therefore, we calculate the second derivative of $s/(s^2 + \Omega^2)$ directly here.
- The region of convergence remains $Re(s) > 0$, because the factor t^2 grows polynomially, while convergence is still determined by the exponential factor e^{-st} in the Laplace transform.

📖 Example 5

Find, where applicable, the initial and final values of the signals corresponding to the transforms:

$$X_1(s) = \frac{2s+5}{s(s+1)(s+2)}$$

and

$$X_2(s) = \frac{\Omega}{s^2+\Omega^2}$$

Check whether the final value theorem can be applied in both cases.

Answer: For $X_1(s)$, the initial value theorem gives:

$$x_1(0^+) = \lim_{s\to\infty} sX_1(s) = \lim_{s\to\infty} \frac{2s+5}{(s+1)(s+2)} = 0$$

From the final value theorem:

$$\lim_{t\to\infty} x_1(t) = \lim_{s\to 0}\, sX_1(s) = \lim_{s\to 0} \frac{2s+5}{(s+1)(s+2)} = \frac{5}{2}$$

The poles of $sX_1(s)$ lie in the left half-plane, so the final value theorem applies.

For $X_2(s)$:

$$x_2(0^+) = \lim_{s\to\infty} sX_2(s) = \lim_{s\to\infty} \frac{s\Omega}{s^2+\Omega^2} = 0$$

If the final value theorem is applied, we obtain:

$$\lim_{s\to 0} sX_2(s) = \lim_{s\to 0} \frac{s\Omega}{s^2+\Omega^2} = 0$$

However, the theorem cannot be applied here, because the poles of $sX_2(s)$ lie on the imaginary axis. Indeed:

$$x_2(t) = \sin(\Omega t)\, u(t)$$

and this signal has no final value, because it oscillates indefinitely.

Comments:

- The final value theorem does not always apply. It applies only under specific conditions on the poles of $sX(s)$, which we must check.
- Correct use of the final value theorem is crucial when studying the steady state of circuits and control systems.

2. Inverse Laplace Transform

📖 Example 6

Find the inverse Laplace transform of the function:

$$X(s) = \frac{s^4+8s^3+15s^2-6s-15}{s^3+2s^2-s-2}$$

with region of convergence $-1 < \sigma < 1$.

Answer: Since the degree of the numerator polynomial exceeds that of the denominator, we first perform polynomial division to reduce the numerator's degree below that of the denominator. We have:

$$X(s) = \frac{s^4+8s^3+15s^2-6s-15}{s^3+2s^2-s-2} = \cdots = s+6+\frac{4s^2+2s-3}{s^3+2s^2-s-2}$$
$$= s+6+\frac{4s^2+2s-3}{(s+1)(s-1)(s+2)}$$

The roots ($\lambda_1 = -1$, $\lambda_2 = 1, \lambda_3 = -2$) are simple and real, so the fraction can be decomposed into partial fractions:

$$\frac{C_1}{s+1} + \frac{C_2}{s-1} + \frac{C_2}{s+2}$$

We calculate the constants C_1, C_2 and C_3 using equation (3.50), obtaining:

$$C_1 = (s+1)\frac{4s^2+2s-3}{(s+1)(s-1)(s+2)}\Big|_{s=-1} = \frac{4s^2+2s-3}{(s-1)(s+2)}\Big|_{s=-1} = \frac{1}{2}$$

$$C_2 = (s-1)\,\frac{4s^2+2s-3}{(s+1)(s-1)(s+2)}\Big|_{s=1} = \frac{4s^2+2s-3}{(s+1)(s+2)}\Big|_{s=1} = \frac{1}{2}$$

$$C_3 = (s+2)\frac{4s^2+2s-3}{(s+1)(s-1)(s+2)}\Big|_{s=-2} = \frac{4s^2+2s-3}{(s+1)(s-1)}\Big|_{s=-2} = 3$$

Therefore, the partial-fraction expansion of the given function $X(s)$ is:

$$X(s) = s + 6 + \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s-1} + 3\cdot\frac{1}{s+2}$$

We now use the region of convergence $-1 < \sigma < 1$ to determine the form and duration of the time-domain signals. Since the poles are located at $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = -2$, the given region of convergence can be written as:

$$R_x = \{-1 < \sigma < 1\} = \{\sigma > -1\} \cap \{\sigma < 1\} \cap \{\sigma > -2\}$$

The inverse Laplace transform is the sum of the inverse transforms of the individual fractions. Two individual regions of convergence extend to the right and one extends to the left, as determined by the position of the ROC relative to the corresponding poles. Thus:

$$x(t) = \delta'(t) + 6\delta(t) + \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{t}u(-t) + 3e^{-2t}u(t)$$

or, equivalently:

$$x(t) = \delta'(t) + 6\delta(t) + \left(\frac{1}{2}e^{-t} + 3e^{-2t}\right)u(t) - \frac{1}{2}e^{t}u(-t)$$

We emphasize that the region of convergence (extending to the right or left) helps us determine whether the individual signal components are right-sided or left-sided, respectively.

Comments:

- When $X(s)$ is an improper rational function, polynomial division must be performed first. Polynomial terms in s correspond to singular terms in time, such as $\delta(t)$ and $\delta'(t)$.
- The algebraic form of $X(s)$ alone is insufficient to determine the time-domain signal. The region of convergence determines whether each exponential component is right-sided or left-sided.
- In this example, the ROC $-1 < Re(s) < 1$ yields a mixed signal: right-sided terms from the poles -1 and -2, a left-sided term from the pole 1, and singular terms resulting from polynomial division.

📖 Example 7

Find the inverse Laplace transform of the function:

$$X(s) = \frac{2s+3}{s^2+2s+4}$$

with region of convergence $\sigma > -1$.

Answer: Factoring the denominator polynomial gives the complex-conjugate roots $\lambda_1 = -1 + j\sqrt{3}$ and $\lambda_2 = -1 - j\sqrt{3}$. We have $\lambda_2 = \lambda_1^*$. Therefore, the function $X(s)$ can be written as:

$$X(s) = \frac{C_1}{s-\lambda_1} + \frac{C_1^*}{s-\lambda_1^*}$$

The coefficient C_1 is calculated using equation (3.57):

$$C_1 = (s-\lambda_1)X(s)|_{s=\lambda_1} = (s-\lambda_1)\frac{2s+3}{(s-\lambda_1)(s-\lambda_1^*)}\bigg|_{s=\lambda_1} = \frac{2s+3}{(s-\lambda_1^*)}\bigg|_{s=\lambda_1}$$

$$= \frac{2\lambda_1+3}{\lambda_1-\lambda_1^*} = \cdots = 1-\frac{j}{2\sqrt{3}} = \sqrt{\frac{13}{12}}e^{-j\varphi}$$

Clearly:

$$C_1^* = 1+\frac{j}{2\sqrt{3}} = \sqrt{\frac{13}{12}}e^{j\varphi}$$

where:

$$\varphi = tan^{-1}\left(\frac{1}{2\sqrt{3}}\right)$$

Since the real part of the poles is -1 and the region of convergence lies to the right of the poles, both signal components are right-sided. Therefore, the inverse Laplace transform is:

$$x(t) = \left[C_1 e^{\lambda_1 t} + C_1^* e^{\lambda_1^* t}\right] u(t)$$

$$= \left[\left(1-\frac{j}{2\sqrt{3}}\right)e^{(-1+j\sqrt{3})t} + \left(1+\frac{j}{2\sqrt{3}}\right)e^{(-1-j\sqrt{3})t}\right]u(t)$$

$$= e^{-t}\left[\left(1-\frac{j}{2\sqrt{3}}\right)e^{j\sqrt{3}t} + \left(1+\frac{j}{2\sqrt{3}}\right)e^{-j\sqrt{3}t}\right]u(t)$$

Let:

$$\theta = \sqrt{3}t$$

Then:

$$x(t) = e^{-t}\left[\left(1-\frac{j}{2\sqrt{3}}\right)e^{j\theta} + \left(1+\frac{j}{2\sqrt{3}}\right)e^{-j\theta}\right]u(t)$$

Expanding:

$$e^{-t}\left[\left(e^{j\theta}+e^{-j\theta}\right) - \frac{j}{2\sqrt{3}}e^{j\theta} + \frac{j}{2\sqrt{3}}e^{-j\theta}\right]u(t)$$

We use the identities:

$$e^{j\theta}+e^{-j\theta} = 2cos\theta \quad \text{and} \quad e^{-j\theta}-e^{j\theta} = -2jsin\theta$$

so:

$$-\frac{j}{2\sqrt{3}}e^{j\theta} + \frac{j}{2\sqrt{3}}e^{-j\theta} = \frac{j}{2\sqrt{3}}\left(e^{-j\theta}-e^{j\theta}\right) = \frac{j}{2\sqrt{3}}(-2jsin\theta) = \frac{1}{\sqrt{3}}sin\theta$$

Thus:

$$x(t) = e^{-t}\left(2cos\theta + \frac{1}{\sqrt{3}}sin\theta\right)u(t)$$

and since $\theta = \sqrt{3}t$, we arrive at the result:

$$x(t) = e^{-t}\left(2\cos\left(\sqrt{3}t\right) + \frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t\right)\right)u(t)$$

Comments:

- When the poles occur in complex-conjugate pairs $\lambda_{1,2} = -1 + j\sqrt{3}$, the system response is oscillatory. The oscillation frequency is determined by the imaginary part $(\sqrt{3})$ of the poles, while the real part (-1) creates the exponential envelope (e^{-t}) that determines how quickly the oscillation decays.
- Partial-fraction decomposition with complex numbers is mathematically correct, but also time-consuming. Alternatively, we can complete the square in the denominator:

$$X(s) = \frac{2s+3}{s^2+2s+4} = \frac{2s+3}{(s+1)^2+3} = \frac{2(s+1)+1}{(s+1)^2+\left(\sqrt{3}\right)^2}$$

Splitting the fraction immediately gives the forms of an exponentially damped cosine and sine:

$$X(s) = 2\frac{(s+1)}{(s+1)^2+\left(\sqrt{3}\right)^2} + \frac{1}{(s+1)^2+\left(\sqrt{3}\right)^2}$$

The inverse transform immediately gives exactly the same solution.

📖 Example 8

Find the inverse Laplace transform of the function:

$$X(s) = \frac{1}{(s+a)(1-e^{-Ts})^2},\ \alpha{>}0 \text{ and } T{>}0$$

Answer: We use the expansion:

$$\frac{1}{(1-z)^2} = \sum_{k=0}^{\infty}(k+1)\,z^k, \qquad |z|<1$$

Let $z = e^{-Ts}$, so:

$$\frac{1}{(1-e^{-Ts})^2} = \sum_{k=0}^{\infty}(k+1)\,e^{-kTs}$$

Thus:

$$X(s) = \sum_{k=0}^{\infty}(k+1)\frac{e^{-kTs}}{s+a}$$

For the region of convergence $Re(s) > 0$, we have:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}u(t)$$

and with a time delay:

$$\mathcal{L}^{-1}\left\{\frac{e^{-kTs}}{s+a}\right\} = e^{-a(t-kT)}u(t-kT)$$

Therefore:

$$x(t) = \sum_{k=0}^{\infty} (k+1)\, e^{-a(t-kT)} u(t-kT)$$

For the series to converge, we require:

$$|e^{-Ts}| < 1 \Rightarrow Re(s) > 0$$

Since the term $1/(s+a)$ requires $Re(s) > -a$ (for $a > 0$), the common region of convergence is $Re(s) > 0$.

Comments:

- Delays in the denominator often lead to infinite series of time-shifted copies of a basic signal.
- The power $(1 - e^{-T_s})^{-2}$ produces coefficients $(k+1)$, so the result is not a simple train of delayed copies, but a sequence of delayed terms with linearly increasing weights.
- Similar forms appear in models of multiple reflections or repeated excitations with progressively increasing contributions.

Note: Examples marked with an asterisk (*) also appear in the theory lecture slides.