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DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Signals and Systems

Lecture 7: Properties of the Laplace Transform
The Inverse Laplace Transform

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Lecture Contents

1. Properties of the Laplace transform
2. Symmetries of the Laplace transform
3. Convolution theorems in time and frequency
4. Initial and final value theorems
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    - Heaviside method
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1. Properties of the Laplace Transform

Properties of the Laplace Transform

1. Linearity
2. Time shifting
3. Complex frequency shifting
4. Scaling in time and complex frequency
5. Complex conjugation
6. Differentiation
7. Integration
8. Differentiation in complex frequency
9. Integration in complex frequency

Properties of the Laplace Transform (1/6)

1. Linearity

If $x_1(t) \xrightarrow{L} \mathrm{X}_1(s)$ has region of convergence (ROC) R_1, and $x_2(t) \xrightarrow{L} \mathrm{X}_2(s)$ has ROC R_2, then for every α_1, α_2:

$$a_1 x_1(t) + a_2 x_2(t) \xrightarrow{L} a_1 X_1(s) + a_2 X_2(s) \quad \text{with ROC } R' \supseteq R_1 \cap R_2$$

2. Time Shifting

If $x(t) \xrightarrow{L} X(s)$ has ROC R, then for every t_0:

$$x(t - t_0) \xrightarrow{L} e^{-st_0} X(s) \quad \text{with the same ROC } R' = R$$

Properties of the Laplace Transform (2/6)

3. Complex Frequency Shifting

If $x(t) \xrightarrow{L} X(s)$ has ROC R, then for every s_0:

$$x(t)\, e^{s_0 t} \xrightarrow{L} X(s - s_0) \qquad \text{with ROC } R' = R + Re(s_0)$$

For $s_0 = j\Omega_0$, this relation becomes:

$$x(t)\, e^{j\Omega_0 t} \xrightarrow{L} X(s - j\Omega_0) \qquad \text{with the same ROC } R' = R$$

4. Scaling in Time and Complex Frequency

If $x(t) \xrightarrow{L} X(s)$ has ROC R, then for every $b > 0$:

$$x(bt) \xrightarrow{L} \frac{1}{|b|} X\left(\frac{s}{b}\right) \qquad \text{with ROC } R' = R/b$$

For $b = -1$, the signal is reversed (reflected), and its Laplace transform is:

$$x(-t) \xrightarrow{L} X(-s) \qquad \text{with ROC } R' = -R$$

Properties of the Laplace Transform (3/6)

5. Complex Conjugation

If $x(t) \xrightarrow{L} X(s)$ has ROC R, then the conjugate function $x^*(t)$ satisfies:

$$x^*(t) \xrightarrow{L} X^*(s^*) \text{ with the same ROC } R$$

This property implies that a real signal has either real poles and zeros or complex conjugate pairs of poles and zeros.

Properties of the Laplace Transform (4/6)

6. Differentiation

If $x(t) \xrightarrow{L} X(s)$ has ROC R, and $x(t)$ is piecewise continuous with a piecewise continuous derivative on every interval $0 \leq t \leq T$, then:

$$\frac{dx(t)}{dt} \xrightarrow{L} sX(s) - x(0^-) \qquad \text{with the ísame ROC } R' = R$$

For the nth derivative of $x(t)$, the property extends to:

$$\frac{d^n x(t)}{dt^n} \xrightarrow{L} s^n X(s) - \sum_{k=0}^{n-1} s^{n-1-k} \left.\frac{d^k x(t)}{dt^k}\right|_{t=0^-} \qquad \text{with the ísame ROC } R' = R$$

With zero initial conditions, the differentiation property is the same for the unilateral and bilateral Laplace transforms:

$$\frac{d^n x(t)}{dt^n} \xrightarrow{L} s^n X(s) \quad \text{with the ísame ROC } R' = R$$

Differentiation and linearity turn a differential equation into a simple polynomial, making it easier to solve.

Properties of the Laplace Transform (5/6)

7. Integration

If $x(t) \xrightarrow{L} X(s)$, then:

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xrightarrow{L} \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)\, d\tau \qquad \text{with ROC } R' = R \cap \{Re(s) > 0\}$$

With zero initial conditions, the integration property is the same for the unilateral and bilateral Laplace transforms.

Properties of the Laplace Transform (6/6)

8. Differentiation in Complex Frequency

If $x(t) \xrightarrow{L} X(s)$ has ROC R, and $x(t)$ is piecewise continuous with a piecewise continuous derivative on every interval $0 \leq t \leq T$, then:

$$-t\,x(t) \xrightarrow{L} \frac{dX(s)}{ds} \qquad \text{with ROC } R' = R$$

Extending to higher derivatives gives:

$$(-t)^n\,x(t) \xrightarrow{L} \frac{d^n X(s)}{ds^n} \qquad \text{with ROC } R' = R$$

9. Integration in Complex Frequency

If $x(t) \xrightarrow{L} X(s)$, then:

$$\frac{1}{t}x(t) \xrightarrow{L} \int_0^\infty X(u)du \qquad \text{with ROC } R' = R$$

2. Symmetries of the Laplace Transform

Symmetries of the Laplace Transform

1. Symmetry between differentiation and integration

2. Symmetry between time shifting and frequency shifting

3. Symmetry between time compression/expansion and reflection

Symmetry between Differentiation and Integration

For a causal signal $x(t)$ with $x(0^-) = 0$ and Laplace transform $x(t) \overset{L}{\longleftrightarrow} X(\mathrm{s})$, the following pairs hold:

$$\frac{dx(t)}{dt} \overset{L}{\longleftrightarrow} s\, X(\mathrm{s})$$

$$t\, x(t) \overset{L}{\longleftrightarrow} -\frac{dX(\mathrm{s})}{ds}$$

We observe that:

- Multiplication by s of a function $X(\mathrm{s})$ in the complex frequency domain is equivalent to taking the first derivative in the time domain.
- Multiplying the time function $x(t)$ by t is equivalent to differentiating $-X(\mathrm{s})$.

Symmetry between Differentiation and Integration

Similarly, for integration:

$$\int_{0^-}^{t} x(\tau)d\tau \overset{L}{\longleftrightarrow} \frac{X(\mathrm{s})}{s}$$

$$\frac{x(t)}{t} \overset{L}{\longleftrightarrow} \int_{-\infty}^{-s} X(-\lambda)d\lambda$$

Thus:

- Division by s of a function $X(\mathrm{s})$ in the complex frequency domain is equivalent to integrating the signal in the time domain.
- Division by t of the time function $x(t)$ is equivalent to integrating $X(\mathrm{s})$.

Symmetry between Time and Frequency Shifting

For a signal $x(t)$ with $x(t) \overset{L}{\longleftrightarrow} X(\mathrm{s})$, the following pairs hold:

$$x(t-a)\,u(t-a) \overset{L}{\longleftrightarrow} X(\mathrm{s})\,e^{-as}$$

$$x(t)\,e^{-at} \overset{L}{\longleftrightarrow} X(\mathrm{s}+\alpha)$$

- These relations show that a shift in one domain (time or frequency) corresponds to multiplication by an exponential term in the other domain, and vice versa.
- This symmetry helps calculate the Laplace transform without using its integral definition.

Symmetry between Time Scaling and Reflection

For a signal $x(t)$ with $x(t) \overset{L}{\longleftrightarrow} X(\mathrm{s})$, the following pairs hold:

$$x(bt)\, u(t) \overset{L}{\longleftrightarrow} \left(\frac{1}{|b|}\right) X\left(\frac{s}{b}\right)$$

$$\left(\frac{1}{|b|}\right) x\left(\frac{s}{b}\right)\, u(t) \overset{L}{\longleftrightarrow} X(\alpha s)$$

These relations show the inverse relationship between the time and frequency domains. For example, compression in time corresponds to expansion in frequency, and vice versa. For $b = -1$, in particular:

$$x(-t)\, u(t) \overset{L}{\longleftrightarrow} X(-s)$$

If x(t) is anticausal, its Laplace transform equals the transform of the causal signal $x(-t)u(t)$, with s replaced by −s.

3. Convolution Theorems in Time and Frequency

Convolution in Time

If $x_1(t) \xrightarrow{L} \mathrm{X}_1(s)$ with ROC R_1 and $x_2(t) \xrightarrow{L} \mathrm{X}_2(s)$ with ROC R_2, then the convolution $x_1(t) * x_2(t)$ has Laplace transform:

$$x_1(t) * x_2(t) \xrightarrow{L} \mathrm{X}_1(s)\, X_2(s) \qquad \text{with ROC } R' = R_1 \cap R_2$$

Convolution in Complex Frequency

If $x_1(t) \xrightarrow{L} \mathrm{X}_1(s)$ with ROC R_1 and $x_2(t) \xrightarrow{L} \mathrm{X}_2(s)$ with ROC R_2, then the product $x_1(t)\, x_2(t)$ has Laplace transform:

$$x_1(t)\, x_2(t) \xrightarrow{L} \frac{1}{2\pi j} X_1(s) * X_2(s)$$

Conversely, the property can be written as:

$$X_1(s) * X_2(s) \xrightarrow{L^{-1}} 2\pi j\, [\, x_1(t)\, x_2(t)]$$

4. Initial and Final Value Theorems

Initial and Final Value Theorems

To calculate $x(t)$ for $t \to 0^+$ and $t \to \infty$, use the initial and final value theorems.

Initial Value Theorem: Let $x(t)$ be a causal signal with no impulses at $t = 0$, Laplace transform $X(s)$, and ROC $Re(s) > s_0$. Then:

$$\lim_{s\to\infty} s\,X(s) = \lim_{t\to 0^+} x(t) = x(0)$$

Final Value Theorem: For a causal signal $x(t)$ with a finite limit as $t \to \infty$:

$$\lim_{s\to 0} s\,X(s) = \lim_{t\to\infty} x(t) = x(\infty)$$

These theorems require the following conditions:

- **Initial value theorem**: The signal must contain no impulse at $t = 0$.
- **Final value theorem**: All poles of $sX(s)$ must lie in the left half-plane. Equivalently, $X(s)$ may have at most one simple pole at $s = 0$.

The final value theorem therefore fails if $sX(s)$ has (a) a higher-order pole at $s = 0$, (b) poles on the imaginary axis, or (c) poles in the right half-plane.

Properties of the Laplace Transform

Property	Time Domain	Frequency Domain	Region of Convergence
Bilateral Laplace Transform			
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$	subset of $R_x \cap R_y$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R_x
Frequency shifting	$e^{s_0t}x(t)$	$X(s - s_0)$	Shifted R_x
Convolution	$x_1(t) * x_2(t)$	$X_1(s)\,X_2(s)$	subset of $R_x \cap R_y$
Time conjugation	$x^*(t)$	$X^*(s^*)$	R_x
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}\,X\left(\frac{s}{a}\right)$	Scaled R_x
Reflection	$x(-t)$	$X(-s)$	$-R_x$
Time integration	$\int_{-\infty}^{\infty} x(\tau)d\tau$	$\frac{1}{s}\,X(s)$	subset of $R_x \cap \{Re(s) > 0\}$
Time differentiation	$\frac{dx(t)}{dt}$	$sX(s)$	subset of R_x
Time differentiation	$\frac{d^nx(t)}{dt^n}$	$s^nX(s)$	subset of R_x
Frequency differentiation	$-t\,x(t)$	$dX(s)/ds$	R_x
Unilateral Laplace Transform			
Time integration	$\int_{-\infty}^{\infty} x(\tau)d\tau$	$\frac{1}{s}\,X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)d\tau$	subset of $R_x \cap \{Re(s) > 0\}$
Time differentiation	$dx(t)/dt$	$sX(s) - x(0^-)$	subset of R_x

Exercises

Exercise 1

Calculate the Laplace transform (LT) of $x(t) = e^{-t}u(t) + e^{-2t}u(t)$.

Answer: The LT of $e^{-at}\,u(t)$ is $1/(s+a)$, with ROC $Re(s) > -a$. Thus, the LT of each term of $x(t)$ is:

$$e^{-t}\,u(t) \xrightarrow{L} \frac{1}{s+1}, \qquad Re(s) > -1$$

and:

$$e^{-2t}\,u(t) \xrightarrow{L} \frac{1}{s+2}, \qquad Re(s) > -2$$

By linearity:

$$X(s) = \frac{1}{s+1} + \frac{1}{s+2} = \frac{2s+3}{s^2+3s+2}$$

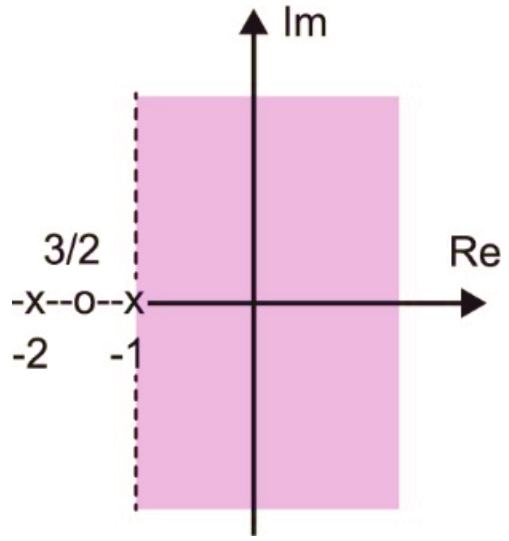
The ROC is the intersection of the individual ROCs: $Re(s) > -1$.

Exercise 1 (continued)

The rational form of the Laplace transform shows:

- One zero at $Re(s) = -3/2$.
- Two poles at $Re(s) = -1$ and $Re(s) = -2$.

Since the transform poles lie
in the left half of the complex plane,
the system is **stable.**



Exercise 2

Calculate the Laplace transform of $x(t) = \sum_{n=0}^{\infty} \delta\,(t - nt_0),\ t_0 > 0$.

Answer: The time-shifting property gives:

$$x(t - nt_0) \xrightarrow{L} e^{-nst_0} X(s)$$

We know that $\delta(t) \xrightarrow{L} 1$, with the entire complex frequency plane as the ROC. Thus:

$$\delta(t - nt_0) \xrightarrow{L} e^{-snt_0} X(s) = e^{-snt_0}$$

with the same ROC. By linearity of the Laplace transform:

$$x(t) = \sum_{n=0}^{\infty} \delta\,(t - nt_0) \xrightarrow{L} 1 + e^{-st_0} + e^{-2st_0} + \cdots + e^{-nst_0} \Rightarrow X(s) = \sum_{n=0}^{\infty} e^{-nst_0}$$

This infinite geometric series converges when $|e^{-st_0}| < 1$, that is, when $Re(s) > 0$. In this case:

$$X(s) = \frac{1}{1 - e^{-st_0}}, \qquad Re(s) > 0$$

Exercise 3

Calculate the Laplace transform of $x(t) = [1 - (1 - t)e^{-3t}]\, u(t)$.

Answer: Write the given function as the sum $x(t) = x_1(t) + x_2(t) + x_3(t)$, where:

$$x_1(t) = u(t), x_2(t) = -e^{-3t}u(t), \text{ and } x_3(t) = te^{-3t}u(t)$$

The Laplace transforms of these functions are:

$$X_1(s) = \frac{1}{s}, \qquad Re(s) > 0$$

$$X_2(s) = \frac{1}{s+3}, \qquad Re(s) > -3$$

$$X_3(s) = \frac{1}{(s+3)^2}, \qquad Re(s) > -3$$

Exercise 3 (continued)

By linearity, the LT is:

$$X(s) = \frac{1}{s} + \frac{1}{s+3} + \frac{1}{(s+3)^2} = \frac{4s+9}{s(s+3)^2}$$

The ROC is the intersection of the three individual ROCs, so $Re(s) > 0$.

The result shows that the given function $x(t)$ has:

- One zero at $s = -9/4$.
- One simple pole at $s = 0$.
- One double pole at $s = -3$.

Since the poles lie in the left half-plane (and at zero), we conclude that the function describes a **stable** system.

Exercise 3 (continued)

- The ROC follows the strictest condition. Some terms require $Re(s) > 0$ and others require $Re(s) > -3$. The region where all terms converge is their **intersection**: $Re(s) > 0$.
- For a causal LTI system to be **BIBO stable**, the ROC of $H(s)$ must include the imaginary axis $\boldsymbol{j\Omega}$. Equivalently, all poles must lie strictly in the left half-plane.
- Here, the pole at $s = 0$ lies on the imaginary axis, so the ROC starts to its right: $Re(s) > 0$.
- The pole at zero, arising from $u(t)$, indicates that the system acts partly as an integrator. A constant bounded input makes an integrator output accumulate continuously and tend to infinity. This unbounded output shows intuitively why the system is **not BIBO stable**.

Exercise 4

Calculate the Laplace transform of $x(t) = t\cos(\Omega t)\,u(t)$.

Answer: Use differentiation in complex frequency:

$$-t\,x(t) \xrightarrow{L} \frac{dX(s)}{ds}$$

to obtain:

$$L\{t\cos(\Omega t)\} = -\frac{d}{ds}\{L\{\cos(\Omega t)\}\}$$

The table of known LTs gives $cos(\Omega t) \xrightarrow{L} s/(s^2 + \Omega^2)$, so:

$$t\cos(\Omega t) \xrightarrow{L} -\frac{d}{ds}\left(\frac{s}{s^2+\Omega^2}\right) = -\frac{(s^2+\Omega^2)-2s^2}{(s^2+\Omega^2)^2} = \frac{s^2-\Omega^2}{(s^2+\Omega^2)^2}, \qquad Re(s) > 0$$

Exercise 4 (continued)

- Multiplication by t in the time domain corresponds to differentiating the Laplace transform with respect to s, with a negative sign.
- Differentiation in complex frequency avoids evaluating the integral $\int t\ cos(\Omega t)\ e^{-st}dt$ by integration by parts.
- The ROC remains $Re(s) > 0$, because t introduces polynomial growth without changing the exponential order of the signal.

Exercise 5

Calculate the Laplace transform of $y(t) = x(t-2) * g(3-t)$, where $x(t) = e^{-2t}u(t)$ and $g(t) = e^{-4t}u(t)$.

Answer: By the convolution property:

$$y(t) = x(t-2) * g(3-t) \Rightarrow L\{y(t)\} = L\{x(t-2)\}\, L\{g(3-t)\}$$

We have:

$$X(s) = \frac{1}{s+2}, \qquad Re(s) > -2$$

$$G(s) = \frac{1}{s+4}, \qquad Re(s) > -4$$

Applying the time-shifting property gives:

$$L\{x(t-2)\} = \frac{e^{-2s}}{s+2}, \qquad Re(s) > -2$$

$$L\{g(3-t)\} = \frac{e^{-3s}}{4-s}, \qquad Re(s) < 4$$

Exercise 5 (continued)

By the convolution property:

$$y(t) = x(t-2) * g(3-t) \Rightarrow L\{y(t)\} = L\{x(t-2)\}\, L\{g(3-t)\}$$

Thus:

$$Y(s) = \frac{e^{-2s}}{s+2} \cdot \frac{e^{-3s}}{4-s} = -\frac{e^{-5s}}{(s+2)(s-4)}, \qquad -2 < Re(s) < 4$$

- The original signal $g(t)$ is right-sided, so its ROC is to the right of the pole $Re(s) > -4$.
- Time reversal in $g(3-t)$ makes the signal left-sided, reversing the inequality to $Re(s) < 4$.
- The intersection $-2 < Re(s) < 4$ forms a closed strip in the complex plane, showing that the output $y(t)$ is two-sided, extending from $-\infty$ to $+\infty$.

Exercise 6

Calculate the value of $x(t) = e^{-t}u(t) + e^{-3t}sin(2t)\,u(t)$ at $t = 0$.

Answer: By linearity and Table 3.1:

$$e^{-t}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+1}, \qquad Re(s) > -1$$

$$e^{-3t}sin(2t)\,u(t) \overset{L}{\longleftrightarrow} \frac{2}{(s+3)^2 + 2^2}, \qquad Re(s) > -3$$

Thus:

$$X(s) = \frac{1}{s+1} + \frac{2}{s^2+6s+13} = \cdots = \frac{s^2+8s+15}{s^3+7s^2+19s+13}, \qquad Re(s) > -1$$

Exercise 6 (continued)

Using the final value theorem and applying L'Hôpital's rule three times gives:

$$x(0^-) = \lim_{s\to\infty} s\,X(s) = \lim_{s\to\infty} \frac{s^3 + 8s^2 + 15s}{s^3 + 7s^2 + 19s + 13} = \lim_{s\to\infty} \frac{s^3(1 + 8s^{-1} + 15s^{-2})}{s^3(1 + 7s^{-1} + 19s^{-2} + 13s^{-3})}$$

$$= \lim_{s\to\infty} \frac{(1 + 8s^{-1} + 15s^{-2})}{(1 + 7s^{-1} + 19s^{-2} + 13s^{-3})} = \frac{1}{1} = 1$$

The answer agrees with $x(0^+)$, , the signal value immediately after $t = 0$:

$$x(0+) = e^{-0}u(0) + e^{0}sin(0)\,u(0) = 1 + 0 = 1$$

- The initial value theorem $x(0^+) = \lim_{s\to\infty} s\,X(s)$ shows the symmetry between time and frequency. To find the signal value at its onset ($t = 0^+$), examine its behavior at infinite frequency ($s \to \infty$).
- The onset of a signal, an abrupt transition from zero to a given value, involves rapid change and is therefore dominated by very high frequencies.

Exercise 7

Where permitted, calculate the initial and final values of the signals with transforms:

$$\text{(a) } X_1(s) = \frac{2s+5}{s(s+1)(s+2)}, \qquad \text{(b) } X_2(s) = \frac{\Omega}{s^2+\Omega^2}$$

Check whether the final value theorem applies in both cases.

Answer: For $X_1(s)$, the initial value theorem gives:

$$x_1(0^+) = \lim_{s\to\infty} sX_1(s) = \lim_{s\to\infty} \frac{2s+5}{(s+1)(s+2)} = 0$$

From the final value theorem:

$$\lim_{t\to\infty} x_1(t) = \lim_{s\to 0} sX_1(s) = \lim_{s\to 0} \frac{2s+5}{(s+1)(s+2)} = \frac{5}{2}$$

The poles of $sX_1(s)$ lie in the left half-plane, so the final value theorem applies.

Exercise 7 (continued)

For $X_2(s)$:

$$x_2(0^+) = \lim_{s\to\infty} sX_2(s) = \lim_{s\to\infty} \frac{s\Omega}{s^2+\Omega^2} = 0$$

Applying the final value theorem would give:

$$\lim_{s\to 0} sX_2(s) = \lim_{s\to 0} \frac{s\Omega}{s^2+\Omega^2} = 0$$

However, the theorem does not apply because the poles of $sX_2(s)$ lie on the imaginary axis. Indeed:

$$x_2(t) = \sin(\Omega t)\, u(t)$$

This signal has no final value because it oscillates indefinitely.

Caution: The final value theorem requires specific conditions on the poles of $sX(s)$, . Always check these conditions.

5. The Inverse Laplace Transform

Inverse Laplace Transform

The inverse LT returns from complex frequency s to time t. Symbolically:

$$x(t) = L^{-1}\{X(s)\}$$

Its mathematical expression is:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, dt$$

- This contour integral, the Bromwich integral, is evaluated along a line $Re(s) = c$ parallel to the imaginary axis of the complex plane.
- The inverse LT integral converges when this line lies to the right of all transform poles: the real constant c must exceed the real parts of all poles of $X(s)$.

Methods for Evaluating the Inverse LT

Several methods can evaluate the inverse Laplace transform:

1) **Complex integration**: Evaluate a contour integral along a suitable path in the complex plane.
2) Power series expansion: Use this when the Laplace transform can be expressed as an infinite power series.
3) **Partial fraction expansion**: The most widely used method for evaluating the inverse Laplace transform, presented in detail in the next section.
4) Heaviside's theorem.

Partial Fraction Expansion

Partial Fraction Expansion (1/7)

This method applies to transforms that can be written in **rational form**:

$$X(s) = B(s)/A(s)$$

It uses LT linearity to express X(s) as a linear combination of elementary LTs:

$$X(s) = \alpha_1 X_1(s) + \alpha_2 X_2(s) + \cdots + \alpha_N\, X_N(s) = \sum_{n=1}^{N} \alpha_n X_n(s)$$

The functions $X_n(s), n = 1,2 \ldots N$ are considered LTs of elementary functions of the form $x_n(t)$, $n = 1,2 \ldots n$, which may correspond to familiar transform pairs.

Once $X_n(s)$ are found, determine $x_n(t)$ and then the signal $x(t)$ from:

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) + \cdots + \alpha_N\, x_N(t) = \sum_{n=1}^{N} \alpha_n x_n(t)$$

Partial Fraction Expansion (2/7)

The LT of a signal often has rational form:

$$X(s) = \frac{B(s)}{A(s)} = \frac{b_M S^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

$\alpha_n, b_m \in R$, so the system is LTI with a real impulse response.

For the system to be realizable, $X(s)$ must be strictly proper: $M < N$, where M is the degree of the numerator $B(s)$ and N the degree of the denominator $A(s)$. The Laplace transform then contains no terms such as s, s^2, so the time function contains no impulses or discontinuous functions.

If $M \geq N$, first perform polynomial long division, then apply the method. Write $X(s)$ as:

$$X(s) = g_0 + g_1 s \ldots + g_k s^k + \frac{\tilde{B}(s)}{A(s)}$$

Partial Fraction Expansion (3/7)

Next, expand the rational expression into partial fractions. The fraction $\tilde{B}(s)/A(s)$ is strictly proper and can now be expanded.

If λ_n are the roots of $A(s)$, it can be written as:

$$\mathrm{A}(s) = \prod_{n=1}^{N}(s - \lambda_n)$$

Depending on the roots of $A(s)$, distinguish:

1)Distinct real roots

2)Repeated real roots

3)Complex roots

Partial Fraction Expansion (4/7)

(1) Distinct Real Roots

Write $X(s)$ as:

$$\mathrm{X}(s) = \frac{C_1}{s-\lambda_1} + \cdots + \frac{C_N}{s-\lambda_N}$$

The coefficients $C_1, \ldots, C_N$ are given by:

$$C_n = \lim_{s \to \lambda_n} (s-\lambda_n)\, X(s)$$

Using known LT pairs and transform properties, the inverse Laplace transform $x(t)$ is:

$$x(t) = L^{-1}\{X(s)\} = \left[C_1 e^{\lambda_1 t} + \cdots + C_N e^{\lambda_N t}\right] u(t)$$

Partial Fraction Expansion (5/7)

(2) Repeated Real Roots

Suppose λ_1 has multiplicity r, while all other roots are distinct. Factor the denominator as:

$$\mathrm{A}(s) = (s-\lambda_1)^r \prod_{i=r+1}^{N} (s-\lambda_i)$$

Then $X(s)$ can be written as:

$$X(s) = \frac{C_1}{s-\lambda_1} + \frac{C_2}{(s-\lambda_1)^2} + \cdots + \frac{C_r}{(s-\lambda_1)^r} + \frac{C_{r+1}}{s-\lambda_{r+1}} + \cdots + \frac{C_N}{s-\lambda_N}$$

The coefficients $C_i, i = 1, \ldots r$ for the repeated root are:

$$C_i = \lim_{s\to\lambda_i} \frac{1}{(r-i)!} \frac{d^{r-i}}{ds^{r-i}} \left[(s-\lambda_1)^r\, X(s)\right]\Big|_{s=\lambda_i}, \qquad i = 1, \ldots, r$$

Partial Fraction Expansion (6/7)

Calculate the remaining coefficients $C_i, \quad i = r+1, \ldots, n$ as in case (1).

Once all coefficients $C_i, i = 1, \ldots n$ are known, the inverse Laplace transform is:

$$x(t) = L^{-1}\{X(s)\} =$$

$$= \left[C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t} + \cdots + C_r \frac{t^{r-1}}{(r-1)!} e^{\lambda_1 t} + C_{r+1} e^{\lambda_{r+1} t} + \cdots + C_N e^{\lambda_N t} \right] u(t)$$

Partial Fraction Expansion (7/7)

(3) Complex Roots

If $A(s)$ has complex conjugate roots $\lambda_1 = \sigma + j\Omega$ and $\lambda_2 = \lambda_1^* = \sigma - j\Omega$, write $X(s)$ as:

$$\mathrm{X}(s) = \frac{C_1}{s-\lambda_1} + \frac{C_1}{s-\lambda_2} + \cdots + \frac{C_N}{s-\lambda_N} = \frac{C_1}{s-\lambda_1} + \frac{C_1}{s-\lambda_1^*} + \cdots + \frac{C_N}{s-\lambda_N}$$

The coefficients $C_1, \ldots, C_N$ are given by:

$$C_n = \lim_{s\to\lambda_n} (s-\lambda_n)\, X(s)\Big|_{s=\lambda_i}$$

The coefficients C_1 and C_2 satisfy $C_2 = C_1^*$, and the inverse Laplace transform is:

$$x(t) = L^{-1}\{X(s)\} = \left[C_1 e^{\lambda_1 t} + C_1^* e^{\lambda_1^* t} + \cdots + \sum_{i=3}^{N} C_i e^{\lambda_i t}\right] u(t)$$

Complex conjugate roots in a signal Laplace transform correspond to sinusoidal terms multiplied by e^{st} in the time signal.

Heaviside Method

Heaviside Method for Simple Poles

If the Laplace transform is in rational form:

$$X(s) = \frac{B(s)}{A(s)}$$

and the degree N of $A(s)$ is greater ($N > M$) than the degree M of $B(s)$, while $A(s)$ has N distinct roots $\lambda_1, \lambda_2, \dots, \lambda_N$,, then the inverse Laplace transform is:

$$x(t) = \sum_{n=1}^{N} \frac{B(\lambda_n)}{A'(\lambda_n)} e^{\lambda_n t}\, u(t)$$

This relation gives the residues at the simple poles, the partial fraction coefficients:

$$X(s) = \sum_{n=1}^{N} \frac{B(\lambda_n)}{A'(\lambda_n)} \frac{1}{s - \lambda_n}$$

Terms with Exponential Delays

Terms with Exponential Delays

Use the time-shifting property: if $x(t) \xrightarrow{L} X(\mathrm{s})$ has ROC R, then for every t_0:

$$x(t - t_0) \xrightarrow{L} e^{-st_0}X(s) \quad \text{with the same ROC } R' = R$$

(a) Exponential terms in the numerator: If the numerator contains e^{as}, write the rational function as:

$$X(s) = \frac{B(s)(1 - e^{as})}{A(s)} = \frac{B(s)}{A(s)} - e^{as}\frac{B(s)}{A(s)}$$

Apply partial fractions or the Heaviside method to find the inverse LT $x_1(t)$ of the first fraction.

By time shifting, the inverse LT of the second fraction is $x_2(t) = x_1(t - a)$.. By linearity, the complete inverse transform of $X(s)$ is:

$$x(t) = x_1(t) + x_2(t) = x_1(t) + x_1(t - \alpha)$$

Terms with Exponential Delays

(b) Exponential terms in the denominator: Use the infinite series:

$$\sum_{i=0}^{\infty}(e^{-as})^i = \frac{1}{1-e^{-as}} \quad \text{and} \quad \sum_{i=0}^{\infty}(-e^{-as})^i = \frac{1}{1+e^{-as}}$$

and distinguish two cases:

(b1) Case $1 - e^{-as}$

For the transform:

$$X(s) = \frac{B(s)}{A(s)(1-e^{-as})} = \frac{B(s)}{A(s)} + \frac{B(s)e^{-as}}{A(s)} + \frac{B(s)e^{-2as}}{A(s)} + \cdots$$

if $f(t)$ is the inverse Laplace transform of $B(s)/A(s)$, then:

$$x(t) = f(t) + f(t-\alpha) + f(t-2\alpha) + \cdots$$

Terms with Exponential Delays

(b2) Case $1 + e^{-as}$

For the transform:

$$X(s) = \frac{B(s)}{A(s)(1 + e^{-as})} = \frac{B(s)}{A(s)} - \frac{B(s)e^{-as}}{A(s)} + \frac{B(s)e^{-2as}}{A(s)} + \cdots$$

if $f(t)$ is the inverse transform of $B(s)/A(s)$, then:

$$x(t) = f(t) - f(t - \alpha) + f(t - 2\alpha) - f(t - 3a) + \cdots$$

In both cases, $x(t)$ is a combination of time-shifted copies of the basic signal $f(t)$.

Exercises

Exercise 8

Calculate the inverse Laplace transform of:

$$X(s) = \frac{3s + 7}{s^2 + 4s + 3}$$

Consider the ROCs (a) $\sigma < -3$ and (b) $\sigma > -1$.

Answer: Factor the denominator:

$$s^2 + 4s + 3 = (s + 1)(s + 3)$$

Thus, $X(s)$ becomes:

$$X(s) = \frac{C_1}{s + 1} + \frac{C_2}{s + 3}$$

Calculate the constants C_1 and C_2:

$$C_1 = \lim_{s \to -1} [(s + 1)X(s)] = \lim_{s \to -1} \frac{3s + 7}{s + 3} = 2$$

$$C_2 = \lim_{s \to -3} [(s + 3)\, X(s)] = \lim_{s \to -3} \frac{3s + 7}{s + 1} = 1$$

Exercise 8 (continued)

Thus:

$$X(s) = \frac{2}{s+1} + \frac{1}{s+3}$$

The inverse transform is the sum of the inverse transforms of the individual fractions. Use the given ROC to determine them.

(a) For the ROC $\boldsymbol{\sigma < -3}$, the signal is left-sided. With poles at $\lambda_1 = -1$ and $\lambda_2 = -3$, write the ROC as:

$$R_x = \{\sigma < -3\} = \{\sigma < -1\} \cap \{\sigma < -3\}$$

Both individual ROCs are left-sided. Adding the inverse transforms of the fractions gives:

$$x(t) = -2e^{-t}u(-t) - e^{-3t}u(-t) = -[2e^{-t} + e^{-3t}]\, u(-t)$$

Exercise 8 (continued)

(b) For the ROC $\boldsymbol{\sigma > -1}$, the signal is right-sided. Given the pole locations, write the ROC as:

$$R_x = \{\sigma > -1\} = \{\sigma > -1\} \cap \{\sigma > -3\}$$

Both individual ROCs are right-sided, so the inverse Laplace transform is:

$$x(t) = 2e^{-t}u(t) + e^{-3t}u(t) = [2e^{-t} + e^{-3t}]\, u(t)$$

The ROC is essential for determining whether each signal component is left-sided or right-sided.

Exercise 8 (continued)

Verify the result using the initial value theorem:

$$\lim_{s\to\infty} s\,X(s) = x(0^+)$$

For case (b), the left-hand side is:

$$\lim_{s\to\infty} s\,X(s) = \lim_{s\to\infty} \frac{3s^2+7s}{s^2+4s+3} = \lim_{s\to\infty} \frac{3+7/s}{1+4/s+3/s^2} = 3$$

and the right-hand side is:

$$x(0^+) = [2e^{-t}+e^{-3t}]\,u(t) = 3$$

Thus, the result is correct.

Exercise 8 (continued)

Alternative ways to calculate C_1 and C_2:

(a) Evaluate $X(s)$ at two values of s, for example $s = 0$ and $s = 1$:

$$s = 0 \qquad X(0) = C_1 + \frac{1}{3}C_2$$

$$s = 1 \qquad X(1) = \frac{1}{2}C_1 + \frac{1}{4}C_2$$

This gives two linear equations in the two unknowns C_1 and C_2. Cramer's rule yields $C_1 = 2$ and $C_2 = 1$.

Exercise 8 (continued)

(b) Put the fractions over a common denominator:

$$X(s) = \frac{C_1}{s+1} + \frac{C_2}{s+3} = \frac{s(C_1 + C_2) + 3C_1 + C_2}{s^2 + 4s + 3} \equiv \frac{3s + 7}{s^2 + 4s + 3}$$

The last two fractions are equal and have the same denominator, so their numerators are equal:

$$s(C_1 + C_2) + 3C_1 + C_2 = 3s + 7$$

Thus, $C_1 + C_2 = 3$ and $C_2 = 1$.

Exercise 9

Calculate the impulse response $h(t)$ of an LTI system with transfer function:

$$H(s) = \frac{1}{s^2(s-1)}$$

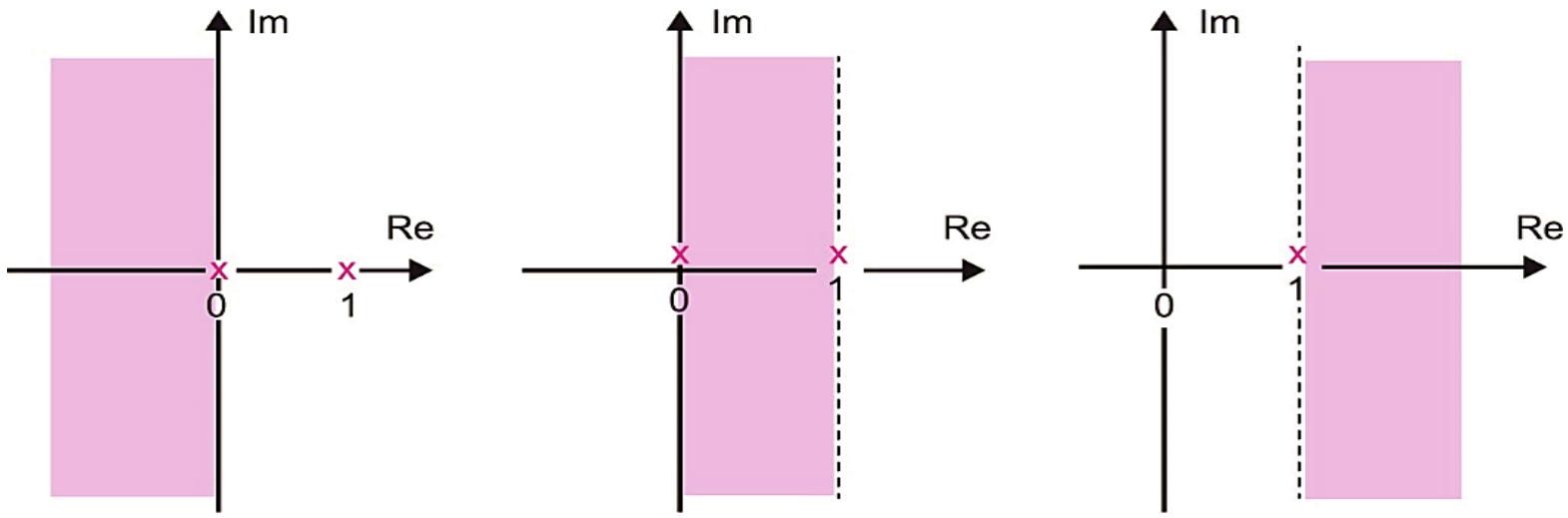
Answer: Find $h(t)$ by taking the inverse LT of $H(s)$. Express $H(s)$ as a sum of rational functions:

$$H(s) = \frac{1}{s^2(s-1)} = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}$$

The function $H(s)$ has a double pole at $s = 0$ and a simple pole at $s = 1$. No ROC is specified.
The poles divide the complex plane into three possible ROCs, shown on the next slide:

(a) $Re(s) < 0$ (b) $0 < Re(s) < 1$ (c) $Re(s) > 1$

Exercise 9 (continued)



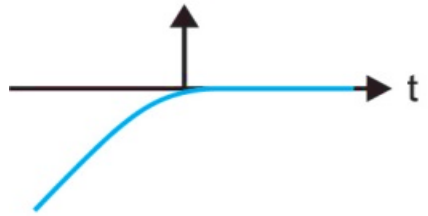
(a) $Re(s) < 0$ (b) $0 < Re(s) < 1$ (c) $Re(s) > 1$

Possible ROCs of the transfer function $H(s)$: (a) marginally stable; (b) and (c) causal system.

Exercise 9 (continued)

(a) For $Re(s) < 0$, all terms are left-sided, so:

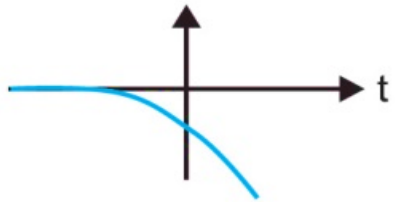
$$h(t) = -e^t u(-t) + u(-t) + t\,u(-t) = (\boldsymbol{-e^t + 1 + t})\boldsymbol{u(-t)}$$



(a)

(b) For $0 < Re(s) < 1$, the exponential term is left-sided and the remaining terms are right-sided, so:

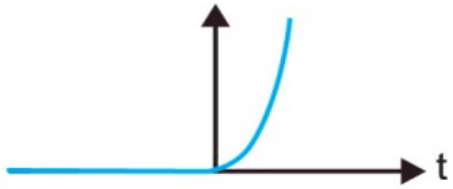
$$h(t) = -e^t u(-t) - u(t) - t\,u(t) = \boldsymbol{-e^t u(-t)} - \boldsymbol{(1 + t)\,u(t)}$$



(b)

(c) For $Re(s) > 1$, all terms are right-sided, so:

$$h(t) = e^t u(t) - u(t) - t\,u(t) = \boldsymbol{(e^t - 1 - t)u(t)}$$



The system is stable only in case (a).

Exercise 10

Calculate the inverse Laplace transform of:

$$X(s) = \frac{s}{(s+1)^3\,(s+2)}$$

Answer: The denominator has a real triple root $\lambda_1 = -1$ and a simple root $\lambda_2 = -2$. Thus, $X(s)$ can be written as:

$$X(s) = \frac{C_1}{s+1} + \frac{C_2}{(s+1)^2} + \frac{C_3}{(s+1)^3} + \frac{C_4}{s+2}$$

Calculate the constants C_1, C_2, C_3 and C_4:

- $C_1 = \lim\limits_{s\to -1} \frac{1}{(3-1)!} \frac{d^2}{ds^2}[(s+1)^3 X(s)] = \lim\limits_{s\to -1} \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{s}{s+2}\right] = -2$

Exercise 10 (continued)

- $C_2 = \lim_{s\to -1} \frac{1}{(3-2)!} \frac{d}{ds}[(s+1)^3 X(s)] = 2$
- $C_3 = \lim_{s\to -1} [(s+1)^3 X(s)] = -1$
- $C_4 = \lim_{s\to -2} [(s+2)\, X(s)] = 2$

Thus:

$$X(s) = -\frac{2}{s+1} + \frac{2}{(s+1)^2} - \frac{1}{(s+1)^3} + \frac{2}{s+2}$$

The ROC is right-sided and can be written as:

$$R_x = \{\sigma > -1\} = \{\sigma > -2\} \cap \{\sigma > -1\}$$

Hence, all signal components are right-sided. The inverse transform is the sum of the inverse transforms of the individual fractions:

$$x(t) = [-2e^{-t} + 2te^{-t} - \frac{t^2}{2} e^{-t} + 2e^{-2t}]\, u(t)$$

Exercise 10 (continued)

Verify the solution using the initial value theorem:

$$\lim_{s\to\infty} s\,X(s) = x(0^+)$$

The left-hand side is:

$$\lim_{s\to\infty} s\,X(s) = \lim_{s\to\infty} \frac{s^2}{s^4 + 5s^3 + 9s^2 + 7s + 2} = \lim_{s\to\infty} \frac{1}{s^2 + 5s + 9 + 7/s + 2/s^2} = 0$$

and the right-hand side is:

$$x(0^+) = \lim_{t\to 0^+} \left[-2e^{-t} + 2\,t\,e^{-t} - \frac{t^2}{2}e^{-t} + 2e^{-2t}\right] u(t) = 0$$

Thus, the solution is correct.

Exercise 11

Calculate the inverse Laplace transform of:

$$X(s) = \frac{s-3}{s^2+4s+3}$$

with ROC $\sigma > -1$.

Answer: The denominator roots are $\lambda_1 = -1$ and $\lambda_2 = -3$.

The denominator derivative is $A'(s) = 2s + 4$. Heaviside's theorem gives:

$$x(t) = \sum_{i=1}^{2} \frac{B(\lambda_i)}{A'(\lambda_i)} e^{\lambda_i t} = \frac{B(-1)}{A'(-1)} e^{-t} + \frac{B(-3)}{A'(-3)} e^{-3t} = \cdots = -2e^{-t} + 3e^{-3t}$$

The ROC shows that the signal is right-sided. Thus:

$$x(t) = [-2e^{-t} + 3e^{-3t}]u(t)$$

Exercise 12

Calculate the inverse Laplace transform with ROC $\sigma > -1$:

$$X(s) = \frac{1 - e^{-2s}}{s^2 + 3s + 2}$$

Answer: The denominator roots are $\lambda_1 = -1$ and $\lambda_2 = -2$. Expand $X(s)$:

$$X(s) = \frac{1 - e^{2s}}{(s+1)(s+2)} = \frac{1}{(s+1)(s+2)} - \frac{e^{-2s}}{(s+1)(s+2)}$$

The ROC shows that the signal is right-sided. The inverse LT of the first fraction is:

$$x_1(t) = [e^{-t} - e^{-2t}]u(t)$$

The inverse LT of the second fraction is:

$$x_2(t) = x_1(t-2) = \left[e^{-(t-2)} - e^{-2(t-2)}\right]u(t-2)$$

Thus, the inverse Laplace transform of $X(s)$ is:

$$x(t) = x_1(t) - x_2(t) = [e^{-t} - e^{-2t}]u(t) - \left[e^{-(t-2)} - e^{-2(t-2)}\right]u(t-2)$$

Exercise 13

Calculate the inverse Laplace transform with ROC $\sigma > -1$:

$$X(s) = \frac{1 - e^{-2s}}{(s+1)(1 - e^{-s})}$$

Answer: Rewrite $X(s)$ as:

$$X(s) = \frac{1 - e^{-2s}}{(s+1)(1 - e^{-s})} = \frac{(1 - e^{-s})(1 + e^{-s})}{(s+1)(1 - e^{-s})} = \frac{1 + e^{-s}}{s+1}$$

Since $\sigma > -1$, the signal is right-sided. Thus:

$$L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}u(t)$$

By the time-delay property:

$$L^{-1}\left\{\frac{e^{-s}}{s+1}\right\} = e^{-(t-1)}u(t-1)$$

Exercise 13 (continued)

Thus, the inverse Laplace transform of $X(s)$ is:

$$x(t) = e^{-t}u(t) + e^{-(t-1)}u(t-1)$$

- Recognizing the difference of squares $1 - e^{-2s} = (1 - e^{-s})(1 + e^{-s})$ simplifies the fraction. The signal has two right-sided exponential components, so the result follows directly from time shifting.
- The factor e^{-s} in the Laplace domain corresponds to a delay of 1. The second term therefore becomes $e^{-(t-1)}u(t-1)$.