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SIGNALS AND SYSTEMS
Solved Examples – Tutorial Handout 6

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LAPLACE TRANSFORM

- Calculating the Laplace transform and region of convergence
- Poles and zeros of the Laplace transform

1. Calculating the Laplace Transform and Region of Convergence

📖Example 1

For $a > 0$, calculate the bilateral Laplace transform of the signal and determine its region of convergence:

$$x(t) = e^{-a|t|}$$

Answer: Write the signal in two parts:

$$x(t) = \begin{cases} e^{-at}, & t \geq 0 \\ e^{at}, & t < 0 \end{cases}$$

The bilateral Laplace transform is:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{0} e^{(a-s)t}\, dt + \int_{0}^{\infty} e^{-(a+s)t}\, dt$$

For the two integrals to converge, the following conditions must hold, respectively:

$$Re(s) < a \quad \text{and} \quad Re(s) > -a$$

Thus, the region of convergence is:

$$-a < Re(s) < a$$

Evaluating the two integrals gives:

$$X(s) = \frac{1}{a-s} + \frac{1}{a+s} = \frac{2a}{a^2 - s^2}$$

Comments:

- This example illustrates the transition from a half-plane ROC to a strip ROC, which occurs for two-sided signals.
- The signal $e^{-a|t|}$ combines symmetry and infinite duration in a simple expression. Its symmetry is also reflected in $X(s)$, which is an even function of s.

- The region of convergence is determined by the compatibility of the convergence conditions for the two sides of the signal and is their intersection.
- Two-sided exponential envelopes are used in theoretical models of transient phenomena with symmetric temporal localization.

📖Example 2 (*)
Calculate the Laplace transforms of the functions:
(a) $x_1(t) = u(t)\cos(\Omega t)$ (b) $x_2(t) = u(t)\sin(\Omega t)$

Answer:[1] (a) Using the definition of the Laplace transform and Euler's formula for the cosine, we obtain:

$$X_1(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} [\cos(\Omega t)]\, u(t)\, e^{-st}\, dt$$

$$= \int_0^{\infty} \cos(\Omega t)\, e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{j\Omega t}\, e^{-st}\, dt + \frac{1}{2}\int_0^{\infty} e^{-j\Omega t}\, e^{-st} dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{(j\Omega - s)t} dt + \frac{1}{2}\int_0^{\infty} e^{-(j\Omega + s)t} dt$$

$$= \frac{1}{2}\left[\frac{e^{(j\Omega - s)t}}{j\Omega - s}\right]_0^{\infty} + \frac{1}{2}\left[-\frac{e^{-(j\Omega + s)t}}{j\Omega + s}\right]_0^{\infty}$$

For the transform to converge, we require:

$$\lim_{t\to\infty}\left[\frac{e^{(j\Omega - s)t}}{j\Omega - s}\right] = \lim_{t\to\infty}\left[-\frac{e^{-(j\Omega + s)t}}{j\Omega + s}\right] = 0$$

For the first limit to vanish, the exponent must be negative, i.e.:

$$j\Omega - s < 0 \Rightarrow j\Omega - \sigma - j\Omega < 0 \Rightarrow \boldsymbol{\sigma > 0}$$

Similarly, for the second limit to vanish, we require:

$$-(j\Omega + s) < 0 \Rightarrow j\Omega + s > 0 \Rightarrow j\Omega + \sigma + j\Omega > 0 \Rightarrow \boldsymbol{\sigma > 0}$$

Thus, the region of convergence is $\boldsymbol{Re\{s\} = \sigma > 0}$, i.e. the part of the complex plane to the right of the line $\sigma > 0$, excluding the line itself. This is expected because the signal is right-sided. There is a pole at 0. In this region of convergence, the transform is:

$$X_1(s) = \frac{1}{2}\left[\frac{1}{j\Omega - s} - \frac{1}{j\Omega + s}\right] = \frac{1}{2}\frac{j\Omega + s - j\Omega + s}{s^2 - (j\Omega)^2} = \frac{s}{s^2 + \Omega^2}$$

[1] $\cos\varphi = \frac{1}{2}[e^{j\varphi} + e^{-j\varphi}]$

(b) Proceeding similarly and using Euler's formula for the sine, we obtain:[2]

$$X_2(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} [\sin(\Omega t)]\, u(t) e^{-st}\, dt$$

$$= \int_0^{\infty} \sin(\Omega t)\, e^{-st}\, dt = \frac{1}{2j}\int_0^{\infty} e^{j\Omega t}\, e^{-st}\, dt - \frac{1}{2j}\int_0^{\infty} e^{-j\Omega t}\, e^{-st} dt$$

$$= \cdots = \frac{1}{2j}\left[\frac{1}{s - j\Omega} - \frac{1}{s + j\Omega}\right] = \frac{\Omega}{s^2 + \Omega^2}$$

The region of convergence is $\boldsymbol{\sigma \equiv Re\{s\} > 0}$, and the signal $x(t)$ is right-sided.

📖Example 3 (*)
Calculate the Laplace transforms of the functions:

(a) $x_1(t) = e^{-\alpha t}\cos(\Omega t)\, u(t)$ (b) $x_2(t) = e^{-\alpha t}\sin(\Omega t)\, u(t)$

Answer: (a) Using the definition of the Laplace transform, we obtain:

$$X_1(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-\alpha t}\cos(\Omega t)\, u(t)\, e^{-st}\, dt$$

$$= \int_0^{\infty} e^{-\alpha t}\cos(\Omega t)\, e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-\alpha t} e^{j\Omega t}\, e^{-st}\, dt + \frac{1}{2}\int_0^{\infty} e^{-\alpha t} e^{-j\Omega t}\, e^{-st} dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{(j\Omega - s - \alpha)t} dt + \frac{1}{2}\int_0^{\infty} e^{-(j\Omega + s + \alpha)t} dt$$

$$= \frac{1}{2}\left[\frac{e^{(j\Omega - s - \alpha)t}}{j\Omega - s - \alpha}\right]_0^{\infty} + \frac{1}{2}\left[-\frac{e^{-(j\Omega + s + \alpha)t}}{j\Omega + s + \alpha}\right]_0^{\infty}$$

For the transform to converge, we require:

$$\lim_{t\to\infty}\left[\frac{e^{(j\Omega - s - \alpha)t}}{j\Omega - s - \alpha}\right] = \lim_{t\to\infty}\left[-\frac{e^{-(j\Omega + s + \alpha)t}}{j\Omega + s + \alpha}\right] = 0$$

For the first limit to vanish, the exponent must be negative, i.e.:

$$j\Omega - s - \alpha < 0 \Rightarrow j\Omega - \sigma - j\Omega - \alpha < 0 \Rightarrow \boldsymbol{\sigma > -\alpha}$$

Similarly, for the second limit to vanish, we require:

$$-(j\Omega + s + \alpha) < 0 \Rightarrow j\Omega + s + \alpha > 0 \Rightarrow j\Omega + \sigma + j\Omega + \alpha > 0 \Rightarrow \boldsymbol{\sigma > -\alpha}$$

Thus, the region of convergence is $\boldsymbol{\sigma \equiv Re\{s\} > -\alpha}$, i.e. the part of the complex plane to the right of the line $\sigma > -\alpha$, excluding the line itself. This is expected because the signal is right-sided. There is

[2] $\sin\varphi = \frac{1}{2}[e^{j\varphi} - e^{-j\varphi}]$

a pole at $-\alpha$. In this region of convergence, the transform is:

$$X_1(s) = \frac{1}{2}\left[\frac{1}{j\Omega - s - \alpha} - \frac{1}{j\Omega + s + \alpha}\right] = \cdots = \frac{s + \alpha}{(s + \alpha)^2 + \Omega^2}$$

(b) Proceeding similarly, we obtain:

$$X_2(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-\alpha t}\sin(\Omega t)\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-\alpha t}\sin(\Omega t)\, e^{-st}\, dt$$

$$= \frac{1}{2j}\int_0^{\infty} e^{-\alpha t} e^{j\Omega t}\, e^{-st}\, dt - \frac{1}{2j}\int_0^{\infty} e^{-\alpha t} e^{-j\Omega t}\, e^{-st} dt = \cdots$$

$$= \frac{\Omega}{(s + \alpha)^2 + \Omega^2}$$

The region of convergence is $\boldsymbol{\sigma \equiv Re(s) > -a}$, and the signal $x(t)$ is right-sided.

Comments:

The following paragraphs summarize the basic properties of the Laplace transform's region of convergence in relation to the form and duration of the signal.

A continuous-time signal $x(t)$ either has no Laplace transform or has a transform whose region of convergence belongs to one of the following basic categories:

- **Finite-duration signal**: If the signal is nonzero only over a finite interval, the region of convergence is generally the entire complex s-plane, possibly excluding $s = \infty$ and, in special cases, isolated points where the integral is undefined. In most practical cases, the Laplace transform of such a signal converges for every finite value of s.
- **Right-sided signal**: The region of convergence has the form $\boldsymbol{Re(s) > \sigma_R}$, where σ_R is the real part of the **rightmost pole**, i.e. the pole with the largest real part. Thus, the ROC lies to the **right** of a vertical line in the complex plane.
- **Left-sided signal**: The region of convergence has the form $\boldsymbol{Re(s) < \sigma_L}$, where σ_L is the real part of the **leftmost pole**, i.e. the pole with the smallest real part. In this case, the ROC lies to the **left** of a vertical line.
- **Two-sided signal of infinite duration**: The region of convergence has the form $\boldsymbol{\sigma_R < Re(s) < \sigma_L}$, where σ_R and σ_L are determined by the poles bounding the ROC. In this case, the ROC is a vertical strip of the complex plane **between** two vertical lines.
- The boundary lines are excluded from the region of convergence. Thus, the lines $Re\{s\} = \sigma_R$ and $Re\{s\} = \sigma_L$ do not belong to the ROC.

If the Laplace transform has poles, none can lie within the region of convergence. The transform becomes infinite at the poles, so they are excluded from the ROC. Poles and zeros will be studied in more detail in a later section.

2. Poles and Zeros of the Laplace Transform

📖Example 4

Consider the rational function:

$$H(s) = \frac{s+1}{(s+2)(s^2+4)}$$

For each of the following regions of convergence, determine whether the corresponding system is causal, anti-causal or two-sided, and whether it is stable:

i. $Re(s) > 0$
ii. $-2 < Re(s) < 0$
iii. $Re(s) < -2$

Answer: The poles of $H(s)$ are at $s = -2,\ s = \pm j2$, and the zero is at $s = -1$. Consider the following cases:

- For $\boldsymbol{Re(s) > 0}$, the ROC lies to the right of the rightmost pole, so the signal or impulse response is right-sided and the system is causal. The imaginary axis is excluded from the ROC, so the system is unstable.
- For $\boldsymbol{-2 < Re(s) < 0}$, the ROC is a strip, so the signal is two-sided. The imaginary axis is again excluded from the ROC, so the system is unstable.
- For $\boldsymbol{Re(s) < -2}$, the ROC lies to the left of the leftmost pole, so the signal is left-sided and the system is anti-causal. The imaginary axis is excluded, so the system is unstable.

Comments:

- The same algebraic function $H(s)$ can correspond to different time-domain signals, depending on the region of convergence (ROC). The side on which the ROC lies is related to the signal's temporal sidedness.
- The imaginary axis is fundamental to the stability of causal LTI systems.
- A sound understanding of the ROC, poles and causality is essential when designing and interpreting analog filters.
- Internal or natural stability concerns the system's natural behavior and is determined by the poles governing its natural response. The system is asymptotically stable when all poles have strictly negative real parts; marginally stable when the imaginary-axis poles are all simple and the remaining poles lie in the left half-plane; and unstable when a pole lies in the right half-plane or a repeated pole lies on the imaginary axis.
- BIBO stability concerns a bounded response to every bounded input. For a causal rational LTI system, it is equivalent to the ROC of $H(s)$ including the imaginary axis. Equivalently, all poles of $H(s)$ must lie strictly in the left half-plane. Thus, simple poles on the $j\Omega$ axis are compatible with marginal internal stability, but not with BIBO stability.

Note: Examples marked with an asterisk (*) also appear in the lecture slides.