

UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Signals and Systems

Lecture 6: Laplace Transform

Michalis Paraskevas
Professor

Lecture Contents

1. Eigenvalues of an LTI system
2. Mathematical definition of the Laplace transform
3. Region of convergence of the Laplace transform
4. Poles and zeros of the Laplace transform
5. Useful Laplace transform pairs

Introduction

- The Laplace transform (LT) is a useful tool for moving from the **time domain *t*** to the **complex angular frequency domain *s***, and back.
- It gives **simple algebraic solutions** to linear differential equations with constant coefficients (LCCDEs), which describe LTI systems.
- The Laplace transform is used mainly to study **continuous-time systems**, whereas the Fourier transform is used to study **continuous-time signals.**

Advantages of the Laplace Transform

The Laplace transform is widely used because:

- It applies to **more classes** of functions than the Fourier transform.
- It can analyze systems that are **not initially at rest** by incorporating their initial conditions.
- The complex frequency domain and **poles and zeros** enrich and simplify the analysis of linear systems.
- It describes both the **steady-state and transient behavior** of systems, whereas the Fourier transform is limited to their steady-state behavior.

1. Eigenvalues of a Linear Time-Invariant System

Eigenvalues of an LTI System

A complex signal $x(t) = e^{s_0 t},$ where $s_0 = \sigma_0 + j\Omega$ and $-\infty < t < \infty$ is applied to an LTI system with impulse response $h(t)$. . The output is $y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$. Thus:

$$\boldsymbol{y(t)} = \int_{-\infty}^{\infty} h(\tau)\, e^{s_0(t-\tau)}\, d\tau = e^{s_0 t} \int_{-\infty}^{\infty} h(\tau)\, e^{-\tau s_0}\, d\tau = \boldsymbol{x(t)\, H(s_0)} \qquad (1)$$

where: $H(s_0) = \int_{-\infty}^{\infty} h(\tau)\, e^{-\tau s_0}\, d\tau$

Since $e^{s_0 t}$ appears in both the input and output, it is called an **eigenfunction** of the system. The output $y(t)$ equals the input multiplied by the complex function $\boldsymbol{H(s_0)}$ **(the eigenvalue)**. This function depends on the system through its impulse response $h(t)$ and is evaluated at frequency s_0.

For each complex frequency s,, the input eigenfunction is multiplied by:

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-\tau s}\, d\tau$$

This function is the **Laplace transform** of the LTI system impulse response, called the system **transfer function**.

Eigenvalues of an LTI System

The function $H(s)$ can be viewed as an infinite combination of complex exponentials weighted by the impulse response $h(t)$. If $x(t)$ is a weighted sum of infinitely many complex exponentials:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds$$

then, by linearity, the output $y(t)$ is:

$$y(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, [H(s)\, e^{st}]\, ds = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} Y(s)\, e^{st} ds$$

where $Y(s) = X(s)\, H(s)$.

For LTI systems, the output is the convolution $y(t) = x(t) * h(t)$,. Thus, the Laplace transform provides an equivalent way to calculate the output: it replaces the computationally demanding convolution of $x(t)$ and $h(t)$ with a simple multiplication of the complex functions $X(s)$ and $H(s)$.

This analysis does not apply to nonlinear or time-varying systems.

2. Mathematical Definition of the Laplace Transform

Bilateral Laplace Transform

The **bilateral** Laplace transform is defined as:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

The variable s is a complex quantity of the form $s = \sigma + j\Omega$, so the integral becomes:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t} e^{-j\Omega t}\, dt$$

- For $\sigma = 0$, the Laplace transform **coincides** with the Fourier transform.
- The factor $e^{-\sigma t}$, absent from the Fourier transform, ensures convergence of the integral. The Laplace transform can therefore exist even when the Fourier transform diverges.
- The quantity σ is the **damping factor**, and Ω (rad/sec) is the real angular frequency.
- The values of s for which the Laplace transform exists form its **region of convergence** (ROC).

Unilateral Laplace Transform

The **unilateral** Laplace transform is defined as:

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

A more mathematically complete definition of the unilateral Laplace transform is:

$$X(s) = \lim_{\alpha \to 0^-} \int_{0^-}^\infty x(t)\, e^{-st} dt$$

Observations:

- An unqualified reference to the Laplace transform usually means the unilateral transform.
- The unilateral LT is used more often because (a) we usually analyze causal systems and (b) it can analyze systems that are not initially at rest.

Inverse Laplace Transform

The inverse Laplace transform is given by:

$$x(t) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} X(s)\, e^{st}\, ds, \qquad s \in ROC$$

Unilateral and Bilateral Laplace Transforms

- The unilateral transform is used more often because we usually study how causal signals affect causal systems, whether or not these systems are initially at rest. A set of linear differential equations with constant coefficients (LCCDEs) describes the causal system.
- The bilateral transform does not directly incorporate initial conditions in the same way as the unilateral transform in the analysis of dynamic systems.
- Two continuous-time signals that differ only on $t < 0$ and are identical for $t > 0$ have different bilateral transforms but the same unilateral Laplace transform.
- For a causal signal (that is, $x(t) = 0$ for $t < 0$), the unilateral and bilateral transforms have the same functional expression and most of the same properties. We will explicitly identify any differences.

Unilateral and Bilateral Laplace Transforms

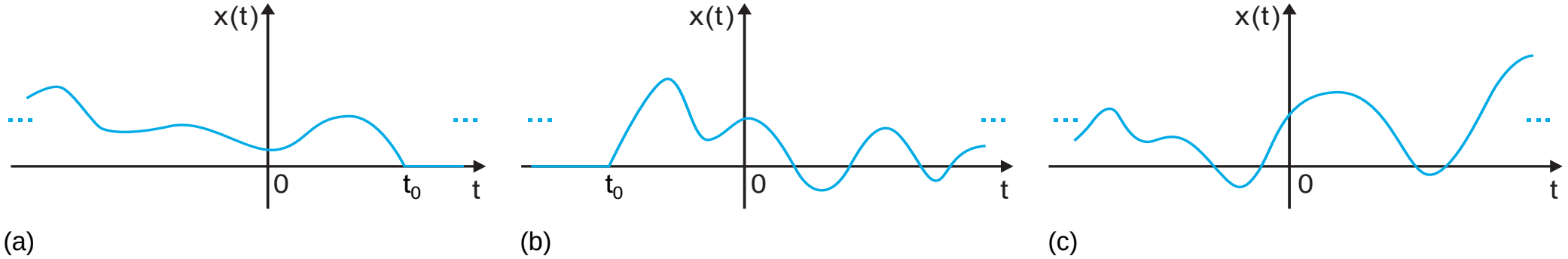
- The properties of the unilateral and bilateral transforms generally coincide when the signal is causal and the system is initially at rest.
- They differ substantially when initial conditions are nonzero or the signal is nonzero for $t < 0$. The unilateral transform then considers only the signal for $t \geq 0$.
- Two signals that differ only on $(-\infty, 0)$ generally have the same unilateral transform but different bilateral Laplace transforms.
- This distinction is crucial when studying physical systems with initial conditions or previously stored energy.

3. Region of Convergence of the Laplace Transform

Signal Sidedness

Recall that a signal $x(t)$ is called:

- **Left-sided** if $x(t) = 0$ for $t > t_0$
- **Right-sided** if $x(t) = 0$ for $t < t_0$
- **Two-sided** if $x(t) \neq 0$ for $-\infty < t < \infty$
- **Finite-duration** if $x(t) = 0$ for $t > t_1$ and $t < t_2$



Signals: (a) left-sided, (b) right-sided, (c) two-sided

Region of Convergence of the Laplace Transform

- For the unilateral Laplace transform, the values of $X(s)$ for which the integral **converges absolutely** over $[0, +\infty)$ form a region of the type $Re\{s\} > a$ or $Re\{s\} \geq a$, where $a \in R$ and $-\infty \leq a \leq \infty$.
- The constant a is the abscissa of absolute convergence and depends on the function $x(t)$.
- The subset of the complex plane s for which the Laplace transform converges, absolutely or conditionally, is its **region of convergence** (ROC).
- The ROC depends on the **duration** and **form** of the signal: whether its duration is finite or infinite and whether it is left-sided, right-sided, or two-sided.

Properties of the Region of Convergence

A continuous-time signal $x(t)$ either has no Laplace transform or falls into one of these four cases:

- For a **finite-duration** signal, the ROC is the **entire complex plane**, possibly excluding $s = 0$ or $s = \infty$..
- For a right-sided signal, the ROC has the form $\boldsymbol{\sigma > \sigma_R}$, where σ_R is the pole with the largest real part.
- For a left-sided signal, the ROC has the form $\boldsymbol{\sigma < \sigma_L}$, , where σ_L is the pole with the smallest real part.
- For an **infinite-duration** signal, the ROC has the form $\boldsymbol{\sigma_R < \sigma < \sigma_L}$,, where σ_R is the pole with the largest real part and σ_L the pole with the smallest real part. The ROC is a **strip** between two vertical lines whose positions are determined by the poles.
- Poles cannot lie within the ROC. This is because the integration limits do not include infinity, but specific time values.

4. Poles and Zeros of the Laplace Transform

Poles and Zeros of the Laplace Transform (1/3)

Consider the Laplace transform of a signal in rational form:

$$X(s) = \frac{B(s)}{A(s)} = \frac{b_M S^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

Factoring the numerator and denominator gives:

$$X(s) = K \frac{\prod_{m=1}^{M}(s - z_m)}{\prod_{n=1}^{N}(s - p_n)}$$

$\alpha_i, i = 1,2, \ldots, N$ and $b_j, j = 1,2, \ldots, M$: real constants; **M** and **N**: positive integers.

- **Zeros**: roots of the **numerator** $(z_1, z_2, \ldots, z_M)$
- **Poles**: roots of the **denominator** $(p_1, p_2, \ldots, p_N)$
- There are M zeros and N poles. The transform also has $|N - M|$ additional zeros when $N > M$, or poles when $N < M$..
- A plot of the poles and zeros is called a **pole-zero map**.

Poles and Zeros of the Laplace Transform (2/3)

- If $M < N$, the rational function is **strictly proper.** If $M > N$, it can be written as the sum of a polynomial and a strictly proper rational function.
- The ROC consists of the values of σ for which the product $x(t)e^{-\sigma t}$ is absolutely integrable.
- The ROC never includes poles: the Laplace transform tends to **infinity** at these roots of the denominator.
- **Zero**s make the transform zero. They do not cause convergence problems, but strongly affect the form and behavior of the function.
- The ROC extends **parallel** to the imaginary axis $j\Omega$.
- The ROC depends only on the **damping factor σ**, not on the frequency Ω. The magnitude of $e^{j\Omega}$ is one ($\left|e^{j\Omega}\right| = 1$)„ so it does not affect convergence. Every ROC therefore includes infinitely many frequencies Ω, that is, $-\infty < \Omega < \infty$.

Poles and Zeros of the Laplace Transform (3/3)

- Let $\{\sigma_i\}$ be the real parts of the poles of the Laplace transform $F(s)$ of a time function $f(t)$,, representing a signal or a system impulse response. The **pole positions** define the ROC as follows:
  - For a **right-sided, causal** function $f(t)$, the ROC is the half-plane to the **right of the rightmost pole**:

$$R_c = \{(\sigma, \Omega):\ \sigma > max\{\sigma_i\}, -\infty < \Omega < \infty\}$$

  - For a left-sided, anticausal function $f(t)$, the ROC is the half-plane to the **left of the leftmost pole**:

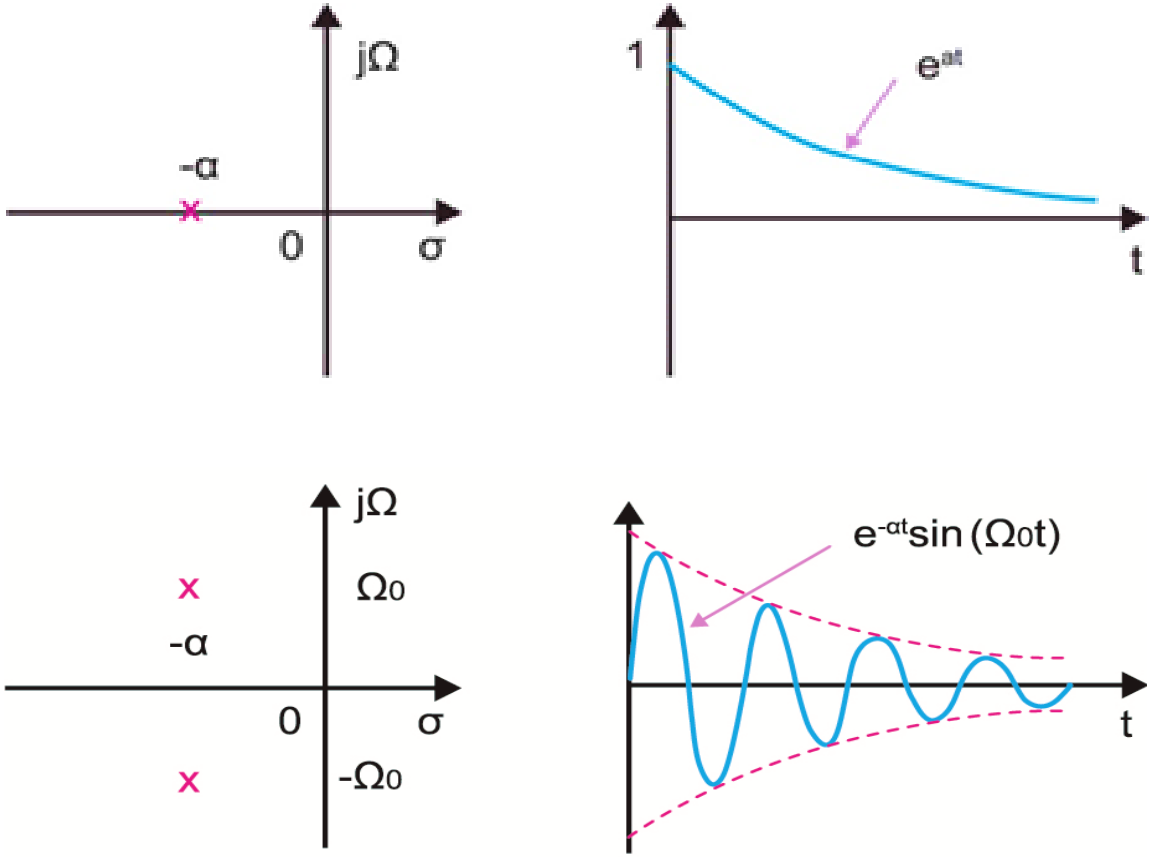
$$R_{ac} = \{(\sigma, \Omega):\ \sigma < min\{\sigma_i\}, -\infty < \Omega < \infty\}$$

  - For a **two-sided, noncausal** function $f(t)$, the ROC is a vertical strip between poles: the intersection of the ROCs of the right-sided (causal) and left-sided (anticausal) parts:

$$R_{nc} = R_c \cap R_{ac}$$

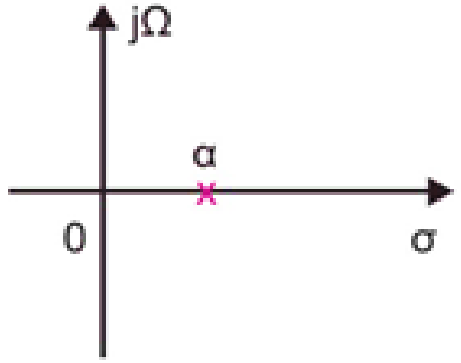
Pole Locations and System Stability (1/4)

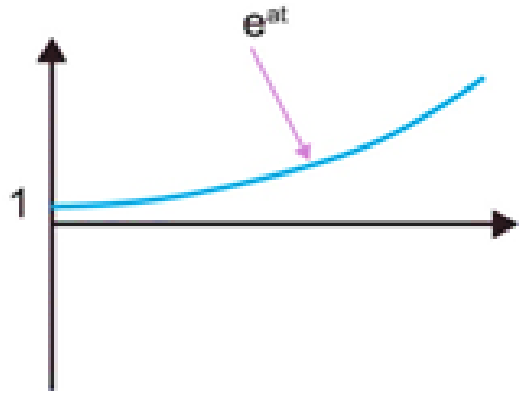
- If the conjugate poles have **negative** real parts, the oscillation decays exponentially and the system is **stable.**

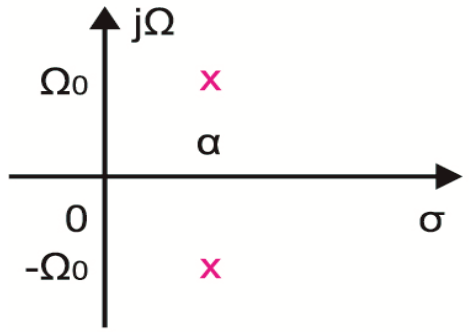


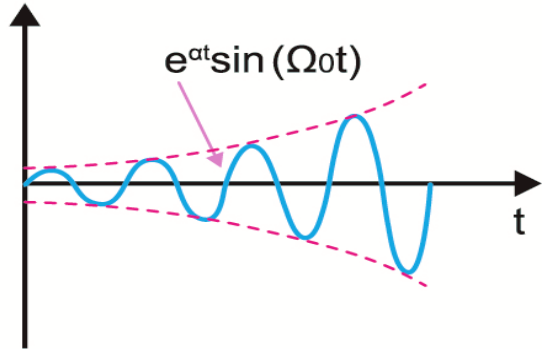
Pole Locations and System Stability (2/4)

- If the conjugate poles have **positive** real parts, the oscillation grows exponentially and the system is **unstable.**



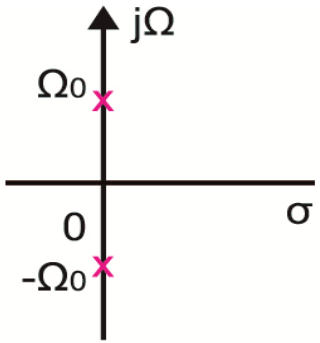


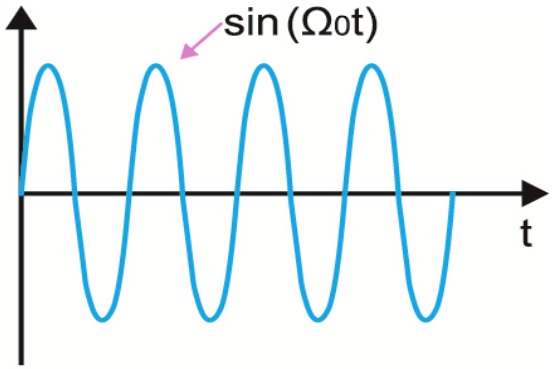




Pole Locations and System Stability (3/4)

- If the conjugate poles have **zero** real parts, the oscillation is sustained and the system is marginally stable.





- Signals containing **sinusoidal terms** have **complex conjugate poles.** If their real parts are **zero**, the **oscillation is sustained** (marginal stability).
- Otherwise, the oscillation decays or grows exponentially, depending on whether the poles lie in the left or right half of the complex frequency plane.

Pole Locations and System Stability (4/4)

- Signals multiplied by **powers of t** have poles in the right half-plane of s and are **unstable**, since their values tend to **infinity** over time.
- Response terms of the form $t^n e^{-\alpha t} u(t)$ produce **repeated poles.** Stability still depends on pole locations relative to the imaginary axis, rather than simply on the presence of powers of t.
- A causal LTI system is **stable** when **all poles** of its transfer function, equivalently the Laplace transform of its impulse response, lie in the **left half-plane** and have strictly **negative real parts.**
- Equivalently, the ROC of the Laplace transform of the impulse response must include the imaginary axis $Re(s) = 0$.

Summary of System Stability

- The same algebraic function $H(s)$ can represent different time signals, depending on its ROC. The sidedness of the ROC is related to the time sidedness of the signal.
- Understanding the ROC, poles, and causality is crucial for designing and interpreting analog filters.
- **Internal or natural stability** concerns the **natural behavior** of the system, determined by the poles of its natural response. The system is (a) asymptotically stable when all poles have negative real parts, (b) marginally stable when imaginary-axis poles are simple and all other poles lie in the left half-plane, and (c) **unstable** when a pole lies in the right half-plane or a repeated pole lies on the imaginary axis.
- BIBO stability means a **bounded response** to every bounded input. For a causal rational LTI system, it requires the ROC of $\boldsymbol{H(s)}$ to include **the imaginary axis**. Equivalently, all poles of $H(s)$ must lie strictly in the left half-plane. Simple poles on the $j\Omega$ axis allow marginal internal stability, but not BIBO stability.

5. Useful Laplace Transform Pairs

Useful Laplace Transform Pairs (1/4)

Signal $x(t)$	Laplace Transform $X(s)$	Region of Convergence
$\delta(t)$	1	$Re(s) > -\infty$ (entire s plane)
$\mathrm{A}\,\delta(t \pm t_0)$	$\mathrm{A}\,e^{\mp t_0 s}$	$Re(s) > -\infty$ (entire s plane)
$u(t)$	$1/s$	$Re(s) > 0$
$-u(-t)$	$1/s$	$Re(s) < 0$
$u(t-t_0), t_0 > 0$	$\frac{1}{s} e^{-t_0 s}$	$Re(s) > 0$
$r(t) = t\,u(t)$	$\frac{1}{s^2}$	$Re(s) > 0$
$\frac{d^n \delta(t)}{dt^n}$	s^n	$Re(s) > -\infty$ (entire the plane s)

Useful Laplace Transform Pairs (2/4)

Signal $x(t)$	Laplace Transform $X(s)$	Region of Convergence
$e^{-at}\,u(t), a > 0$	$\frac{1}{s+a}$	$Re(s) > -a$
$-e^{-at}\,u(-t), a > 0$	$\frac{1}{s+a}$	$Re(s) < -a$
$e^{j\Omega_0 t}\,\mathrm{u}(t)$	$\frac{1}{s-j\Omega_0}$	$Re(s) > 0$
$\cos(\Omega_0 t)\,\mathrm{u}(t)$	$\frac{s}{s^2+\Omega_0^2}$	$Re(s) > 0$
$\sin(\Omega_0 t)\,\mathrm{u}(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$	$Re(s) > 0$

Useful Laplace Transform Pairs (3/4)

Signal $x(t)$	Laplace Transform $X(s)$	Region of Convergence
$e^{-at}cos(\Omega_0 t)\,u(t), a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$	$Re(s) > -a$
$e^{-at}sin(\Omega_0 t)\,u(t), a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$	$Re(s) > -a$
$t^n\,u(t)$	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$-t^n\,u(-t)$	$\frac{n!}{s^{n+1}}$	$Re(s) < 0$

Useful Laplace Transform Pairs (4/4)

Signal $x(t)$	Laplace Transform $X(s)$	Region of Convergence
$e^{-at}\, t^n\, u(t), a > 0$	$\frac{n!}{(s+a)^{n+1}}$	$Re(s) > -a$
$-e^{-at}\, t^n\, u(-t), a > 0$	$\frac{n!}{(s+a)^{n+1}}$	$Re(s) < -a$
$t\, cos(\Omega_0 t)\, u(t)$	$\frac{s^2 - \Omega_0^2}{(s^2 + \Omega_0^2)^2}$	$Re(s) > 0$
$t\, sin(\Omega_0 t)\, u(t)$	$\frac{2s\Omega_0^2}{(s^2 + \Omega_0^2)^2}$	$Re(s) > 0$

Exercises

Exercise 1

Calculate the Laplace transform of $x(t) = u(t)$.

Answer: From the definition of the Laplace transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-st}\, dt = \left[-\frac{1}{s} e^{-st}\right]_{0}^{\infty} = -\frac{1}{s}\left[\lim_{t\to\infty} e^{-st} - e^{0}\right] = -\frac{1}{s}[0-1] = \frac{1}{s}$$

The ROC consists of the values of s for which this integral exists. The limit must converge, so the exponent $-st$ must be negative for the limit to approach zero. Thus, $s > 0$ must hold.

Therefore, the ROC is $\boldsymbol{Re(s) > 0}$.

Exercise 2

Calculate the Laplace transforms of (a) $x_1(t) = \delta(t)$ and (b) $x_2(t) = \delta(t - t_0)$.

Answer: (a) From the definition of the Laplace transform:

$$X_1(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} \delta(t) e^{-st}\, dt = e^{-st}|_{t=0} = e^0 = 1$$

The ROC is $Re(s) > -\infty$, that is, the entire complex plane.

Comment: We used the sifting property of $\delta(t)$ to evaluate the integral:

$$\int_{-\infty}^{\infty} \delta(t)\, x(t)\, dt = x(0)$$

Exercise 2 (continued)

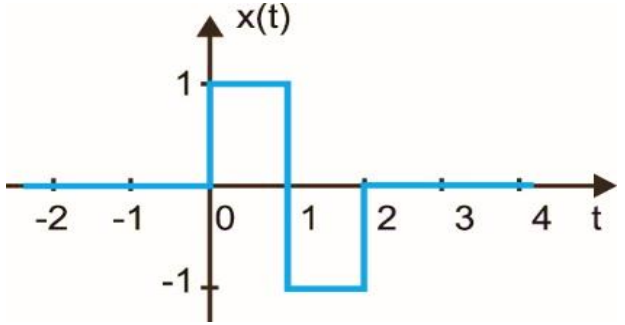
(b) Similarly, we obtain:

$$X_2(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} \delta(t - t_0) e^{-st}\, dt = e^{-st}|_{t=t_0} = e^{-st_0}$$

The ROC is $Re(s) > -\infty$, that is, the entire complex plane.

Exercise 3

Calculate the Laplace transform of this function:



Answer: From the definition
of the Laplace transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0}^{1} e^{-st}dt - \int_{1}^{2} e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_{0}^{1} - \frac{1}{s}e^{-st}\Big|_{1}^{2}$$

$$= -\frac{1}{s}[e^{-s} - 1 + e^{-2s} + e^{-s}] = \frac{e^{-2s} - 2e^{s} + 1}{s} = \frac{(1 - e^{-s})^2}{s}$$

Since the integrals converge for every value of s, the ROC is the entire complex plane s.

The signal has finite duration: it exists only over $0 \leq t \leq 2$.

Exercise 4

Calculate the Laplace transforms of
(a) $x_1(t) = e^{-\alpha t}\, u(t)$ (right-sided) and (b) $x_2(t) = -e^{-\alpha t}\, u(-t)$ (left-sided).

Answer: (a) From the definition of the Laplace transform:

$$X_1(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-at}\, e^{-st}\, dt = \int_{0}^{\infty} e^{-(\alpha+s)t}\, dt =$$

$$= \left[\frac{1}{\alpha+s} e^{-(\alpha+s)t}\right]_{0}^{\infty} = \frac{1}{\alpha+s}\left[\lim_{t\to\infty} e^{-(\alpha+s)t} - e^{0}\right]$$

The limit exists only when the exponent is negative. An exponent $-\infty$ makes the limit approach zero. If the exponent is positive, the limit tends to $+\infty$ and the integral diverges. Thus, the Laplace transform exists only if:

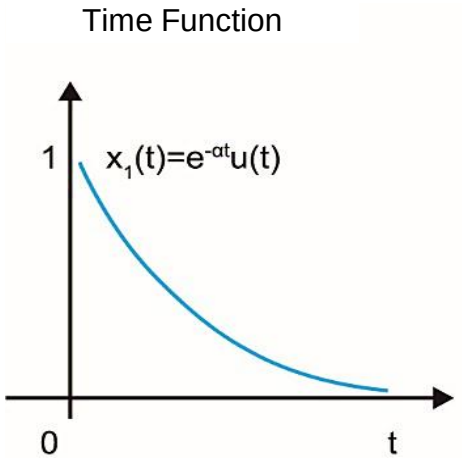
$$\alpha + s > 0 \Rightarrow (a+\sigma) + j\Omega > 0 \Rightarrow (a+\sigma) > 0 \Rightarrow \sigma > -\alpha$$

Exercise 4 (continued)

Since $\sigma = Re(s)$, the ROC is defined by $\boldsymbol{\sigma \equiv Re(s) > -\alpha}$, and the transform is:

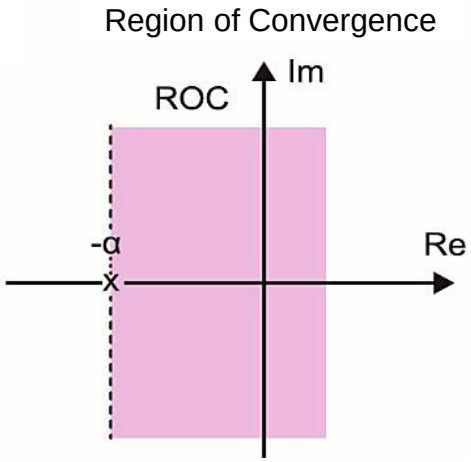
$$\boldsymbol{X_1(s) = \frac{1}{s + \alpha}}$$

- The ROC is the part of the complex plane to the **right** of the line $\sigma > -\alpha$, excluding the line itself. This is expected because the signal is right-sided.
- There is a pole at $-\alpha$.
- The line $\sigma = -\alpha$ is the boundary of the ROC.



Laplace Transform

$x_1(s) = \frac{1}{\alpha+s}$



Exercise 4 (continued)

Consider different values of the exponent α:.

- If $\alpha > 0$, for example $\alpha = 3$, then $x_1(t) = e^{-3t}\,u(t)$ is a **decaying causal** (right-sided) exponential. Its Laplace transform pole is at $s = -\alpha = -3$, on the real axis σ in the left half of the s-plane.
- If $\alpha < 0$, for example $\alpha = -3$, then $x_1(t) = e^{3t}u(t)$ is a **growing causal** (right-sided) exponential. Its pole is at $s = -\alpha = 3$, on the real axis σ in the right half of the s-plane.
- In both cases, the **closer** the pole is to the vertical axis, the **more slowly** the signal decays or grows. The **farther** the pole is from the vertical axis, the **faster** the signal changes.
- Thus, the **distance of the pole from the vertical axis** $\sigma = 0$ determines the **rate of change** of the signal.
- If $\alpha = 0$, then $x_1(t) = u(t)$ is the constant, right-sided step function. Its Laplace transform pole is at $s = 0$.

Exercise 4 (continued)

(b) Proceeding in the same way gives:

$$X_2(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{0} -e^{-at}\, e^{-st}\, dt = \int_{-\infty}^{0} -e^{-(\alpha+s)t}\, dt =$$

$$= \left[-\frac{1}{-(\alpha+s)} e^{-(\alpha+s)t} \right]_{-\infty}^{0} = \frac{1}{\alpha+s} \left[e^{0} - \lim_{t\to-\infty} e^{-(\alpha+s)t} \right] = \frac{1}{s+\alpha}$$

The limit exists only when the exponent is negative. Since t is negative (because $t \to -\infty$), the integral converges only if:

$$\alpha + s < 0 \Rightarrow (a+\sigma) + j\Omega < 0 \Rightarrow (a+\sigma) < 0 \Rightarrow \sigma < -\alpha$$

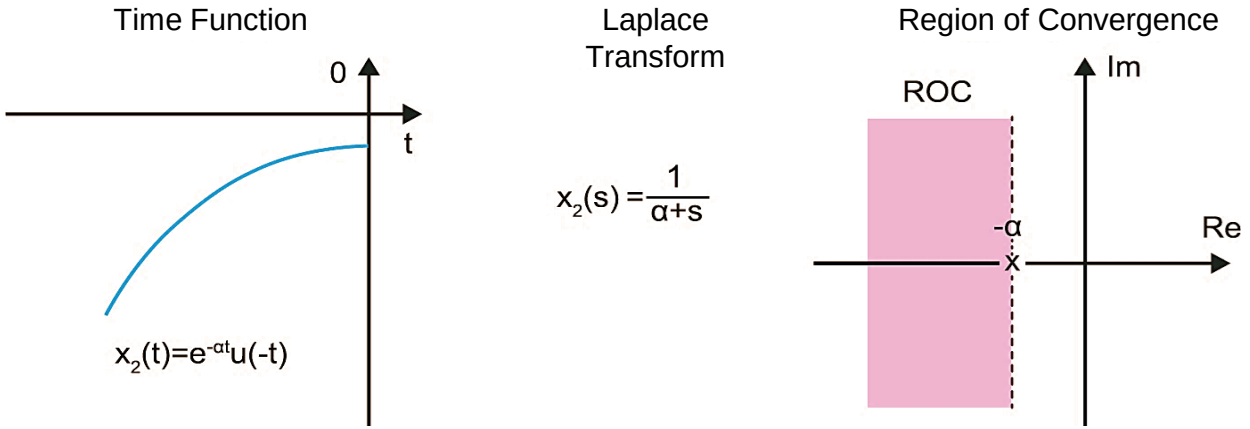
The transform has the same value as in the previous case:

$$\boldsymbol{X_2(s) = \frac{1}{s+\alpha}}$$

but the ROC is $\boldsymbol{Re(s) < -\alpha}$.

Exercise 4 (continued)

- Two different signals have the same LT but different ROCs.



- Both signals are **stable.** In both cases, the boundary value (pole) of the ROC lies in the left half of the complex frequency plane s.
- Pole locations in the *s* plane are related to the stability of an LTI system.

Exercise 5

Calculate the Laplace transforms of:

$$\text{(a) } x_1(t) = [\cos(\Omega_0 t)]\, u(t) \text{ and (b) } x_2(t) = [\sin(\Omega_0 t)]\, u(t).$$

Answer: (a) From the LT definition and Euler's formula $\cos\varphi = \frac{1}{2}\left[e^{j\varphi} + e^{-j\varphi}\right]$, , we obtain:

$$X_1(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} [\cos(\Omega_0 t)]\, u(t) e^{-st}\, dt = \int_{0}^{\infty} \cos(\Omega_0 t)\; e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{j\Omega_0 t}\, e^{-st}\, dt + \frac{1}{2}\int_0^{\infty} e^{-j\Omega_0 t}\, e^{-st} dt = \frac{1}{2} L\left\{e^{j\Omega_0 t}\right\} + \frac{1}{2} L\left\{e^{-j\Omega_0 t}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s - j\Omega_0} + \frac{1}{s + j\Omega_0}\right] = \frac{s}{s^2 + {\Omega_0}^2}$$

For the first limit to vanish, its exponent must be negative:
$j\Omega - s < 0 \Rightarrow j\Omega - \sigma - j\Omega < 0 \Rightarrow \boldsymbol{\sigma > 0}$.

For the second limit to vanish, we require $-(j\Omega + s) < 0 \Rightarrow j\Omega + s > 0 \Rightarrow j\Omega + \sigma + j\Omega > 0 \Rightarrow \boldsymbol{\sigma > 0}$.

Thus, the ROC is $Re(s) > 0$.

Exercise 5 (continued)

(b) Similarly, using Euler's formula for the sine function $\sin\varphi = \frac{1}{2j}\left[e^{j\varphi} - e^{-j\varphi}\right]$, we obtain:

$$X_2(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} [\sin(\Omega_0 t)]\, u(t) e^{-st}\, dt = \int_{0}^{\infty} \sin(\Omega_0 t)\ e^{-st}\, dt$$

$$= \frac{1}{2j}\int_{0}^{\infty} e^{j\Omega_0 t}\, e^{-st}\, dt - \frac{1}{2j}\int_{0}^{\infty} e^{-j\Omega_0 t}\, e^{-st} dt = \frac{1}{2j} L\left\{e^{j\Omega_0 t}\right\} - \frac{1}{2j} L\left\{e^{-j\Omega_0 t}\right\}$$

$$= \frac{1}{2j}\left[\frac{1}{s - j\Omega_0} - \frac{1}{s + j\Omega_0}\right] \quad = \frac{\Omega_0}{s^2 + {\Omega_0}^2}$$

The ROC is $Re(s) > 0$.

Exercise 6

Calculate the Laplace transforms of:

$$\text{(a) } x_1(t) = e^{-\alpha t}cos(\Omega_0 t)u(t) \qquad \text{and (b) } x_2(t) = e^{-\alpha t}sin(\Omega_0 t)u(t)$$

Answer: (a) From the definition of the Laplace transform:

$$X_1(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} [e^{-\alpha t}cos(\Omega_0 t)]\, u(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-\alpha t}\, cos(\Omega_0 t)\, e^{-st}\, dt$$

$$= \frac{1}{2}\int_{0}^{\infty} e^{-\alpha t}e^{j\Omega_0 t}\, e^{-st}\, dt + \frac{1}{2}\int_{0}^{\infty} e^{-\alpha t}e^{-j\Omega_0 t}\, e^{-st} dt = \frac{1}{2}L\{e^{-(a-j\Omega_0)t}\} + \frac{1}{2}L\{e^{-(a+j\Omega_0)t}\}$$

$$= \frac{1}{2}\left[\frac{1}{s+a-j\Omega_0} + \frac{1}{s+a+j\Omega_0}\right] = \frac{s+a}{(s+a)^2 + {\Omega_0}^2}$$

Region of convergence: $Re(s) > -a$.

Exercise 6 (continued)

(b) Similarly, we obtain:

$$X_2(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} [e^{-\alpha t} sin(\Omega_0 t)]\, u(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-\alpha t}\, sin(\Omega_0 t)\, e^{-st}\, dt$$

$$= \frac{1}{2j}\int_{0}^{\infty} e^{-\alpha t} e^{j\Omega_0 t}\, e^{-st}\, dt - \frac{1}{2j}\int_{0}^{\infty} e^{-\alpha t} e^{-j\Omega_0 t}\, e^{-st} dt = \frac{1}{2j} L\{e^{-(a-j\Omega_0)t}\} - \frac{1}{2j} L\{e^{-(a+j\Omega_0)t}\}$$

$$= \frac{1}{2j}\left[\frac{1}{s+a-j\Omega_0} - \frac{1}{s+a+j\Omega_0}\right] = \frac{\Omega_0}{(s+a)^2 + {\Omega_0}^2}$$

Region of convergence: $Re(s) > -a$.

Exercise 7

For $\alpha > 0$ and $b > 0$, consider (a) $x_1(t) = e^{-at}u(t)$ and (b) $x_2(t) = e^{-at}u(t) + e^{bt}u(-t)$. Calculate the unilateral and bilateral Laplace transforms of $x_1(t)$ and $x_2(t)$, and interpret the result.

Answer: (a) Unilateral transform of $x_1(t)$:

$$X_1^+(s) = \int_0^\infty e^{-(s+a)t}\, dt = \frac{1}{s+a}, \qquad Re(s) > -a$$

For $x_2(t)$, the unilateral transform considers only the part for $t \geq 0$, so:

$$X_2^+(s) = \int_0^\infty e^{-(s+a)t}\, dt = \frac{1}{s+a}$$

Bilateral transform of $x_1(t)$:

$$X_1(s) = \frac{1}{s+a}, \qquad Re(s) > -a$$

Exercise 7 (continued)

(b) For $x_2(t)$:

$$X_2(s) = \frac{1}{s+a} + \frac{1}{b-s}, \qquad -a < Re(s) < b$$

- This example shows the difference between the unilateral and bilateral Laplace transforms. The unilateral transform considers the signal only for $t \geq 0$, with the past affecting the result through initial conditions. The bilateral transform describes the signal over all time.
- The unilateral Laplace transform is suitable for causal systems and differential equations with initial conditions. The bilateral transform provides a complete description of the signal over time.
- The distinction is crucial when studying physical systems with initial conditions or previously stored energy.