UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
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SIGNALS AND SYSTEMS
Solved Examples – Tutorial Handout 5

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METHODS FOR EVALUATING THE CONVOLUTION INTEGRAL

- Analytical evaluation of the convolution integral
- Graphical evaluation of the convolution integral
- Interconnections of LTI systems

1. Analytical Evaluation of the Convolution Integral

📖Example 1 (*)

(a) Calculate the convolution of the signals $x(t) = e^{-\alpha t}u(t)$ and $y(t) = e^{-\beta t}u(t)$.

(b) Assume $\alpha \neq \beta, \alpha \neq 0$ and $\beta \neq 0$. (b) Repeat the calculation for $\alpha = \beta$.

Answer: (a) The convolution of the signals $x(t)$ and $y(t)$ is given by:

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-\alpha\tau}\, u(\tau)\, e^{-\beta(t-\tau)}\, u(t-\tau)\, d\tau$$

where the step functions $u(\tau)$ and $u(t-\tau)$ are defined as:

$$u(\tau) = \begin{cases} 1, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases} \quad \text{and} \quad u(t-\tau) = \begin{cases} 1, & \tau \leq t \\ 0, & \tau > t \end{cases}$$

In general, this integral must be evaluated over the infinite interval $(-\infty, +\infty)$. However, the two step functions restrict the calculation to $0 < \tau < t$, since both are nonzero (and equal to one) only over $0 < \tau \leq t$, and their product is zero elsewhere. Thus:

$$z(t) = \int_0^t e^{-\alpha\tau}\, e^{-\beta(t-\tau)}\, d\tau = e^{-\beta t} \int_0^t e^{(\beta-\alpha)\tau}\, d\tau = e^{-\beta t} \left[\frac{e^{(\beta-\alpha)\tau}}{\beta-\alpha}\right]_0^t$$

$$= \frac{e^{-\beta t}}{\beta-\alpha}\left[e^{(\beta-\alpha)t} - 1\right] = \frac{e^{-\alpha t} - e^{-\beta t}}{\beta-\alpha}$$

This relation holds for $t > 0$. For $t < 0$, no value of τ simultaneously satisfies $\tau > 0$ and $\tau < t < 0$. The overlap is therefore empty and the convolution is $z(t) = 0$. We can combine these cases into one equation:

$$z(t) = \frac{e^{-\alpha t} - e^{-\beta t}}{\beta-\alpha} u(t)$$

(b) If $a = b$, then:

$$x(t) = y(t) = e^{-at}u(t)$$

and the convolution becomes:

$$z(t) = \int_0^t e^{-a\tau}\, e^{-a(t-\tau)}\, d\tau = e^{-at} \int_0^t d\tau = te^{-at}u(t)$$

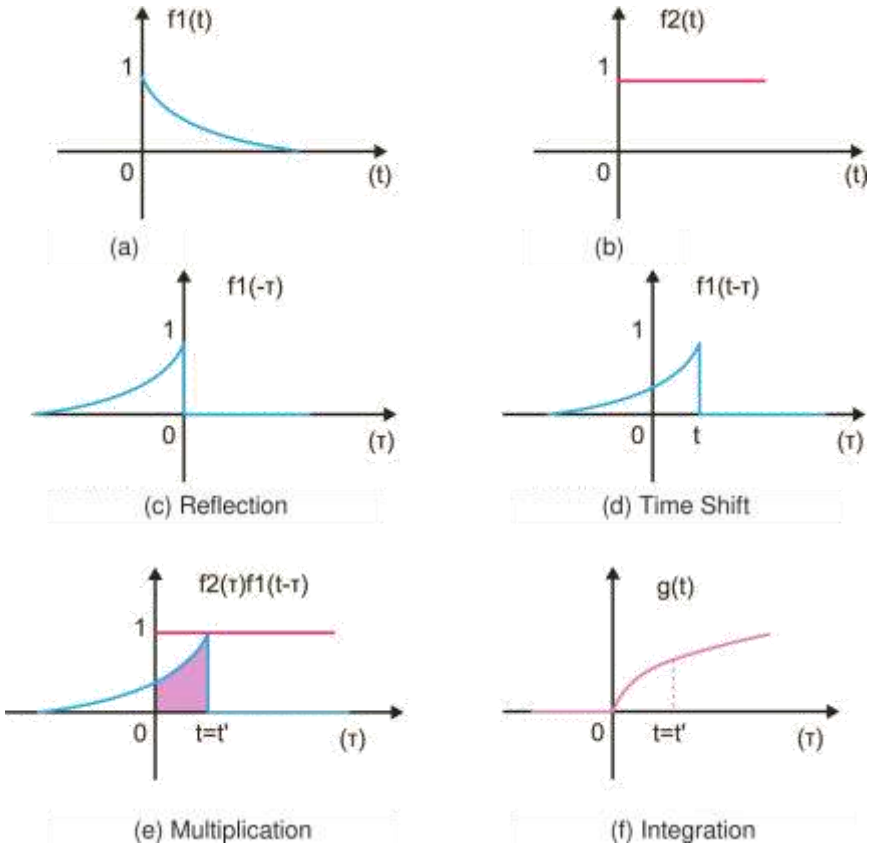
Comments:

- Convolution of two causal exponentials occurs in many practical problems and yields a closed-form result with a clear mathematical interpretation.
- The key step is to determine the integration limits correctly. Although convolution is defined over the entire time axis, the integral is restricted in practice to the interval where both signals are nonzero. When these intervals do not overlap, the convolution is zero.
- When the two exponentials have different rates of change, the result is their difference. When their rates coincide, the term $t\, e^{-at}$ appears. Thus, convolution produces a form similar to that arising from repeated roots of differential equations, as seen in Examples 4 and 5.

2. Graphical Evaluation of the Convolution Integral

📖Example 2
Calculate the convolution of the functions $f_1(t) = e^{-t}u(t)$ and $f_2(t) = u(t)$.

Answer:



Keep $f_2(\tau) = u(\tau)$ fixed, reflect $f_1(t)$ to obtain $f_1(-\tau) = e^{\tau}u(-t)$, and then shift it in time to obtain $f_1(t-\tau) = e^{t+\tau}u(t-\tau)$. Calculate the area between the graphs of the two functions using the integral:

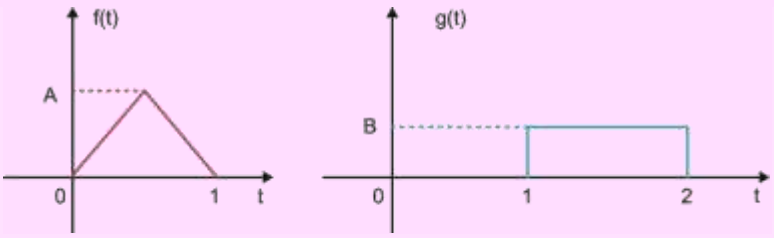
$$g(t) = f_1(t) * f_2(t) = \int_{-\infty}^{+\infty} f_2(\tau) f_1(t-\tau) dt$$

Since the graphical method (panels (c) to (f)) produces areas that cannot be calculated using simple geometric shapes, continue by evaluating the integral:

$$g(t) = \int_0^t 1\, e^{-(t-\tau)} dt = e^{-t}\left([e^{\tau}]_0^t\right) = e^{-t}[e^t - 1] = 1 - e^{-t}$$

Example 3

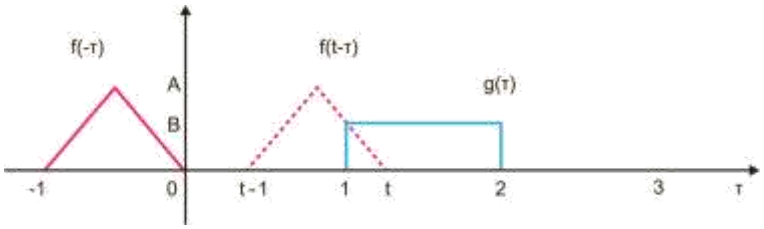
Calculate the convolution integral of the signals $f(t)$ and $g(t)$ shown in the figure.



Answer: The convolution integral is $f(t) * g(t) = \int_{-\infty}^{+\infty} g(\tau) f(t-\tau) dt$. Evaluate it over the following intervals of time (t):

1) $t < 1$: $\quad f(t) * g(t) = 0$

2) $1 < t \leq 3/2$: $\quad f(t) * g(t) = B\int_1^t (-2A\tau + 2At) dt = ABt^2 - 2ABt + AB$ The calculation is based on the diagram:

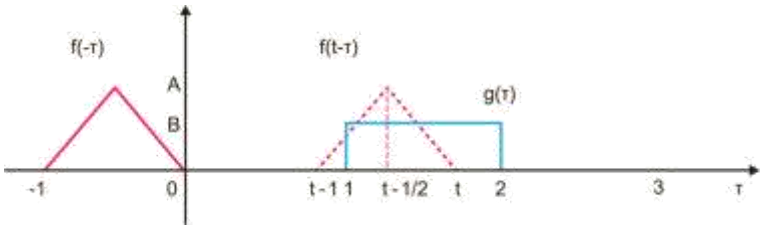


Calculation for $1 < t \leq \frac{3}{2}$

3) $3/2 \leq t \leq 5/2$:

$$f(t) * g(t) = \int_1^t g(\tau) f(t-\tau)\, d\tau =$$

$$= B\int_1^{t-1/2} [2A\tau + 2A(1-t)] d\tau + B\int_{t-1/2}^{t} [-2A\tau + 2At] d\tau = -ABt^2 + 4ABt - \frac{7}{2}ABt$$

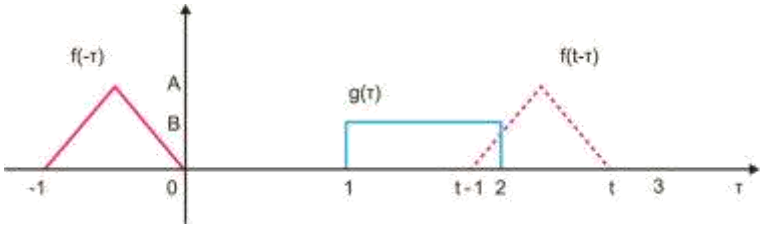


Calculation for $3/2 \leq t \leq 5/2$

For the interval $2 \leq t \leq 5/2$, the integration limits are from $t-1$ to 2, but the result is unchanged. We therefore use this expression for the entire interval $3/2 \leq t \leq 5/2$.

4) $5/2 \leq t \leq 3$:

$$f(t) * g(t) = \int_{t-1}^{2} g(\tau) f(t-\tau) dt = \int_{t-1}^{2} [2A\tau + 2A(1-t)] dt = AB\, t^2 - 6AB\, t + 9AB$$



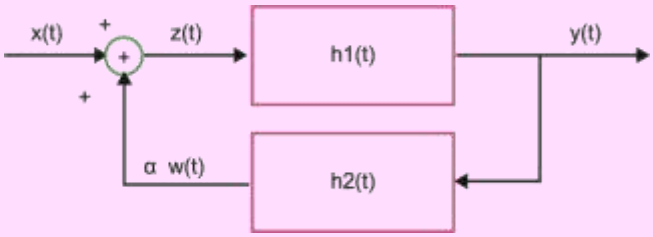
Calculation for $5/2 \leq t \leq 3$

5) $f(t) * g(t) = 0$, for $t \geq 3$

3. Calculating the Equivalent System

Example 4

The figure shows a feedback interconnection of systems initially at rest.



Calculate the equivalent impulse response.

Answer: For negative feedback and zero initial conditions, set:

$$z(t) = x(t) - h_2(t) * y(t)$$

Since the forward system has impulse response $h_1(t)$, the output is:

$$y(t) = h_1(t) * z(t)$$

Thus, the output is:

$$y(t) = h_1(t) * [x(t) - h_2(t) * y(t)]$$

$$y(t) = h_1(t) * x(t) - h_1(t) * h_2(t) * y(t)$$

Applying the Laplace transform, with $H_1(s) = L\{h_1(t)\}$ and $H_2(s) = L\{h_2(t)\}$, gives:

$$Y(s) = X(s)\, H_1(s) - H_1(s)\, H_2(s)\, Y(s)$$

Hence:

$$Y(s)[1 + H_1(s)\, H_2(s)] = H_1(s)\, X(s)$$

and the transfer function of the equivalent system is:

$$H_{eq}(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)\, H_2(s)}$$

The equivalent impulse response therefore cannot be obtained by pointwise division of $h_1(t), h_2(t)$ in the time domain (which is not permitted). Instead, use the inverse Laplace transform:

$$h_{eq}(t) = L^{-1}\left\{\frac{H_1(s)}{1 + H_1(s)\, H_2(s)}\right\}$$

For positive feedback, only the sign in the denominator changes:

$$h_{eq}(t) = L^{-1}\left\{\frac{H_1(s)}{1 - H_1(s)\, H_2(s)}\right\}$$

This is the rigorous result for a feedback interconnection of LTI systems.

A feedback interconnection is also called a closed-loop interconnection. If $h_2(t) = 0$, the feedback loop is absent, the interconnection is open-loop, and its equivalent impulse response is $h_1(t)$.

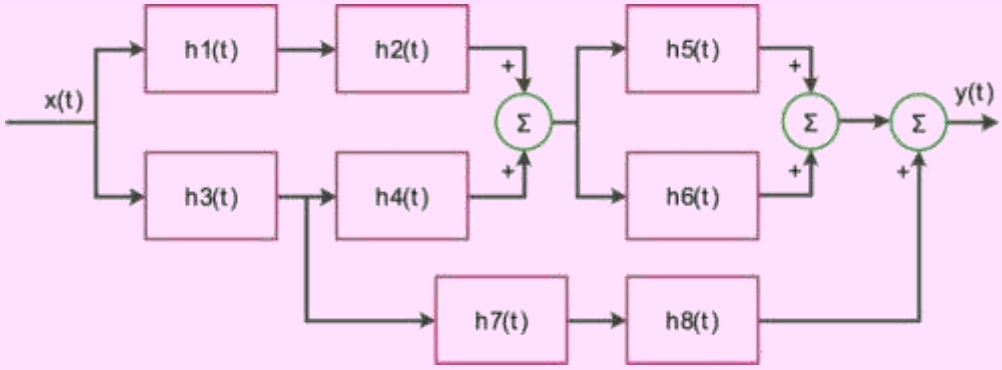
These are the interconnections most commonly encountered in practice. The next example simplifies an interconnection by calculating the equivalent impulse response step by step.

This procedure is useful because it replaces a complex arrangement with a single equivalent system while preserving the overall input–output relation. The analysis becomes simpler and more systematic: each part is evaluated in turn, and the complete system is ultimately described by one equivalent impulse response.

4. Interconnections of LTI Systems

Example 5

Calculate the equivalent impulse response of the following interconnection of LTI systems.



Answer: Divide the given interconnection into three parts:

- The left part consists of systems with impulse responses $h_1(t), h_2(t), h_3(t), h_4(t)$, in a combination of cascade and parallel connections;
- the right part consists of systems with impulse responses $h_5(t), h_6(t)$, connected in parallel; and
- the bottom part consists of systems with impulse responses $h_3(t), h_7(t), h_8(t)$, connected in cascade.

The upper branch of the left part, containing systems with impulse responses $h_1(t), h_2(t)$, simplifies to:

$$h_{12}(t) = h_1(t) * h_2(t)$$

Similarly, the lower branch of the left part simplifies to:

$$h_{34}(t) = h_3(t) * h_4(t)$$

Thus, the entire left part simplifies to:

$$h_{1234}(t) = h_{12}(t) + h_{34}(t) = h_1(t) * h_2(t) + h_3(t) * h_4(t)$$

Similarly, the right part simplifies to:

$$h_{56}(t) = h_5(t) + h_6(t)$$

The left and right parts are connected in cascade, so they reduce to a single equivalent impulse response:

$$h_A(t) = h_{1234}(t) * h_{56}(t) = [h_1(t) * h_2(t) + h_3(t) * h_4(t)] * [h_5(t) + h_6(t)]$$

The bottom part reduces to a single equivalent impulse response:

$$h_B(t) = h_3(t) * h_7(t) * h_8(t)$$

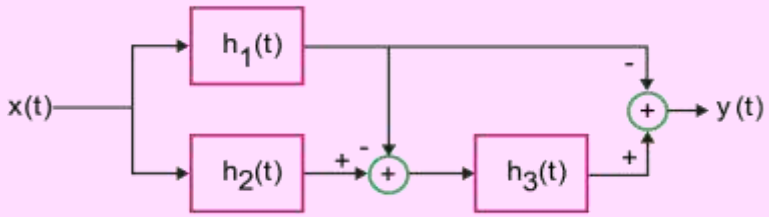
The upper and lower parts, with impulse responses $h_A(t)$ and $h_B(t)$, respectively, are connected in parallel because their outputs are added. Thus, the total equivalent impulse response is the sum of

$h_A(t)$ and $h_B(t)$:

$$h_{eq}(t) = h_A(t) + h_B(t)$$
$$*= [h_1(t) * h_2(t) + h_3(t) * h_4(t)] * [h_5(t) + h_6(t)] + [h_3(t) * h_7(t) * h_8(t)]$$
$$= [h_1(t) * h_2(t)] * [h_5(t) + h_6(t)] + [h_3(t) * h_4(t)] * [h_5(t) + h_6(t)] + [h_3(t) * h_7(t) * h_8(t)]$$
$$= h_1(t) * h_2(t) * h_5(t) + h_1(t) * h_2(t) * h_6(t) + h_3(t) * h_4(t) * h_5(t)$$
$$+h_3(t) * h_4(t) * h_6(t) + h_3(t) * h_7(t) * h_8(t)$$

📖Example 6 (*)

Calculate the equivalent impulse response of the following interconnection of LTI systems with individual impulse responses $h_1(t) = u(t-2)$, $h_2(t) = u(t)$ and $h_3(t) = \delta(t-2)$.



Answer: The systems with impulse responses $h_1(t), h_2(t)$ are connected in parallel, so they simplify to:

$$h_{12}(t) = h_2(t) - h_1(t) = u(t) - u(t-2)$$

The system with impulse response $h_3(t)$ is connected in cascade with the systems $h_1(t), h_2(t)$, so they simplify to:

$$h_{123}(t) = h_{12}(t) * h_3(t) = [u(t) - u(t-2)] * \delta(t-2) = u(t-2) - u(t-4)$$

The equivalent impulse response of the complete system is:

$$h_{eq}(t) = h_{123}(t) - h_1(t) = u(t-2) - u(t-4) - u(t-2) = -u(t-4)$$

Note: Examples marked with an asterisk (*) also appear in the lecture slides.