

UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Signals and Systems

Lecture 5: Methods for Evaluating the Convolution Integral

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Lecture Contents

- Analytical evaluation of the convolution integral
- Graphical evaluation of the convolution integral
- Convolutions of common continuous-time signals
- System step response
- System response to a complex exponential input
- Interconnections of LTI systems

1. Analytical Evaluation of the Convolution Integral

Analytical Evaluation

- Analytical evaluation applies when the functions $x(t)$ and $h(t)$ have simple analytical expressions.
- Special care is needed when $x(t)$ and $h(t)$ are discontinuous or have different analytical expressions over different time intervals.
- For functions $x(\tau)$ and $h(t-\tau)$ defined on $\{L_1, U_1\}$ and $\{L_2, U_2\}$, respectively, choose $max(L_1, L_2)$ as the lower integration limit and $max(U_1, U_2)$ as the upper limit.
- The limits L_1 and U_1 of $x(\tau)$ remain fixed, whereas the limits of $h(t-\tau)$ change as t varies.
- Plotting the functions helps determine the correct integration limits.

2. Graphical Evaluation of Convolution

Graphical Evaluation of Convolution

To calculate the output of an LTI system using the convolution $y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$, follow these steps:

- **Reflection of $h(\tau)$**. Reverse the impulse response $h(\tau)$ to obtain $h(-\tau)$.
- **Time shift of $h(-\tau)$ by t**. Shift $h(-\tau)$ by t to obtain $h(t-\tau)$.
- **Multiply $h(t-\tau)$ by the input $x(\tau)$** to obtain the product $x(\tau)\, h(t-\tau)$.
- **Integration** or area calculation. Integrate this product (or calculate the area between its graph and the time axis). The result equals the system output $y(t)$ at time t, the shift applied in step 2.
- **Repetition**. Repeat these steps for different values of time t, that is, $-\infty < t < \infty$.

Graphical Evaluation of Convolution

By the commutative property of convolution, the relation:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

can be written as:

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Thus, the functions $x(t)$ and $h(t)$ can exchange roles in the iterative procedure described above.

Stages of Graphical Convolution

The graphical method has four stages when a finite-duration signal $x(t)$ passes through an LTI system with a finite-duration impulse response $h(t)$:

- **No overlap**: During these intervals, the signals do not overlap. Their product is zero, so the convolution is zero.
- **Entry**: The input signal $x(t)$ begins to overlap the reflected impulse response $h(-t)$. Evaluating the convolution integral over this interval gives a function describing the system output $y(t)$ for every t in the interval.
- **Full overlap**: The signal $x(t)$ slides while remaining entirely within the impulse response $h(t)$. As in the previous stage, the convolution integral gives a function describing the system output $y(t)$ over this interval.
- **Exit:** Evaluate the convolution integral over the interval in which $x(t)$ leaves the impulse response $h(t)$, up to the time when the signals no longer overlap.

Convolutions of Common Continuous-Time Signals

$x(t)$	$y(t)$	$z(t) = x(t) * y(t)$
$e^{at}\,u(t)$	$e^{\beta t}\,u(t)$	$\dfrac{e^{at} - e^{\beta t}}{a - \beta}u(t), \qquad a \neq \beta$
$t\,e^{at}\,u(t)$	$e^{\beta t}\,u(t)$	$\dfrac{e^{\beta t} - e^{at} + (a-\beta)te^{at}}{(a-\beta)^2}u(t)$
$t^M\,e^{at}\,u(t)$	$t^N e^{at}\,u(t)$	$\dfrac{M!\,N!}{(N+M+1)!}\,t^{M+N+1}\,e^{at}\,u(t)$
$t^M\,u(t)$	$t^N\,u(t)$	$\dfrac{M!\,N!}{(N+M+1)!}\,t^{M+N+1}\,u(t)$
$t^N\,u(t)$	$e^{at}\,u(t)$	$\dfrac{N!\,e^{at}}{a^{N+1}}u(t) - \sum_{k=0}^{N}\dfrac{N!\,t^{N-k}}{a^{k+1}(N-k)!}u(t)$
$e^{at}u(t)$	$e^{\beta t}u(-t)$	$\dfrac{e^{at}u(t) + e^{\beta t}u(-t)}{\beta - \alpha}$
$e^{at}u(t)$	$e^{\beta t}u(-t)$	$\dfrac{e^{at}u(t) + e^{\beta t}u(-t)}{\beta - \alpha}$

3. Step Response of an LTI System

Step Response

- The step response is the output when the input is the unit step function $u(t)$..
- A step input represents the instantaneous application of a constant excitation. The system begins to change its response, but this change cannot occur instantaneously.
- Calculate the step response $s(t)$ using:

$$s(t) = u(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau = \int_{-\infty}^{t} h(\tau) d\tau$$

- For a causal system, the lower integration limit is zero.
- Thus, the step response is the integral of the impulse response.
- The first derivative of the step response gives the impulse response:

$$\frac{ds(t)}{dt} = h(t)$$

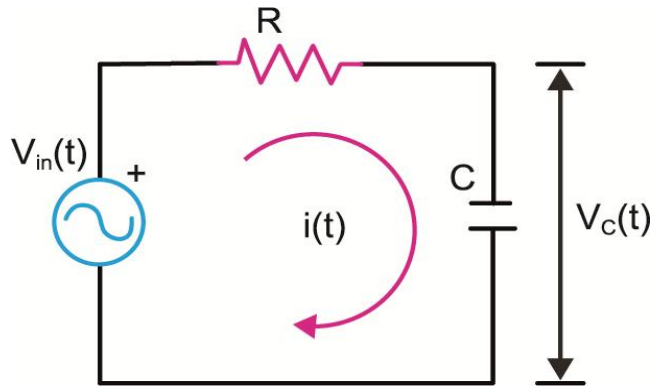
- These relations allow us to obtain the step response from the impulse response, and vice versa.

Exercise 9

Consider the LTI RC circuit with $R = 10\Omega$ and $C = 1F$. Calculate its step response: the capacitor voltage $v_c(t)$, with the source voltage $v_{in}(t)$ as the input. Assume $v_c(0) = 0$.

Answer: The differential equation describing the system is:

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_{in}(t)$$



and the impulse response is:

$$h(t) = \frac{1}{RC}e^{-t/RC}\,u(t)$$

For $R = 10\Omega$ and $C = 1F$, the impulse response becomes:

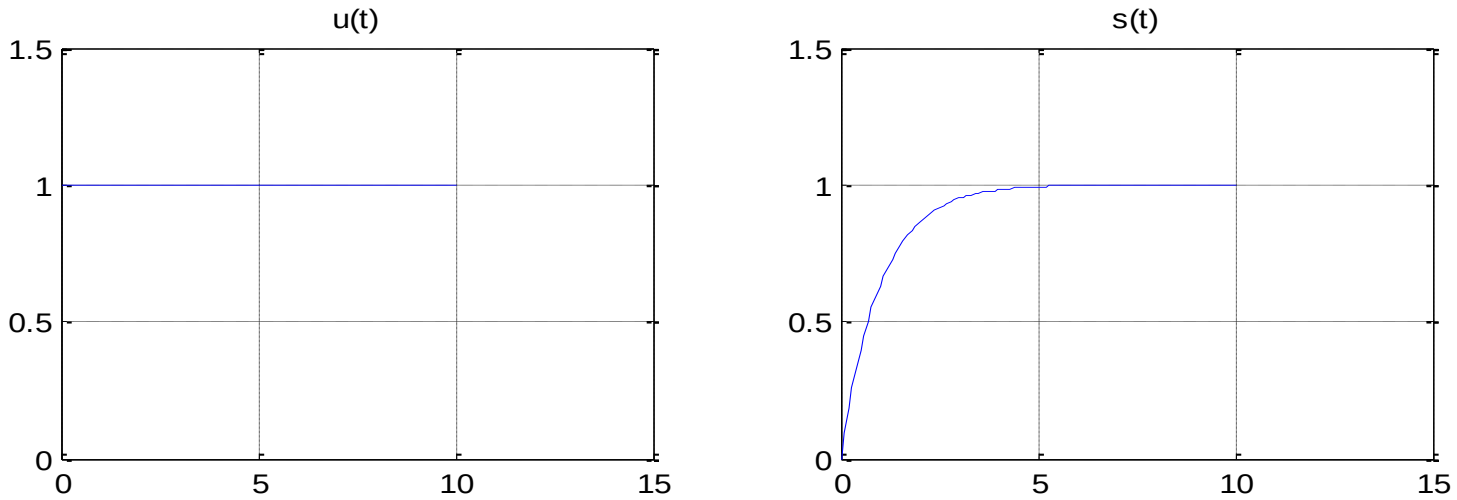
$$h(t) = 0.1\,e^{-0.1t}\,u(t)$$

Integrating the impulse response gives the step response:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau = \int_{-\infty}^{t} 0.1\,e^{-0.1t}\,u(t)d\tau = 0.1\int_{0}^{t} e^{-0.1t}d\tau = -e^{-0.1t}\Big|_{0}^{t} = \cdots = (1 - e^{-0.1t})\,u(t)$$

Exercise 9 (continued)

The figure shows the step input and the step response of the RC circuit.



- An instantaneous constant input does not change the output instantly. The output gradually approaches its steady value. This behavior results from the dynamic nature of the system and energy storage in the capacitor.
- The impulse response and the system eigenvalues determine the response shape. The transition time is directly related to the circuit time constant.

4. Response of an LTI System to a Complex Exponential Input

Response to a Complex Exponential Input

When a stable LTI system receives a complex exponential input (or a combination of sines and cosines) at a given frequency, of the form:

$$x(t) = Ae^{j(\Omega_0 t+\varphi_0)}, \qquad \text{where } A > 0 \text{ and } \varphi_0 \in (-\pi, \pi]$$

the output follows from convolution:

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau = A\int_{-\infty}^{\infty} e^{j(\Omega_0(t-\tau)+\varphi_0)}\, h(\tau)\, d\tau = Ae^{j\Omega_0 t}e^{j\varphi_0}\int_{-\infty}^{\infty} e^{-j\Omega_0\tau}\, h(\tau)\, d\tau$$

Define the integral as:

$$H(\Omega_0) = \int_{-\infty}^{\infty} e^{-j\Omega_0\tau}\, h(\tau)\, d\tau$$

Then the output $y(t)$ is given by:

$$y(t) = Ae^{j(\Omega_0 t+\varphi_0)}\, H(\Omega_0) = x(t)\, H(\Omega_0)$$

Response to a Complex Exponential Input

- When a stable LTI system receives a complex exponential input at frequency Ω_0, its output equals the input $x(t)$ multiplied by a complex constant $H(\Omega_0)$. This constant describes the system response at the input frequency Ω_0. This result provides a foundation for frequency-domain analysis of signals and systems.
- Complex exponential functions are eigenfunctions of LTI systems, with corresponding eigenvalue $H(\Omega_0)$. The constant $H(\Omega_0)$ is the value of the frequency response $H(\Omega)$ at frequency Ω_0. We will introduce **frequency response** and the **transfer function** later.
- Because the system is LTI, this property extends to any number of complex exponentials whose sum forms the input signal.
- If the complex functions are conjugates, $x(t)$ is a real signal. This applies to sinusoidal signals, which Euler's formula expresses as the sum of conjugate complex exponentials.

5. Interconnections of LTI Systems

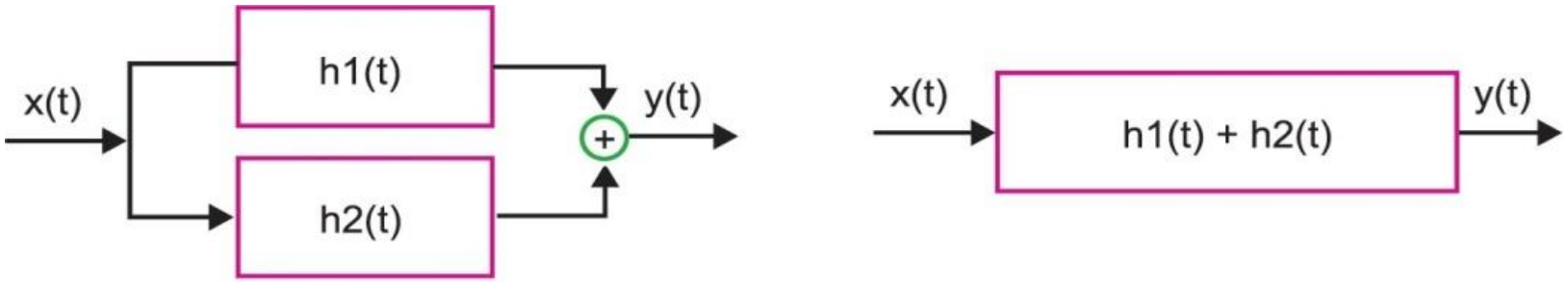
Interconnections of LTI Systems

- Cascade connection of LTI systems:



$$h_{eq}(t) = h_1(t) * h_2(t)$$

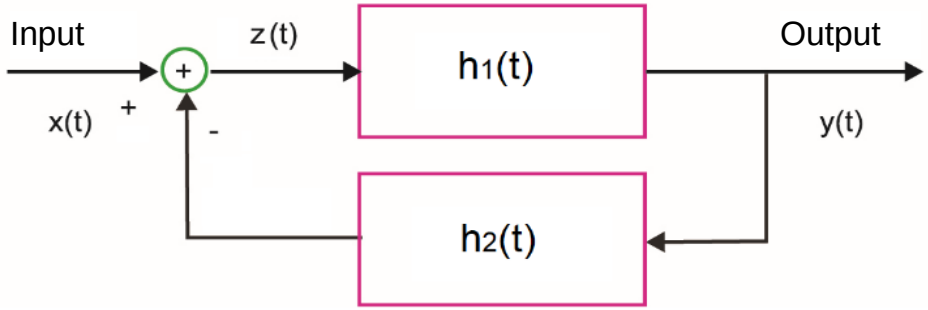
- Parallel connection of LTI systems:



$$h_{eq}(t) = h_1(t) + h_2(t)$$

Interconnections of LTI Systems

- Feedback connection of LTI systems:



Using the Laplace transform, the equivalent impulse response is:

$$h_{eq}(t) = L^{-1}\left\{\frac{H_1(t)}{1 + H_1(t) * H_2(t)}\right\}$$

- A feedback interconnection is also called a **closed-loop** connection.
- If $h_2(t) = 0$, the feedback loop is absent. This is an **open-loop** connection, with equivalent impulse response $h_{eq}(t) = h_1(t)$.

Exercises

Exercise 1

Calculate the impulse response of the moving-average system:

$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\, d\tau$$

Answer: The impulse response of the moving-average system $h(t)$ equals its output when the input is $\delta(t)$, that is:

$$h(t) = \frac{1}{T}\int_{t-T}^{t} \delta(\tau)\, d\tau = \frac{1}{T}\int_{t-T}^{t} du(\tau) = \frac{1}{T}u(t)\Big|_{t-T}^{t} = \frac{1}{T}[u(t) - u(t-T)]$$

where we used:

$$\delta(\tau) = \frac{du(\tau)}{d\tau}$$

Exercise 2

Calculate the impulse response of the **integrator**, described by the input–output relation:

$$y(t) = \boldsymbol{S}[\, x(t)] = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Answer: The integrator impulse response equals its output when the input is $\delta(t)$. Thus:

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

Exercise 3

(a) Calculate the convolution of $x(t) = e^{-\alpha t}u(t)$ and $y(t) = e^{-\beta t}u(t)$. Assume $\alpha \neq \beta, \alpha \neq 0$ and $\beta \neq 0$. (b) Repeat for $\alpha = \beta$.

Answer: (a) The convolution of $x(t)$ and $y(t)$ is:

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-\alpha\tau}\, u(\tau)\, e^{-\beta(t-\tau)}\, u(t-\tau)\, d\tau$$

where the step functions $u(\tau)$ and $u(t-\tau)$ are defined as:

$$u(\tau) = \begin{cases} 1\ for\ \tau > 0 \\ 0\ for\ \tau < 0 \end{cases} \text{ and } \quad u(t-\tau) = \begin{cases} 1\ for\ \tau < t \\ 0\ for\ \tau > t \end{cases}$$

In general, this integral spans the infinite interval $(-\infty,\ \infty)$. However, the two step functions restrict the integration to $0 < \tau < t$.. Both step functions are **nonzero** (and equal to one) only on $0 < \tau < t$, ; their product is **zero** everywhere else.

Exercise 3 (continued)

Using this observation, we obtain:

$$z(t) = \int_0^t e^{-\alpha\tau}\, e^{-\beta(t-\tau)}\, d\tau = e^{-\beta t} \int_0^t e^{(\beta-\alpha)\tau}\, d\tau =$$

$$= e^{-\beta t} \left[\frac{e^{(\beta-\alpha)\tau}}{\beta-\alpha}\right]_0^t = \frac{e^{-\beta t}}{\beta-\alpha}\left[e^{(\beta-\alpha)t} - 1\right] = \frac{e^{-\alpha t} - e^{-\beta t}}{\beta-\alpha}$$

This relation holds for $t > 0$. For $t < 0$, no τ satisfies both $\tau > 0$ and $\tau < t < 0$, so the intersection is empty and the convolution is $z(t) = 0$.. Combining both cases gives:

$$z(t) = x(t) * y(t) = \frac{e^{-\alpha t} - e^{-\beta t}}{\beta-\alpha} u(t)$$

(b) If $a = b$, then $x(t) = y(t) = e^{-at}u(t)$, and the convolution becomes:

$$z(t) = \int_0^t e^{-a\tau}\, e^{-a(t-\tau)}\, d\tau = e^{-at} \int_0^t d\tau = te^{-at}u(t)$$

Exercise 4

Calculate the convolution of $x(t) = u(t)$ and $y(t) = u(t)$.

Answer: This is a special case of the previous exercise with $\alpha = \beta = 0$. However, substituting into the solution:

$$z(t) = \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} u(t)$$

gives the indeterminate form $0/0$. Applying L'Hôpital's rule gives:

$$z(t) = \lim_{\alpha \to \beta} \left\{ \frac{\frac{d}{d\alpha}\left(e^{-\alpha t} - e^{-\beta t}\right)}{\frac{d}{d\alpha}(\beta - \alpha)} u(t) \right\} = -\lim_{\alpha \to \beta} \{-te^{-\alpha t} u(t)\} = te^{-\beta t} u(t) = t\, u(t)$$

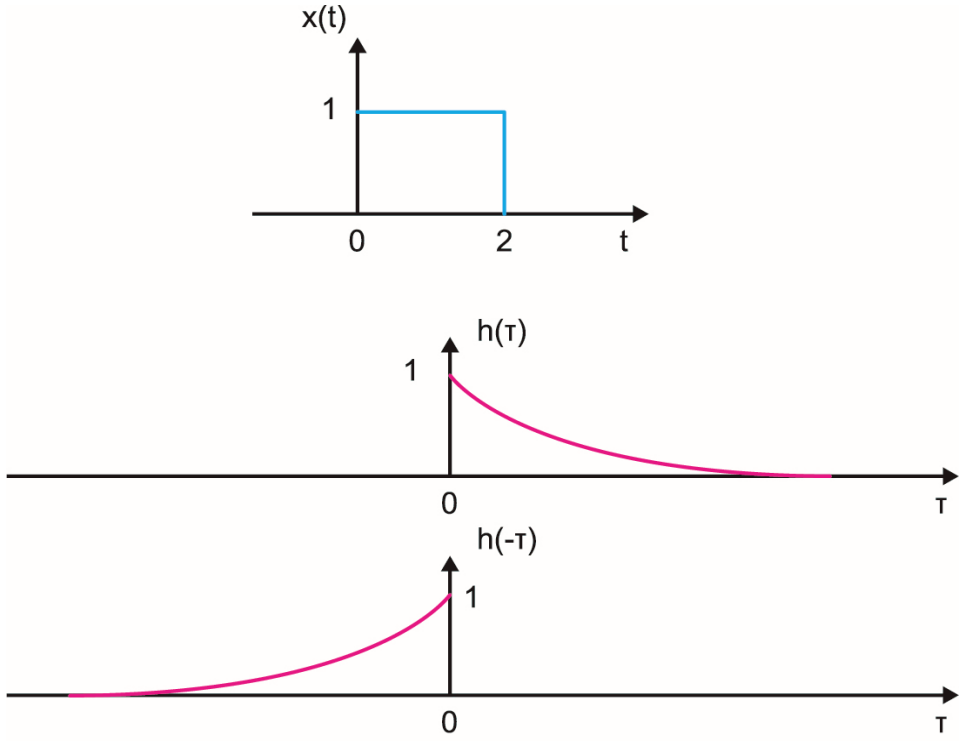
where $t\, u(t) = r(t)$ is the familiar ramp function.

Thus, convolving the step function $u(t)$ with itself gives the ramp function $r(t)$.

Exercise 5

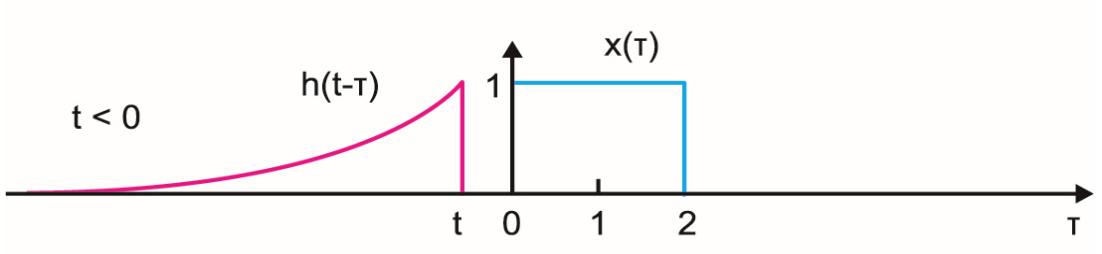
The impulse response of a system is $h(t) = e^{-t}u(t)$. Find its output when the input is $x(t) = u(t) - u(t-2)$.

Answer: Plot the signals $x(t), h(\tau), h(-\tau)$.



Exercise 5 (continued)

Plot $x(\tau)$ and $h(-\tau)$ for $t < 0$.



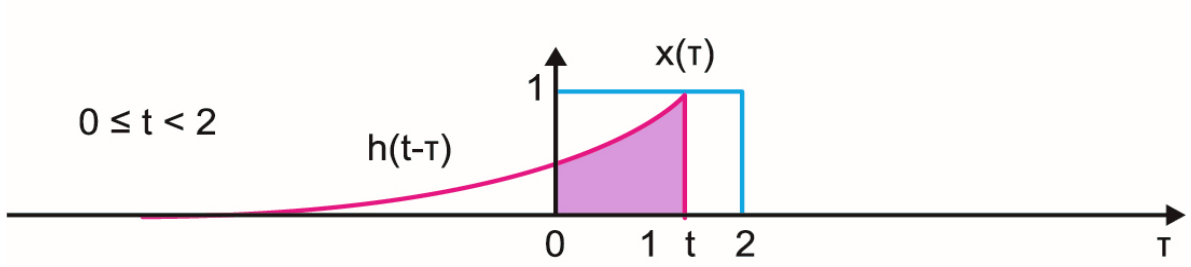
We observe that $h(t-\tau)\, x(\tau) = 0$ for every $t < 0$.

Thus, the system output is $\boldsymbol{y(t) = 0}$ when $t < 0$.

Exercise 5 (continued)

The figure shows the reflected impulse response $h(t-\tau)$, shifted by $0 \le t \le 2$:

The output of the LTI system is:



$$y(t) = \int_{-\infty} x(\tau)h(t-\tau)dt$$

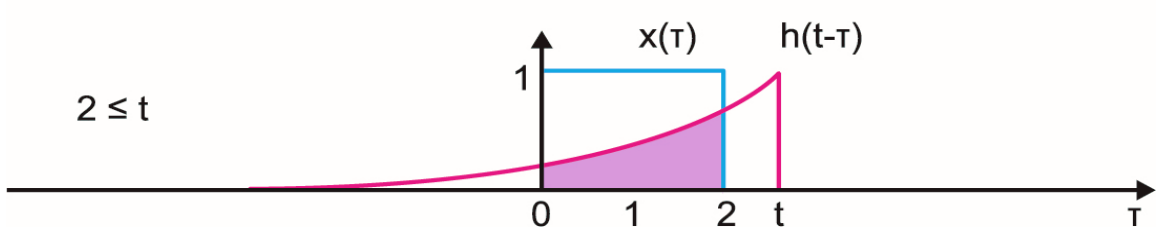
Since $\tau < 0$ gives $x(\tau) = 0$, while $\tau > t$ gives $h(t-\tau) = 0$, the integration limits become:

$$y(t) = \int_0^t h(t-\tau)dt = \int_0^t 1.e^{-(t-\tau)}d\tau = [e^{\tau-t}]_0^t = \mathbf{1} - \boldsymbol{e^{-t}}, \qquad 0 \le t \le 2$$

which equals the area of the shaded region.

Exercise 5 (continued)

For $t > 2$, using the figure:



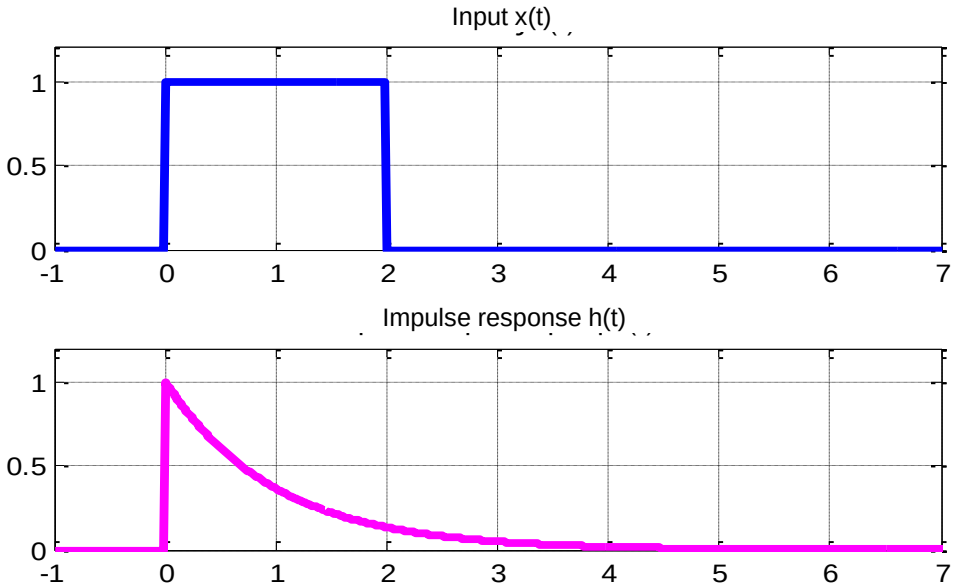
we obtain:

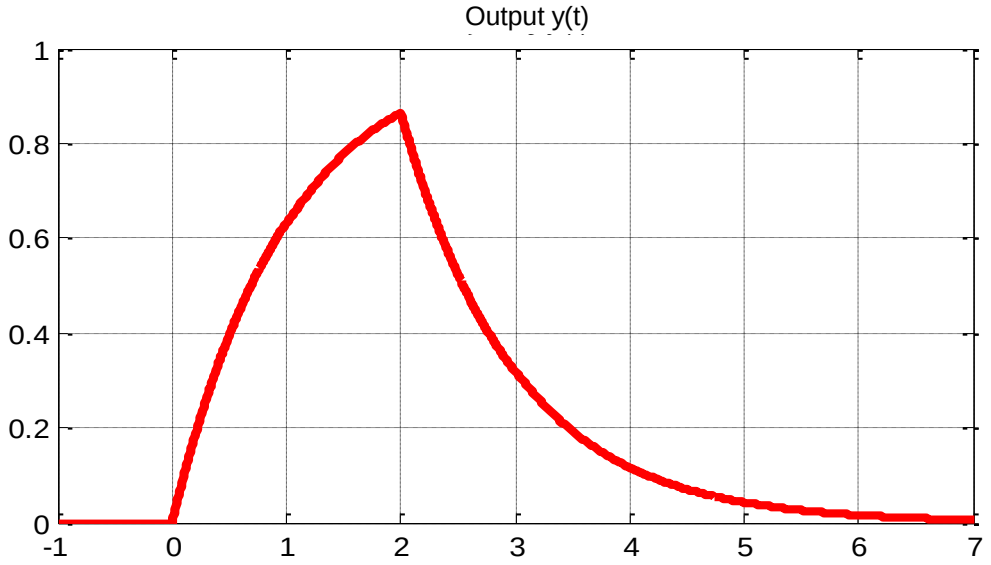
$$y(t) = \int_0^2 1.\, e^{-(t-\tau)} d\tau = e^{\tau - t}\Big|_0^2 = e^{2-t} - e^{-t} = \boldsymbol{\left(e^2 - 1\right)e^{-t}}, \qquad t \geq 2$$

Exercise 5 (continued)

Thus, the system output $y(t)$ is:

$$y(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t}, & 0 \le t < 2 \\ (e^2 - 1)e^{-t}, & t \ge 2 \end{cases}$$





Exercise 6

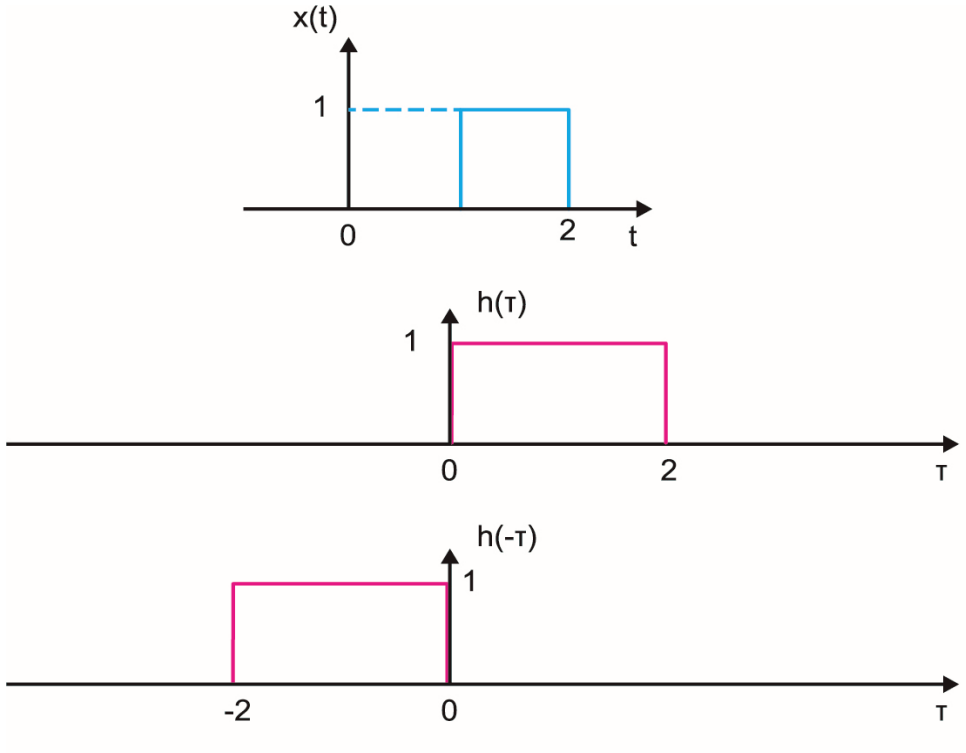
An LTI system has impulse response $h(t) = u(t) - u(t-2)$. Use the convolution integral to calculate its output for the input $x(t) = u(t-1) - u(t-2)$.

Answer: Following the same procedure as in the previous exercise, the system output is:

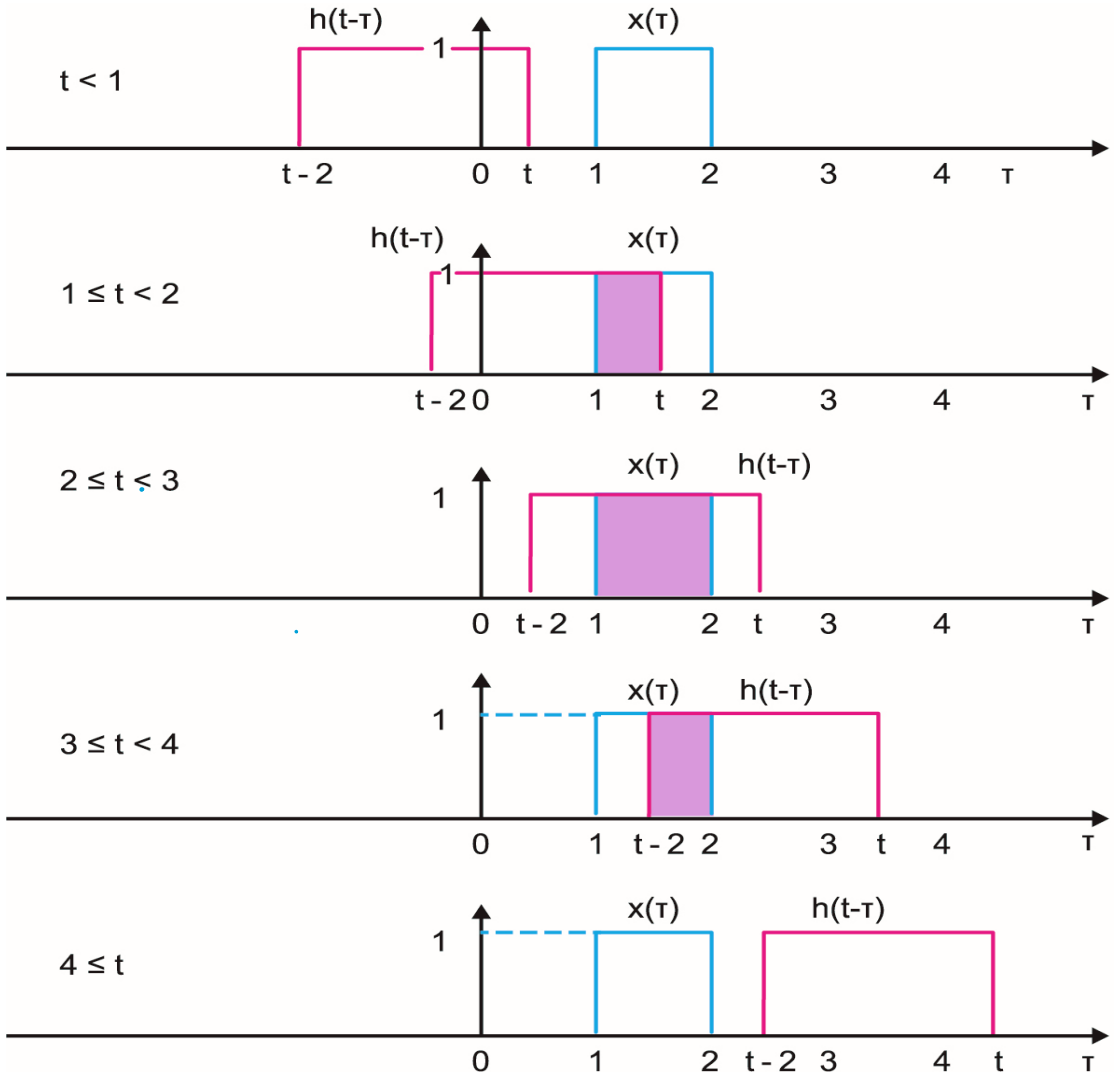
$$y(t) = \begin{cases} t-1 & \text{when } 1 \le t < 2 \\ 1 & \text{when } 2 \le t < 3 \\ 4-t & \text{when } 3 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 6 (continued)

- Plot the signals $x(t), h(\tau), h(-\tau)$.



Exercise 6 (continued)

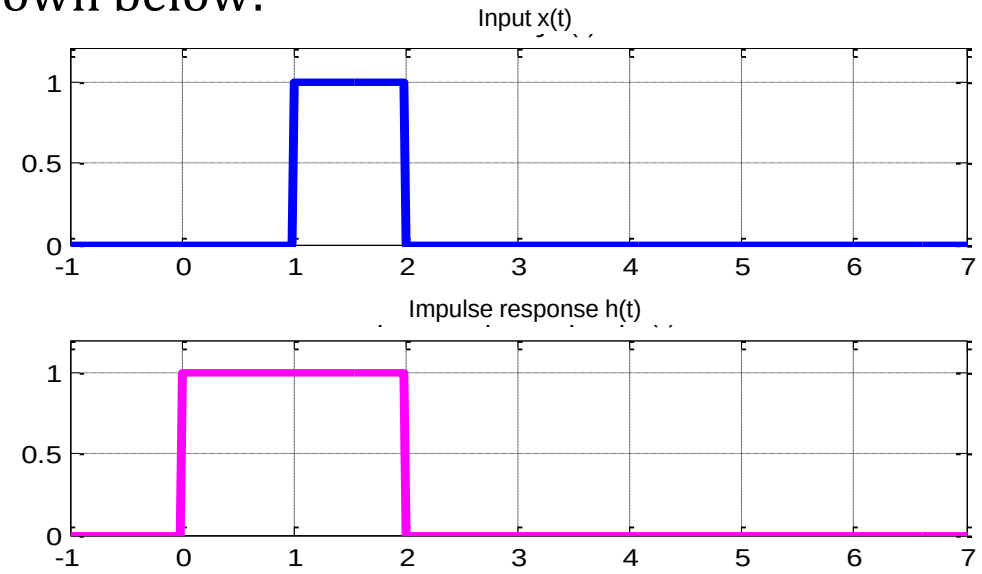


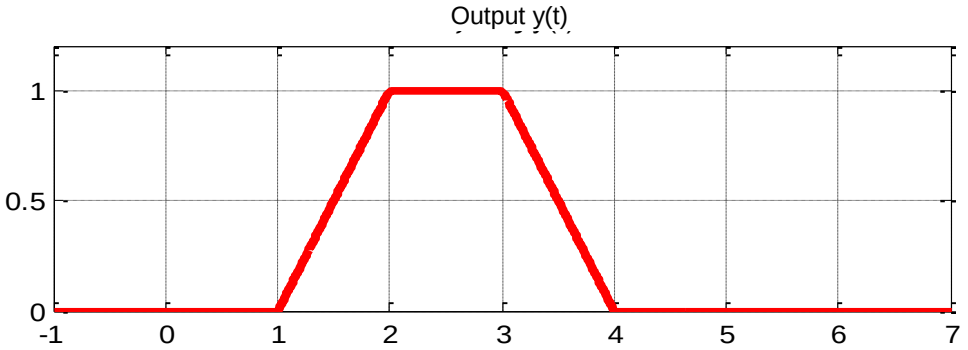
Exercise 6 (continued)

The graphs of the input $x(t) = u(t-1) - u(t-2)$, impulse response $h(t) = u(t) - u(t-2)$, and output $y(t)$, where:

$$y(t) = \begin{cases} t-1 & \text{when } 1 \le t < 2 \\ 1 & \text{when } 2 \le t < 3 \\ 4-t & \text{when } 3 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}$$

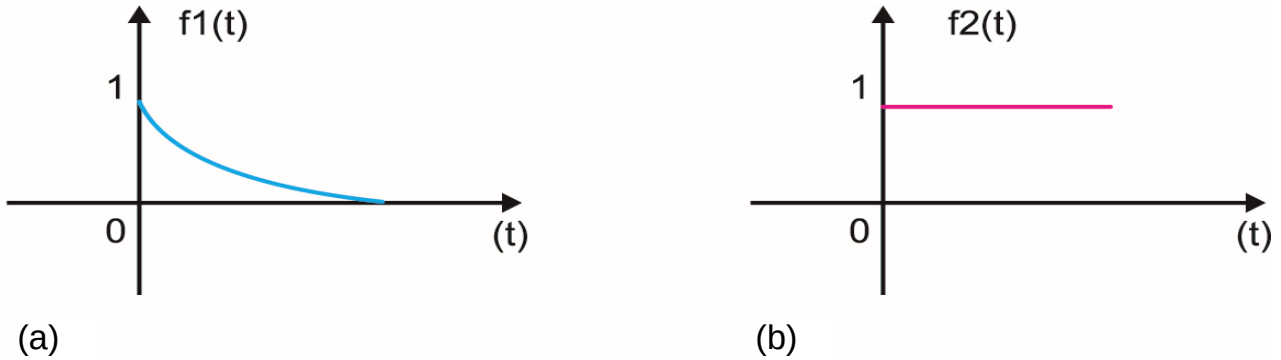
are shown below:





Exercise 7

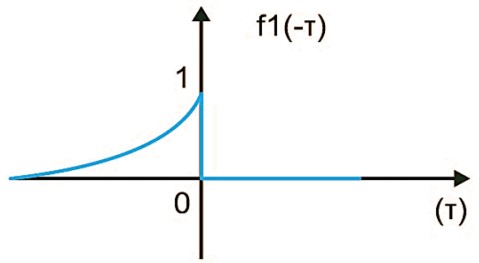
Use the graphical method to calculate the convolution integral $g(t)$ of $f_1(t)$ and $f_2(t)$,, shown in figures (a) and (b).



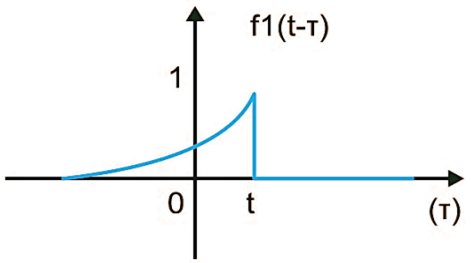
Answer: Figures (c)–(f) on the next slide show the graphical procedure. The result is:

$$g(t) = 1 - e^{-t}$$

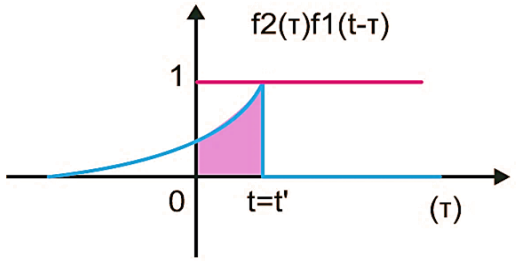
Exercise 7 (continued)



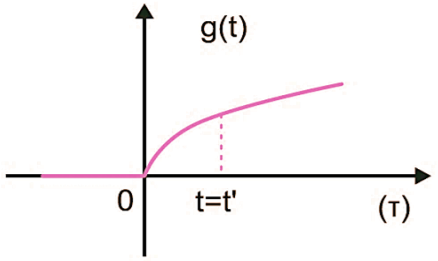
(c) Reflection



(d) Time shift



(e) Multiplication



(f) Integration

Evaluating the convolution integral analytically gives:

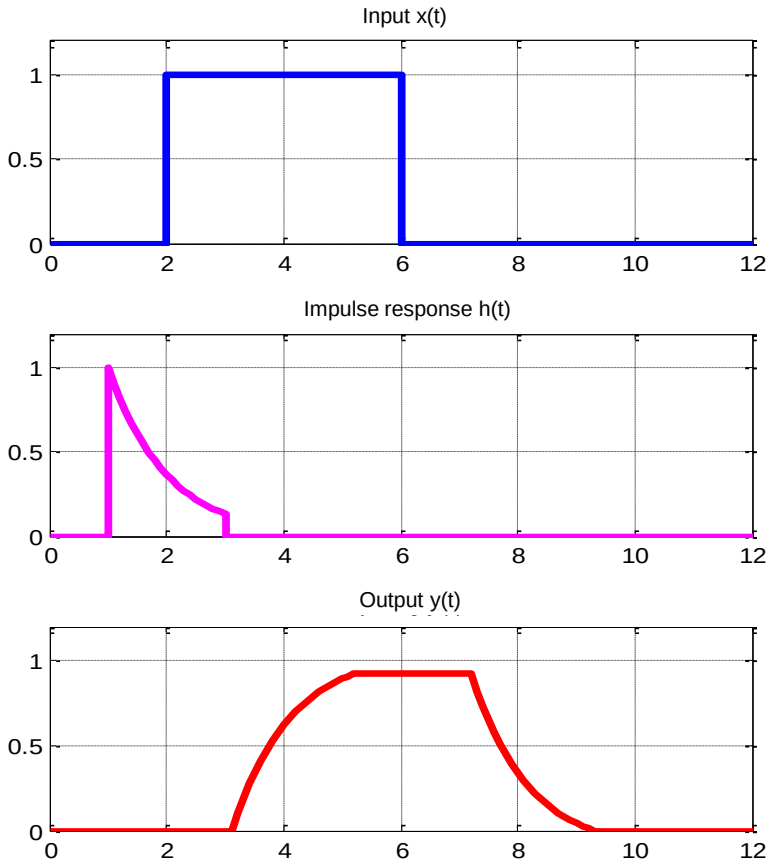
$$g(t) = f_1(t) * f_2(t) = \int_{-\infty}^{+\infty} f_2(\tau) f_1(t-\tau) dt = \int_{0}^{t} 1\, e^{-(t-\tau)} dt = e^{-t}\left([e^{\tau}]_0^t\right) = e^{-t}[e^t - 1] = 1 - e^{-t}$$

Exercise 8

Use MATLAB to calculate the convolution of $x(t) = u(t-2) - u(t-6)$ and $h(t) = e^{-(t-1)}[u(t-1) - u(t-3)]$.

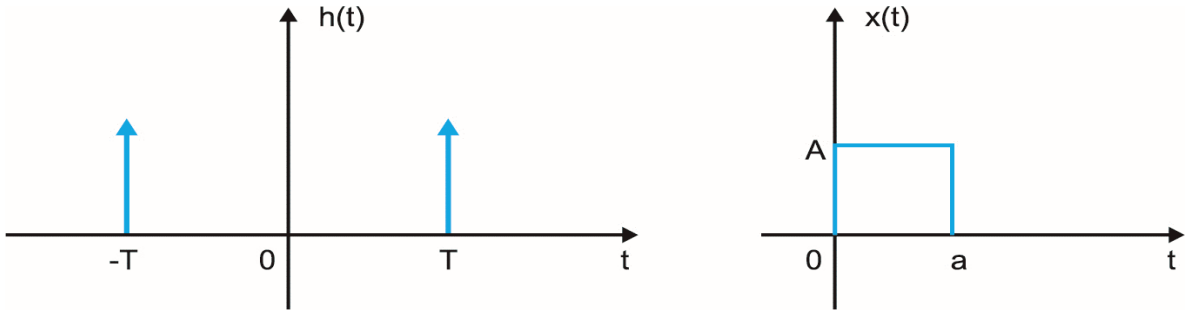
Answer:

```
Step = 0.1;
% Create signals h(t) and x(t)
 th = 1 : Step : 3;
h = exp(-1*(th0-1)).*ones(1,length(th0));
tx = 2 : Step : 6;
x = ones(1,length(tx0));
% Calculate convolution
y = conv(x,h).*Step;
% Set the output time range
ty = min(tx)+min(th) : Step : max(tx)+max(th);
% Plot the convolution result
subplot(311); plot(tx,x,'b','linewidth',3);
grid on; title('Input x(t)');
subplot(312); plot(th,h,'m','linewidth',3);
grid on; title('Impulse response h(t)');
subplot(313); plot(ty,y,'r','linewidth',3);
grid on; title('Output y(t)');
```



Exercise 9

Find the convolution of the functions shown below:



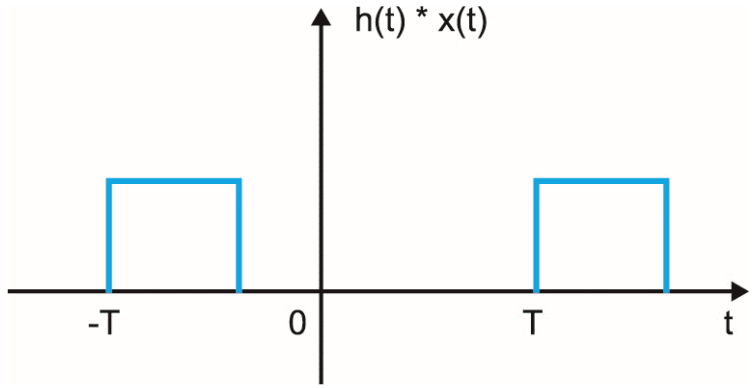
Answer: The figure shows that $h(t)$ can be written as $h(t) = \delta(t - T) + \delta(t + T)$. Thus, the convolution of the two signals is:

$$x(t) * h(t) = \int_{-\infty}^{\infty} [\delta(\tau - T) + \delta(\tau + T)]\, x(t - \tau) d\tau$$

$$= \int_{-\infty}^{\infty} \delta(\tau - T)\, x(t - \tau)\, d\tau + \int_{-\infty}^{+\infty} \delta(\tau + T)\, x(t - \tau)\, d\tau = x(t - T) + x(t + T)$$

Exercise 9 (continued)

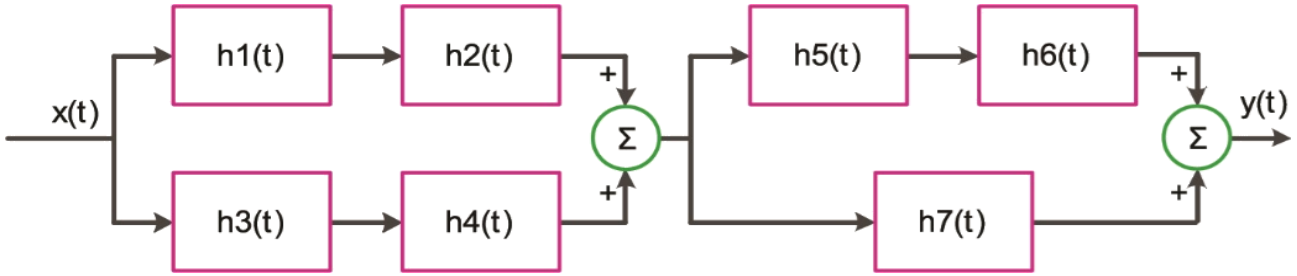
The result, the convolution of $x(t)$ and $h(t)$, is shown below.



- Convolving a signal with the delta function reproduces that signal. Convolving it with a shifted delta function gives a copy shifted by the same amount. Specifically, $x(t) * \delta(t) = x(t)$ and $x(t) * \delta(t - T) = x(t - T)$.
- This result helps explain the repetitions in the spectrum of a sampled signal, as we will see in a later lecture.

Exercise 10

Calculate the equivalent impulse response of this interconnection:



Answer: The left section has equivalent impulse response:

$$h_{eq1}(t) = [h_1(t) * h_2(t)] + [h_3(t) * h_4(t)]$$

The right section has equivalent impulse response:

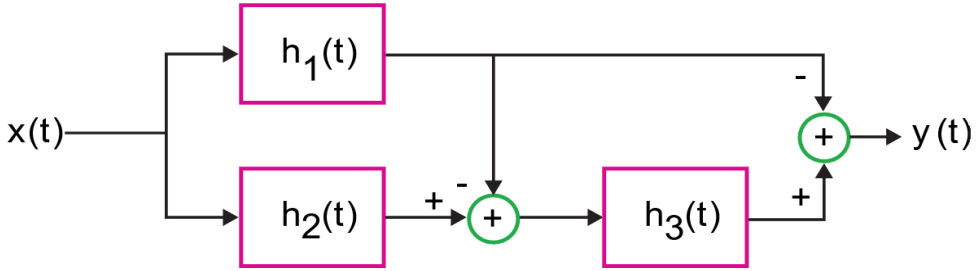
$$h_{eq2}(t) = [h_5(t) * h_6(t)] + h_7(t)$$

Since the two sections are connected in cascade, the equivalent impulse response of the complete system is their convolution:

$$h_{eq}(t) = h_{eq1}(t) * h_{eq2}(t) = \{[h_1(t) * h_2(t)] + [h_3(t) * h_4(t)]\} * \{[h_5(t) * h_6(t)] + h_7(t)\}$$

Exercise 11

Calculate the equivalent impulse response of this interconnection of LTI systems with individual impulse responses $h_1(t) = u(t-2)$, $h_2(t) = u(t)$, and $h_3(t) = \delta(t-2)$.



Answer: The systems with impulse responses $h_1(t), h_2(t)$ are connected in parallel, giving:

$$h_{12}(t) = h_2(t) - h_1(t) = u(t) - u(t-2)$$

The system with impulse response $h_3(t)$ is connected in cascade with $h_1(t), h_2(t)$,, giving:

$$h_{123}(t) = h_{12}(t) * h_3(t) = [u(t) - u(t-2)] * \delta(t-2) = u(t-2) - u(t-4)$$

The equivalent impulse response of the complete system is:

$$h_{eq}(t) = h_{123}(t) - h_1(t) = u(t-2) - u(t-4) - u(t-2) = -u(t-4)$$