UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

SIGNALS AND SYSTEMS
Worked Examples – Tutorial Handout 4

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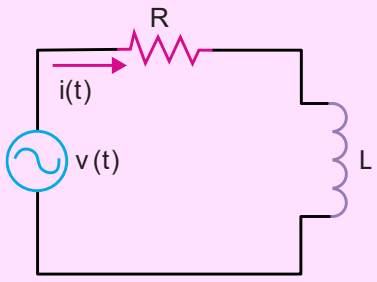
STUDY OF LINEAR TIME-INVARIANT SYSTEMS

- Description of Linear Time-Invariant (LTI) Systems:
  - Using ordinary differential equations
  - Using the convolution integral
- The Impulse Response of LTI Systems
- The Convolution Integral in LTI Systems
- Properties of Convolution

1. Finding the Zero-State and Zero-Input Responses

Example 1

Consider an LTI RL circuit with resistance $R = 1\Omega$ and inductance (coil) $L = 1H$. The source voltage is $v(t) = Bu(t)$ and the initial current in the inductor is I_0 (A).



RL Circuit

(a) Formulate and solve the differential equation for $B = 1$ and initial condition $I_0 = 1$ (A).

(b) Determine the zero-state and zero-input responses.

(c) Investigate the conditions under which the system is linear.

(d) Investigate the conditions under which the system is time-invariant.

Answer: (a) Applying Kirchhoff's law to the circuit above gives:

$$v(t) = v_R(t) + v_L(t)$$

The current $i(t)$ through the inductor and the voltage across it, $v_L(t)$, are related by:

$$v_L(t) = L\frac{di(t)}{dt}$$

Using the relations above and substituting $R = 1\Omega$ and $L = 1H$, we obtain:

$$\frac{di(t)}{dt} + i(t) = v(t)$$

This is a first-order linear constant-coefficient differential equation (LCCDE). Together with the initial value $i(0) = I_0$, it fully describes the RL circuit. Its solution is:

$$i(t) = [I_0\, e^{-t} + B(1 - e^{-t})]u(t) \qquad (1)$$

For $I_0 = 1$ and $B = 1$, the complete solution becomes:

$$i(t) = [e^{-t} + (1 - e^{-t})]u(t) = u(t)$$

(b) The zero-input response, i.e. for $v(t) = 0$, is:

$$i_{zi}(t) = I_0\, e^{-t}u(t)$$

and the zero-state response, i.e. for zero initial conditions and $v(t) = u(t)$, is:

$$i_{zs}(t) = B(1 - e^{-t})\, u(t)$$

(c) Testing linearity:

- For nonzero initial conditions: Double the input by setting $B = 2$ in (1), while keeping the initial condition fixed at $I_0 = 1$. The output is:

  $$i'(t) = [I_0\, e^{-t} + 2(1 - e^{-t})]u(t) = (2 - e^{-t})\, u(t)$$

  The system would be linear if doubling the input produced $i'(t) = 2i(t)$. This clearly does not hold, so the system (RL circuit) is nonlinear when the initial condition is nonzero.

- For zero initial conditions: In this case the output is:

  - For $B = 1$: $\quad i_1(t) = (1 - e^{-t})\, u(t)$
  - For $B = 2$: $\quad i_2(t) = 2(1 - e^{-t})\, u(t) = 2i_1(t)$

Since doubling the input doubles the output, the system is linear. We have established that this occurs only when the initial condition is zero.

(d) Testing time variance: Taking $B = 1$ and shifting the input in time to $(t-1)$, i.e. setting $v(t) = u(t-1)$, gives the output:

$$i''(t) = [I_0\, e^{-t} + (1 - e^{-t})]u(t-1) = I_0\, e^{-t}u(t) + \left(1 - e^{-(t-1)}\right) u(t-1)$$

- For nonzero initial conditions, the output for the time-shifted input is not an equally time-shifted version of the original output, because the initial-condition term is present.
- For zero initial conditions, the output for the time-shifted input is:

$$i''(t) = \left(1 - e^{-(t-1)}\right) u(t-1)$$

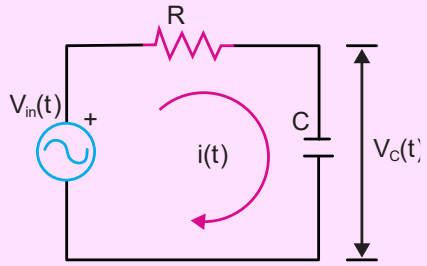
The output for an input shifted to $(t-1)$ equals the original output at $(t-1)$. Thus, $i''(t) = i(t-1)$ holds and the system is time-invariant.

Comments:

- The analysis above shows that a system described by an ordinary linear constant-coefficient differential equation (LCCDE) is linear and time-invariant (LTI) only when its initial conditions are zero (equivalently, its zero-input response is zero).
- In this case, the system's behavior is determined entirely by the input and can be studied efficiently using the impulse response and convolution.
- The next section presents a specific method for efficiently modeling LTI systems through their impulse responses and convolution.

📖Example 2

Consider the LTI RC circuit shown. Calculate the output, i.e. the capacitor voltage $v_c(t)$, taking the source voltage $v_{in}(t)$ as the input. Assume the initial condition $v_c(0) = 0$.



RC Circuit

Answer: The source voltage equals the sum of the voltage drops across the resistor and the capacitor:

$$v_{in}(t) = i(t)R + v_c(t)$$

The current $i(t)$ through the capacitor and the voltage $v_c(t)$ across it are related by:

$$i(t) = C\frac{dv_c(t)}{dt}$$

The relations above yield a first-order linear constant-coefficient differential equation (LCCDE) which, together with the initial value $v_c(0) = 0$, fully describes the given system:

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_{in}(t)$$

To calculate the output $v_c(t)$ for any input $v_{in}(t)$, we need the general solution of the LCCDE. It can be shown that:

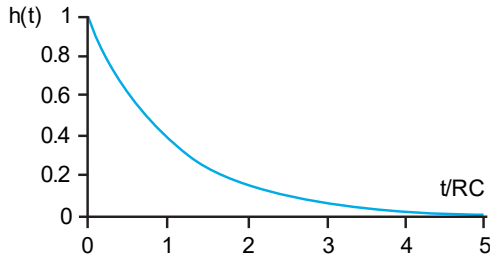
$$v_c(t) = v_c(0)e^{-t/RC} + e^{-t/RC}\frac{1}{RC}\int_{0^-}^{t} e^{\tau/RC}\, v_{in}(\tau)\, d\tau$$

The lower integration limit is 0^-**, rather than simply 0, because we seek the solution for an input** $\delta(t)$. For a general input, 0 would suffice. Since the RC circuit is initially at rest (because $v_c(0) = 0$), the output simplifies to the following zero-state response:

$$v_c(t) = e^{-t/RC}\frac{1}{RC}\int_{0^-}^{t} e^{\tau/RC}\, v_{in}(\tau)\, d\tau \qquad (1)$$

If the delta function is applied to the RC circuit input, the system produces an output $h(t)$, called the impulse response, given by $h(t) = S[\delta(t)]$. To calculate the impulse response $h(t)$ of this RC circuit, use the delta function as the input in (1), i.e. $v_{in}(t) = \delta(t)$, giving:

$$h(t) = e^{-t/RC}\frac{1}{RC}\int_{0^-}^{t} e^{\tau/RC}\, \delta(\tau)\, d\tau$$



Impulse Response of the RC Circuit

Using (1.46) and taking $\varphi(t) = e^{\tau/RC}$ as the test function, we obtain the impulse response of the RC circuit:

$$h(t) = \frac{1}{RC}e^{-t/RC}\, u(t)$$

The impulse response $h(t)$ is plotted in Figure 2.16.

Comment:

- The impulse response describes the output of a system initially at rest when it is excited at $t = 0^-$ by an ideal impulse input $\delta(t)$, i.e. a signal of infinitesimal duration and infinite amplitude.
- This input acts instantaneously, delivering a large amount of energy to the system in an infinitesimal interval and exciting its natural modes. The system therefore produces an output even after the excitation has ended.
- At $t = 0^+$, the input has vanished, but the system continues to respond because of the energy instantaneously stored in its elements. This output tends to resemble the zero-input response, since the signal acts for an infinitesimal interval and then ceases to exist. The impulse response is therefore closely related to the system's natural or homogeneous response.
- We can regard this input signal as establishing new initial conditions at $t = 0^+$. The homogeneous solution for these new initial conditions (which must be calculated) then produces the system output.
- The new initial conditions at $t = 0^+$ are unrelated to those at $t = 0^-$, which were zero, as stated earlier, and were previously used to find the homogeneous solution of the differential equation, i.e. the zero-input response. They arise entirely from the instantaneous effect of the impulse excitation.
- It can be shown that the impulse, step and ramp responses of an LTI system are related by:[1]

$$h(t) = \frac{d}{dt}s(t) \quad \text{and} \quad h(t) = \frac{d^2}{dt^2}r(t)$$

provided that these derivatives exist.

📖Example 3 (*)

Calculate (a) the zero-input response and (b) the zero-state response of the system with LCCDE:

$$\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 15y(t) = 6x(t)$$

with initial conditions $y(0) = 2$ and $y'(0) = 0$, and input $x(t) = e^{-2t}u(t)$.

Answer: (a) Calculating the zero-input response: The characteristic equation is:

$$\lambda^2 + 8\lambda + 15 = (\lambda + 3)(\lambda + 5) = 0$$

The system eigenvalues (roots) are $\lambda_1 = -3$ and $\lambda_2 = -5$, and the zero-input response is:

1 "Signals and Systems using Matlab", L.Chaparro, Elsevier, 2nd edition, 2015, page 146.

$$y_{zi}(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-3t} + C_2 e^{-5t}, \quad \text{for } t \geq 0$$

Determine the coefficients C_1, C_2 from the initial conditions:

$$y_{zi}(0) = C_1 e^{-3.0} + C_2 e^{-5.0} = C_1 + C_2 = 2$$
$$y_{zi}'(0) = (-3C_1 e^{-3t} - 5C_2 e^{-5t})\big|_{t=0} = -3C_1 - 5C_2 = 0$$

Solving the system gives $C_1 = 5,\ C_2 = -3$.
Therefore, the zero-input response is:

$$y_{zi}(t) = (5e^{-3t} - 3e^{-5t})\, u(t)$$

(b) Calculating the zero-state response: Since the input is the exponential function $x(t) = e^{-2t}u(t)$, Table 2.1 suggests the signal Ae^{-2t} as a particular solution of the differential equation, with the same form as the input. Substituting this solution and its derivatives into the LCCDE gives:

$$(-2)(-2)Ae^{-2t} + 8A(-2)e^{-2t} + 15Ae^{-2t} = 6e^{-2t} \Rightarrow \ldots \Rightarrow A = 2$$

Thus, the particular solution is:

$$y_p(t) = 2e^{-2t}$$

The zero-state response includes the natural modes of the system and the particular solution:

$$y_{zs}(t) = A_1 e^{-3t} + A_2 e^{-5t} + 2e^{-2t}$$

We must determine the coefficients A_1, A_2. We use the initial conditions, which are zero by definition for the zero-state response:

$$y_{zs}(0) = A_1 e^{-3.0} + A_2 e^{-5.0} + 2e^{-2.0} = A_1 + A_2 + 2 = 0$$
$$y_{zs}'(0) = (-3A_1 e^{-3t} - 5A_2 e^{-5t} - 4e^{-2t})\big|_{t=0} = -3A_1 - 5A_2 - 4 = 0$$

Solving the system gives $A_1 = -3,\ A_2 = 1$. Therefore, the zero-state response is:

$$y_{zs}(t) = (2e^{-2t} - 3e^{-3t} + e^{-5t})\, u(t)$$

Total response:

$$y(t) = y_{zi}(t) + y_{zs}(t) = (2e^{-2t} + 2e^{-3t} - 2e^{-5t})\, u(t)$$

Comments:

- This example expresses the total system response as the sum of the zero-input and zero-state responses. This distinction is more than a formality: it provides a deeper understanding of how the system operates.
- The zero-input response depends only on the initial conditions and expresses the system's natural behavior. We calculate it by solving the homogeneous equation through the characteristic polynomial. The eigenvalues determine the form of the solution, and the initial conditions determine the coefficients. Thus, the system's dynamic behavior is embodied in the roots rather than the input.
- The zero-state response depends only on the input. To calculate it, we select a trial solution based on the form of the input. Here the input is exponential, so we choose a corresponding exponential form.
- Even in the zero-state response, the final solution includes the system's natural modes. Zero initial conditions do not eliminate these modes; they determine their coefficients so that the initial conditions are satisfied.

📖Example 4

For $t > 0$, an LTI system is described by the linear constant-coefficient differential equation (LCCDE):

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = x(t)$$

The initial conditions are:

$$y(0) = 1, \qquad y'(0) = -1$$

Calculate the system's zero-input response and comment on its form.

Answer: For the zero-input response, set $x(t) = 0$, giving the homogeneous differential equation:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = 0$$

The characteristic equation is:

$$\lambda^2 + 4\lambda + 4 = 0$$

so there is a repeated real root:

$$\lambda_1 = \lambda_2 = -2$$

With a repeated root, the homogeneous solution cannot have the form $C_1e^{-2t} + C_2e^{-2t}$, as in previous examples, because the two terms would be linearly dependent. To obtain linearly independent terms, choose:

$$y_{zi}(t) = (C_1 + C_2t)\, e^{-2t}$$

This is a crucial point, since the second term takes the form te^{-2t}.

Differentiate the solution:

$$y'_{zi}(t) = C_2e^{-2t} - 2(C_1 + C_2t)\, e^{-2t} = (C_2 - 2C_1 - 2C_2t)\, e^{-2t}$$

Apply the initial conditions:

- For $y(0) = 1$, we obtain:
$$y_{zi}(0) = C_1 = 1$$
- For $y'(0) = -1$, we obtain:
$$y'_{zi}(0) = C_2 - 2C_1 = -1$$

and since $C_1 = 1$, we find:

$$C_2 - 2 = -1 \Rightarrow C_2 = 1$$

Thus, the zero-input response is:

$$y_{zi}(t) = (1 + t)\, e^{-2t} u(t)$$

The response consists of an exponentially decaying term e^{-2t} and a term $t\, e^{-2t}$, which arises from the root's multiplicity. Although te^{-2t} initially increases because of the factor t, it eventually decays as exponential damping dominates.

Comments:

- The zero-input response (natural response) is calculated from the homogeneous equation and arises solely from the system's initial conditions. Its form depends on both the values of the roots and their multiplicities.
- The repeated root requires the term te^{-2t} in the result, so that the natural response modes $C_1 e^{-2t}$ and $C_2 te^{-2t}$ are linearly independent. Omitting te^{-2t} leads to linearly dependent solutions and an incorrect response calculation.
- The factor t temporarily increases the output before exponential damping ultimately dominates. The solution therefore remains stable because the repeated root lies in the left half-plane. Multiplicity changes the form of the response, but it still tends to zero.

Example 5

An LTI system with zero initial conditions and LCCDE:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = e^{-2t}u(t)$$

has input:

$$x(t) = e^{-2t}u(t)$$

Calculate the zero-state response $y_{zs}(t)$.

Answer: With zero initial conditions, we seek only the zero-state response. The LCCDE is:

$$\frac{d^2y(t)}{dt} + 4\frac{dy(t)}{dt} + 4y(t) = e^{-2t}u(t)$$

The homogeneous equation has characteristic equation:

$$\lambda^2 + 4\lambda + 4 = 0$$

so there is a repeated real root:

$$\lambda_1 = \lambda_2 = -2$$

and the system's natural mode is therefore associated with e^{-2t}, as in previous examples. Here, however, the input is:

$$x(t) = e^{-2t}u(t)$$

The input has the same exponential form as the system's natural mode e^{-2t}. Since the corresponding pole is repeated, the trial particular solution must be multiplied by t^2. Thus, the exponential

excitation coincides (resonates) with a double system pole: the trial particular solution overlaps with the homogeneous solution. Unlike the previous example, we cannot seek a particular solution of the form:

$$y_p(t) = Ce^{-2t}$$

or of the form:

$$y_p(t) = (C_1 + C_2 t)\, e^{-2t}$$

because these terms already belong to the homogeneous solution. Since the root -2 has multiplicity 2, we must multiply by t^2. Thus, choose the trial solution:

$$y_p(t) = Ct^2 e^{-2t}$$

Its derivatives are:

$$y_p{}'(t) = C(2t - 2t^2)\, e^{-2t}$$

$$y_p{}''(t) = C(2 - 8t + 4t^2)e^{-2t}$$

Substitute into the differential equation:

$$y_p''(t) + 4y_p'(t) + 4y_p(t) = C(2 - 8t + 4t^2)\, e^{-2t} + 4C(2t - 2t^2)\, e^{-2t} + 4Ct^2 e^{-2t}$$

Collecting terms:

$$y_p''(t) + 4y_p'(t) + 4y_p(t) = C[(2 - 8t + 4t^2) + (8t - 8t^2) + 4t^2]e^{-2t}$$

$$y_p{}''(t) + 4y_p{}'(t) + 4y_p(t) = 2Ce^{-2t}$$

To satisfy the equation, we require:

$$2Ce^{-2t} = e^{-2t}$$

hence:

$$C = \frac{1}{2}$$

Thus, the zero-state response is:

$$y_{zs}(t) = \frac{1}{2} t^2 e^{-2t} u(t)$$

The term $t^2 e^{-2t}$ appears because the input coincides with a natural mode associated with a repeated root of the characteristic equation. A simple root would produce a term te^{-2t}, whereas here an additional factor t is required.

Comments:

- When the input coincides with a homogeneous solution, the original trial solution fails and must be modified.
- The root's multiplicity sets the required number of multiplications by t. Here the root -2 is

repeated, so the correct trial form is $t^2 e^{-2t}$.

- Similar cases arise when a system is excited by an input with the same dynamic form as the system's natural decay mode, for example in resonance or limiting transient conditions.

2. Calculating the Impulse Response from a Differential Equation

📖Example 6 (*)

Calculate the impulse response of the system described by the differential equation:

$$\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 15y(t) = 15x(t) - 2\frac{dx(t)}{dt}$$

Answer: Since the order N of the left-hand side (output and its derivatives) exceeds the order M of the right-hand side (input and its derivatives), the impulse response contains only exponential terms. It contains neither the delta function nor its derivatives.

Assuming the system is initially at rest and taking the delta function as the input, i.e. $x(t) = \delta(t)$, the differential equation becomes:

$$\frac{d^2h(t)}{dt^2} + 8\frac{dh(t)}{dt} + 15h(t) = 15\delta(t) - 2\frac{d\delta(t)}{dt}$$

The homogeneous equation is:

$$\frac{d^2h_o(t)}{dt^2} + 8\frac{dh_o(t)}{dt} + 15h_o(t) = 0$$

and the initial conditions are:

$$h_o(0^+) = 0 \text{ and } \frac{dh_o(0^+)}{dt} = \frac{1}{\alpha_2} = 1$$

The characteristic polynomial is:

$$\lambda^2 + 8\lambda + 15 = (\lambda + 3)(\lambda + 5) = 0$$

The system eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -5$, and the homogeneous solution is:

$$h_o(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{-3t} + c_2 e^{-5t}, \quad \text{for } t > 0$$

Determine the coefficients c_1, c_2 from the initial conditions:

$$h_o(0^+) = c_1 e^{-3.0} + c_2 e^{-5.0} = c_1 + c_2 = 0$$

$$\frac{dh_o(0^+)}{dt} = (-3c_1 e^{-3t} - 5c_2 e^{-5t})\Big|_{t=0} = -3c_1 - 5c_2 = \frac{1}{\alpha_N{}^2} = 1$$

Solving the system of equations gives $c_1 = 1/2,\ c_2 = -1/2$. Thus, the homogeneous solution for the impulse response is:

$$h_o(t) = \frac{1}{2}(e^{-3t} - e^{-5t})\, u(t)$$

The total impulse response is:

$$h(t) = 15h_o(t) - 2\frac{dh_o(t)}{dt}$$

[2] Where α_N is the coefficient of the highest-order term of the differential equation.

To simplify the calculation, evaluate the derivative [3] $dh_o(t)/dt$ separately:

$$\frac{dh_o(t)}{dt} = \left[\frac{1}{2}(e^{-3t} - e^{-5t})\,u(t)\right]' = \frac{1}{2}\left[(e^{-3t} - e^{-5t})'u(t) + (e^{-3t} - e^{-5t})u'(t)\right]$$
$$= \frac{1}{2}[(-3e^{-3t} + 5e^{-5t})\,u(t) + (e^{-3t} - e^{-5t})\delta(t)]$$
$$= \frac{1}{2}(-3e^{-3t} + 5e^{-5t})\,u(t)$$

Therefore, the impulse response is:

$$h(t) = 15h_o(t) - 2\frac{dh_o(t)}{dt} = \frac{15}{2}(e^{-3t} - e^{-5t})\,u(t) - 2\frac{1}{2}(-3e^{-3t} + 5e^{-5t})\,u(t)$$
$$= \frac{1}{2}(21e^{-3t} - 25e^{-5t})\,u(t)$$

Example 7 (*)

Calculate the impulse response of each system described by the differential equations:

(a) $\dfrac{d^2y(t)}{dt^2} + 4\dfrac{dy(t)}{dt} + 4y(t) = \dfrac{dx(t)}{dt} + x(t)$

(b) $\dfrac{d^2y(t)}{dt^2} + 4\dfrac{dy(t)}{dt} + 4y(t) = \dfrac{d^2x(t)}{dt^2} + 3\dfrac{dx(t)}{dt} + x(t)$

Answer: (a) Since $N > M$, the impulse response contains only exponential terms. Set the input to $x(t) = \delta(t)$. With initial rest conditions (as required for an impulse response), the LCCDE becomes:

$$\frac{d^2h(t)}{dt^2} + 4\frac{dh(t)}{dt} + 4h(t) = \frac{d\delta(t)}{dt} + \delta(t)$$

For $t > 0$, the terms $\delta(t)$ and $\delta'(t)$ have vanished, so the impulse response follows the system's homogeneous solution:

$$\frac{d^2h_o(t)}{dt^2} + 4\frac{dh_o(t)}{dt} + 4h_o(t) = 0$$

The function $h_o(t)$ is the auxiliary homogeneous solution for the impulse response, rather than the impulse response itself. Define the initial conditions for the homogeneous solution of the normalized system:

$$a_2h_o''(t) + a_1h_o'(t) + a_0h_0(t) = x(t)$$

i.e. the system with input $\delta(t)$ on the right-hand side rather than $\delta'(t) + \delta(t)$. These initial conditions at 0^+ are:

$$h_o(0^+) = 0 \text{ and } h_o'(0^+) = \frac{1}{\alpha_2} = 1$$

The characteristic polynomial is:

$$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$$

[3] We use the property $x(t)\delta(t) = x(0)\delta(0)$ and the relation $\delta(t) = du(t)/dt$.

There is a repeated root $\lambda_{1,2} = -2$, and the homogeneous solution is:

$$h_o(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_2 t} = C_1 e^{-2t} + C_2 t e^{-2t}, \quad \text{for } t > 0$$

Determine the coefficients C_1, C_2 from the initial conditions:

$$h_o(0^+) = C_1 e^{-2\cdot 0} + C_2 \cdot 0 \cdot e^{-2\cdot 0} = C_1 = 0$$
$$h_o'(0^+) = \left(-2C_1 e^{-2t} + C_2 e^{-2t}(1-2t)\right)\Big|_{t=0} = -2C_1 + C_2 = \frac{1}{\alpha_N} = 1$$

Solving the system gives $C_1 = 0, C_2 = 1$. Thus, the homogeneous solution for the impulse response is:

$$h_o(t) = te^{-2t}u(t)$$

As in the previous example, since the system is LTI, the total impulse response is:

$$h(t) = \frac{dh_o(t)}{dt} + h_o(t)$$

Calculate:

$$h(t) = \frac{d}{dt}[te^{-2t}u(t)] + te^{-2t}u(t)$$
$$= \frac{dt}{dt}e^{-2t}u(t) + t\frac{d}{dt}\left(e^{-2t}u(t)\right) + te^{-2t}u(t)$$
$$= e^{-2t}u(t) + t[-2e^{-2t}u(t) + e^{-2t}\delta(t)] + te^{-2t}u(t)$$
$$= e^{-2t}u(t) - 2te^{-2t}u(t) + te^{-2t}\delta(t) + te^{-2t}u(t)$$
$$= e^{-2t}u(t) - te^{-2t}u(t) + te^{-2t}\delta(t)$$
$$= (1-t)e^{-2t}u(t)$$

The relation $te^{-2t}\delta(t) = 0$ holds by the delta-function property $\varphi(t)\delta(t) = \varphi(0)\delta(t)$.

(b) Since $N = M$, the impulse response includes $\delta(t)$ (a singular term) in addition to exponential terms. Set the input to $x(t) = \delta(t)$. With initial rest conditions (as required for an impulse response), the differential equation is:

$$\frac{d^2h(t)}{dt^2} + 4\frac{dh(t)}{dt} + 4h(t) = \frac{d^2\delta(t)}{dt^2} + 3\frac{d\delta(t)}{dt} + \delta(t)$$

The homogeneous equation, initial conditions, characteristic polynomial, eigenvalues and homogeneous solution are the same as in part (a), because the LCCDE has the same left-hand side. Thus, the homogeneous solution is:

$$h_o(t) = te^{-2t}u(t)$$

Calculate the impulse response using:

$$h(t) = \frac{d^2h_o(t)}{dt^2} + 3\frac{dh_o(t)}{dt} + h_o(t) = \frac{d^2}{dt^2}[te^{-2t}u(t)] + 3\frac{d}{dt}[te^{-2t}u(t)] + te^{-2t}u(t)$$

Evaluate each derivative separately. The first derivative is:

$$\frac{d}{dt}[te^{-2t}u(t)] = (te^{-2t})'u(t) + (0\cdot e^{-2\cdot 0})\,\delta(t) = [1\cdot e^{-2t} + t(-2e^{-2t})]u(t) = (1-2t)e^{-2t}u(t)$$

The second derivative is:

$$\frac{d^2}{dt^2}[te^{-2t}u(t)] = \frac{d}{dt}[(1-2t)e^{-2t}u(t)]$$
$$= [(1-2t)e^{-2t}]'u(t) + (1-2t)e^{-2t}u'(t)$$
$$= \left[(1-2t)'e^{-2t} + (1-2t)(e^{-2t})'\right]u(t) + (1-2t)e^{-2t}\delta(t)$$
$$= [-2e^{-2t} - 2(1-2t)e^{-2t}]\,u(t) + (1-2\cdot 0)e^{-2\cdot 0}\delta(t)$$
$$= (4t-4)e^{-2t}\,u(t) + \delta(t)$$

Therefore, the impulse response is:

$$h(t) = (4t-4)e^{-2t}\,u(t) + \delta(t) + 3(1-2t)e^{-2t}u(t) + te^{-2t}u(t) = \cdots$$
$$= \delta(t) - (t+1)e^{-2t}u(t)$$

Comments:

- In the first example, the system is in normal form, so the impulse response is a smooth function containing only exponential terms.
- Take $\delta(t)$ as the input, solve the homogeneous equation and determine the coefficients from the initial conditions at $t = 0^+$. As in the previous example, the initial conditions are not given; they arise from the impulse excitation.
- In the second example, the system is not in normal form, which introduces a singular term containing $\delta(t)$. Thus, the form of the LCCDE determines whether the impulse response contains only exponential terms or also singular terms.
- The final impulse response combines the homogeneous solution with terms related to $\delta(t)$ and its derivatives.
- When a system is not in normal form, part of the input energy passes directly to the output in a modified form (an instantaneous component). Such behavior occurs in ideal amplifiers or circuits with parallel branches without delay.

Note: Examples marked with an asterisk (*) also appear in the lecture slides.