

UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Signals and Systems

Lecture 4: Analysis of Linear and Time-Invariant Systems

Michalis Paraskevas
Professor

Lecture Contents

- Description of Linear Time-Invariant (LTI) Systems:
  - Using ordinary differential equations
  - Using the convolution integral
- Impulse Response of LTI Systems
- The Convolution Integral in LTI Systems
- Properties of Convolution and LTI Systems:
  - Commutativity
  - Associativity
  - Distributivity
  - Identity property
  - Homogeneity
  - Duration property

Ways to Describe LTI Systems

1. **Using ordinary differential equations**
2. **Using the impulse response**
3. Using the Laplace transform
4. State-space analysis

1. Description of an LTI System Using Ordinary Differential Equations

Analysis of LTI Systems Using Differential Equations

In general, a dynamic system with causal input $x(t)$ and output $y(t)$, can be described, for $t \geq 0$, by an order-N linear differential equation with constant coefficients (LCCDE):

$$a_N \frac{d^N y(t)}{dt^N} + \cdots + \alpha_1 \frac{dy(t)}{dt} + \alpha_0 = b_M \frac{d^M x(t)}{dt^M} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

with N known initial conditions: $y(0), y'(0), y''(0), \ldots, y^{(N-1)}(0)$.

Introducing the differential operator:

$$D^n[y(t)] = \frac{d^n y(t)}{dt^n} \quad \text{and} \quad D^0[y(t)] = y(t)$$

the differential equation becomes an algebraic equation of the form:

$$(a_N D^N + \cdots + \alpha_1 D + \alpha_0)\, y(t) = (b_M D^M + \cdots + b_1 D + b_0)\, x(t)$$

$$D^k[y(t)]_{t=0}, \qquad k = 0,1, \ldots N-1$$

Solution of a Differential Equation (LCCDE)

The **complete solution** of the LCCDE, i.e., the response $y(t)$ for $t \geq 0$, decomposes into two components because the system is linear:

$$y(t) = y_{zs}(t) + y_{zi}(t)$$

- $y_{zi}(t)$ **zero-input response**: the output due solely to the initial conditions, with zero input $x(t)$.
- $y_{zs}(t)$ **zero-state response**: the output due solely to the input $x(t)$, with all initial conditions equal to zero.

Note: A system described by an ordinary linear differential equation with constant coefficients (LCCDE) is **linear and time-invariant** (LTI) only when its **initial conditions are zero.** Equivalently, its zero-input response must be zero.

Zero-Input Response (1/5)

(A) The **zero-input response** $y_{zi}(t)$ is due solely to the initial conditions, with zero input $x(t)$. It results from energy stored in the system before the input is applied. The response $y_{zi}(t)$ solves:

$$a_N D^N + \cdots + \alpha_1 D + \alpha_0 = 0 \qquad \text{(I)}$$

with initial conditions:

$$y(0), y'(0), y''(0), \ldots, y^{(N-1)}(0)$$

The solution retains its form under differentiation. One such function is $e^{\lambda t}$. We therefore assume a solution of the form:

$$y_{zi}(t) = ce^{\lambda t}$$

where λ and c are constants, generally complex, to be determined. To find $y_{zi}(t)$, replace the operator D in (I) with the complex variable λ:

$$\alpha_0 + \alpha_1 \lambda + \cdots a_N \lambda^N = 0$$

Zero-Input Response (2/5)

This homogeneous equation is the **characteristic equation** of the system; the corresponding polynomial is the **characteristic polynomial.** Factoring it gives:

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_N) = 0$$

The roots are called **natural frequencies** or **eigenvalues.** They depend only on the **internal structure** of the system, rather than on its input, and characterize its **dynamic behavior.**

There are therefore N different values of λ satisfying the equation:

$$c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}, \dots\dots, c_N e^{\lambda_N t}$$

where $c_i, i = 1,2, \dots, N$ are constants, generally complex.

Each term $\boldsymbol{c_i e^{\lambda_i t}}, i = 1,2, \dots, N,$ is called a **natural mode.**

Zero-Input Response (3/5)

The initial conditions determine the constants c_i, which are generally complex. It can be shown that:

$$y_{zi}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t} = \sum_{k=1}^{N} c_k e^{\lambda_k t} \quad \text{for } t > 0$$

Replacing $t > 0$ with the unit step function $u(t)$ gives the causal form:

$$y_{zi}(t) = \sum_{k=1}^{N} c_k e^{\lambda_k t}\, u(t)$$

This expression applies when the characteristic roots are real and distinct.

Zero-Input Response $(4/5)$

For a real root of multiplicity $r \geq 2$, the characteristic polynomial becomes:

$$(\lambda - \lambda_1)^r (\lambda - \lambda_{r+1}) \dots (\lambda - \lambda_N) = 0$$

and the zero-input response is:

$$y_{zi}(t) = \left((c_1 + c_2 t + \cdots + c_r t^{r-1})\, e^{\lambda_1 t} + c_{r+1} e^{\lambda_{r+1} t} + \cdots c_N e^{\lambda_N t} \right) u(t)$$

In both cases above, the output has an exponential form.

Zero-Input Response (5/5)

- If **all** eigenvalues have negative **real part**s, i.e., $Re\{\lambda_k\} < 0$, $k = 1,2, \ldots . N$, the zero-input response decays to zero as $t \to \infty$, and the system comes to **rest**.
- If **at least one eigenvalue has a positive real part**, i.e., $Re\{\lambda_k\} > 0$, the zero-input response grows **without bound**.
- **Complex eigenvalues** occur in conjugate pairs because the characteristic polynomial has real coefficients. For the conjugate pair $\lambda_{1,2} = \sigma \pm j\Omega$, the solution is:

$$y_{zi,o}(t) = e^{\sigma t}\,(c_1 \cos \Omega t + c_2 \sin \Omega t)$$

- **Complex roots** produce a **decaying oscillation when $\boldsymbol{\sigma < 0}$**, a **growing oscillation when $\boldsymbol{\sigma > 0}$**, and a **sustained oscillation when $\boldsymbol{\sigma = 0}$**.

Zero-State Response

(B) The **zero-state response** $y_{zs}(t)$ is due solely to the input $x(t)$, with zero initial conditions. It solves the following equation with zero initial conditions:

$$(a_N D^N + \cdots + \alpha_1 D + \alpha_0)\, y(t) = (b_M D^M + \cdots + b_1 D + b_0)\, x(t)$$

The particular solution $y_{zs}(t)$ can be found using the method of undetermined coefficients, choosing a trial form appropriate to the input.

The table lists trial forms of the particular solution for some standard input signals.

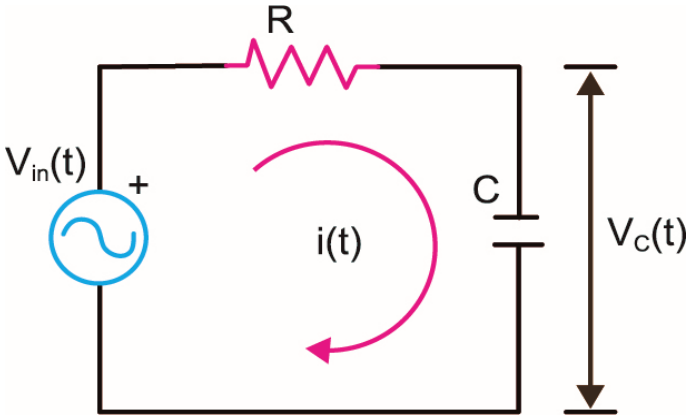
If the trial form matches a term in the homogeneous solution, multiply it by a suitable power of t until it is independent.

Input	Particular Solution
K	C
Kt	$\mathrm{C}_1 t + \mathrm{C}_2$
Kt^2	$C_1 t^2 + \mathrm{C}_2 t + C_3$
Ke^{-at}	Ce^{-at}
$K\,sin(\Omega t)$ or $K\,cos(\Omega t)$	$C_1 \sin(\Omega_0 t) + C_2 \cos(\Omega_0 t)$
$\delta(t)$	0

2. Impulse Response of an LTI System

The Concept of Impulse Response (1/2)

For the LTI system below (an RC circuit), we seek a relation between the source voltage $v_{in}(t)$ (input) and the capacitor voltage $v_c(t)$ (output).



Kirchhoff's law shows that the RC circuit is described by a first-order linear differential equation with constant coefficients:

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_{in}(t)$$

The Concept of Impulse Response (2/2)

Assuming that the RC circuit is initially at rest, i.e., $v_c(0) = 0$, the general solution of the differential equation is:

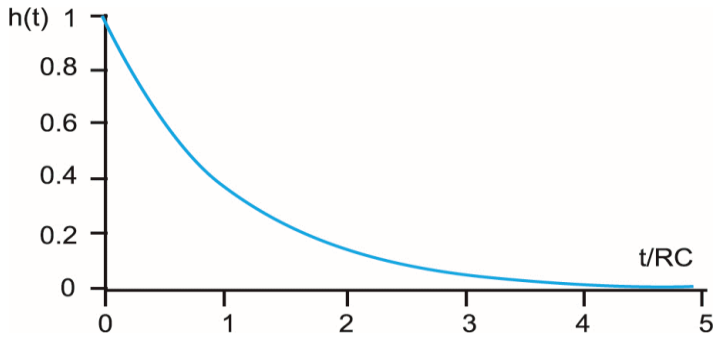
$$v_c(t) = e^{-t/RC} \frac{1}{RC} \int_{0^-}^{t} e^{\tau/RC} \, v_{in}(t) \, dt$$

To find the **impulse response** $h(t)$, apply the unit impulse as the input, i.e., $v_{in}(t) = \delta(t)$:

$$h(t) = e^{-t/RC} \frac{1}{RC} \int_{0^-}^{t} e^{\tau/RC} \, \delta(t) \, dt = \frac{1}{RC} e^{-t/RC} \, u(t)$$

The graph shows that the RC circuit is **dynamic**: it has memory.

Impulse response of an RC circuit



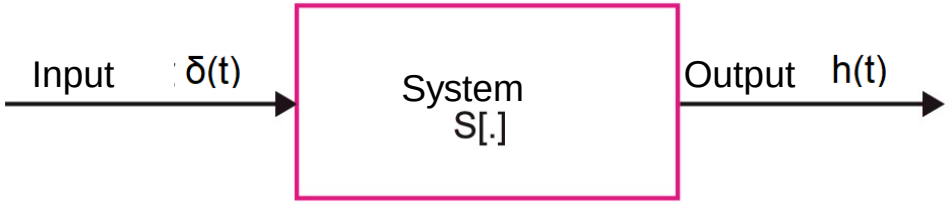
What Does the Impulse Response Describe?

- The **impulse response** is the output of a system initially at rest when an input of infinitesimal duration and infinite amplitude is applied at time $t = 0^-$.
- This input, the delta function, delivers a **large amount of energy** in an infinitesimal interval and **excites the natural modes** of the system, producing an output.
- At time $t = 0^+$, the input has ceased, but the system continues to produce an output. This output resembles the zero-input response because the excitation acts for an infinitesimal interval and then disappears.
- The impulse can be regarded as establishing **new initial conditions** at $t = 0^+$. The homogeneous solution for these new conditions, which must be calculated, produces the system output.
- The new initial conditions at $t = 0^+$ differ from those at $t = 0^-$, which were zero. The earlier conditions were used to find the homogeneous solution, i.e., the zero-input response.

3. Description of an LTI System Using Impulse Response and Convolution

Impulse Response of LTI Systems

- The **impulse response** is the output of a system when the unit impulse function $\delta(t)$ is applied to its input.



- For a **linear time-invariant** (LTI) system, the impulse response $h(t)$ is:

$$h(t) = \boldsymbol{S}[\delta(t)]$$

- For a **linear time-varying** system, the impulse response $h(t, \tau)$ to the unit impulse $\delta(t - \tau)$ applied at time τ is:

$$h(t, \tau) = \boldsymbol{S}[\delta(t - \tau)]$$

The Convolution Integral in LTI Systems

- The response $h(t,\tau)$ **completely characterizes an LTI system**: knowing it allows the output $y(t)$ to be calculated for any input $x(t)$.
- For an LTI system initially at rest, with impulse response $h(t)$,, the output $y(t)$ for input $x(t)$ is:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

- This integral is called the **convolution** or convolution integral of the functions $x(t)$ and $h(t)$. Symbolically:

$$y(t) = x(t) * h(t)$$

- If the **LTI system is also causal**, convolution simplifies to:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{t} h(\tau)\, x(t-\tau)\, d\tau$$

Initial Notes on Convolution

- The convolution integral is a general representation of an LTI system **initially at rest**. In this case, the output equals the zero-state response.
- For a system described by an ordinary linear differential equation with constant coefficients, convolution gives the zero-state response: the response due solely to the input, with all initial conditions equal to zero.
- For an LTI system described by an ordinary linear differential equation with constant coefficients, the **total response** $y(t)$ to input $x(t)$ is:

$$y(t) = y_{zi}(t) + y_{zs}(t) = \sum_{i=1}^{N} C_i e^{\lambda_i t} + x(t) * h(t)$$

  where $h(t)$ is the impulse response, λ_i are the eigenvalues, and C_i are constant coefficients, generally complex, determined by the initial conditions.

- If the LTI system is **initially at rest**, the sum in this expression (the **zero-input response**) **vanishes**. The output then consists only of the zero-state response, i.e., the convolution of the input with the impulse response.

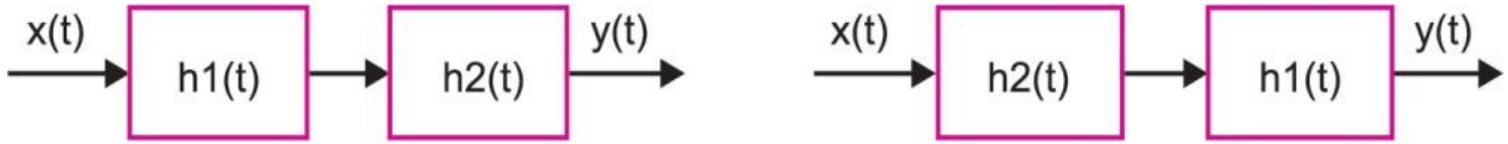
4. Properties of Convolution and LTI Systems

Commutativity

The commutative property is expressed by:

$$h_1(t) * h_2(t) = h_2(t) * h_1(t)$$

The diagram illustrates the physical meaning: two systems connected in cascade can be interchanged without changing the final output $y(t)$.



Associativity

The associative property is expressed by:

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

The diagram shows that two systems connected in cascade can be replaced by one equivalent system. Its **impulse response** equals the **convolution of the impulse responses** of the two original systems.

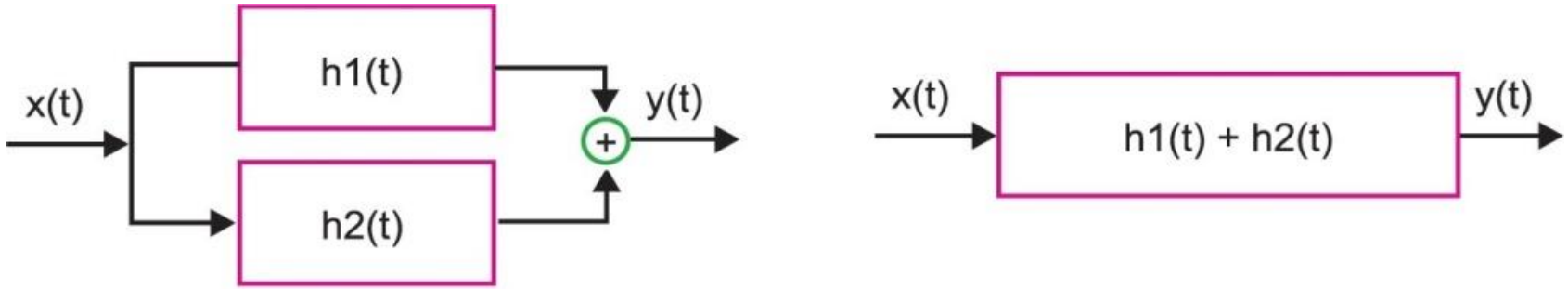


Distributivity

The distributive property is expressed by:

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

The diagram illustrates the physical meaning of distributivity.



By distributivity, two systems connected in parallel can be replaced by one equivalent system whose **impulse response** is the **sum of their impulse responses.**

Identity Property

The identity property is expressed by:

$$x(t) * \delta(t) = x(t)$$

The unit impulse function $\delta(t)$ is the **identity element** for convolution.

The following sampling property is also useful. It defines the sampling of a continuous-time signal to produce a discrete-time signal:

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

Other Properties of Convolution

- **Homogeneity**:

$$[a\,x(t)] * h(t) = a[x(t) * h(t)] = x(t) * [a\,h(t)]$$

- **Duration property:** The duration of the convolution $z(t) = x(t) * y(t)$ of two continuous-time signals $x(t)$ and $y(t)$, of durations T_1 and T_2, equals the sum $T_1 + T_2$.

- **Differentiation property:** For continuous-time signals $x(t)$, $y(t)$ and $z(t) = x(t) * y(t)$, differentiating either signal gives the derivative of their convolution:

$$\frac{dz(t)}{dt} = \frac{dx(t)}{dt} * y(t) = x(t) * \frac{dy(t)}{dt}$$

Other Properties of Convolution

- **Time-shift property:** For signals $x(t)$, $y(t)$ and $z(t) = x(t) * y(t)$, shifting either signal in time produces the same time shift in their convolution:

$$z(t - t_0) = x(t - t_0) * y(t) = \int_{-\infty}^{\infty} x(t - t_0) * y(t - \tau)\, d\tau$$

- **Stability property:** An LTI system is BIBO stable if its impulse response is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Exercises

Exercise 1

Find the zero-input response of the system described by the LCCDE:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = x(t)$$

with initial conditions $y(0) = 0$ and $y'(0) = -4$.

Answer: The characteristic polynomial is $\lambda^2 + 6\lambda + 8$, and the homogeneous equation is:

$$\lambda^2 + 6\lambda + 8 = (\lambda + 2)(\lambda + 4) = 0$$

The natural frequencies are $\lambda_1 = -2$ and $\lambda_2 = -4$. Therefore, the zero-input response is:

$$y_{zi}(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-2t} + C_2 e^{-4t}, \qquad \text{for } t > 0$$

Use the initial conditions to determine the coefficients C_1, C_2:

$$y_{zi}(0) = C_1 e^{-2.0} + C_2 e^{-4.0} = C_1 + C_2 = 0$$

$$y'_{zi}(0) = (-2C_1 e^{-2t} - 4C_2 e^{-4t})\Big|_{t=0} = -2C_1 - 4C_2 = -4$$

We obtain $C_1 = -2,\ C_2 = 2$. Therefore, the zero-input response is: $y_{zi}(t) = (-2e^{-2t} + 2e^{-4t})\, u(t)$

Exercise 1 (continued)

- To find the **zero-input response** from the eigenvalues, start with the characteristic equation and find the roots of the characteristic polynomial. These roots determine the form of the homogeneous solution, i.e., the zero-input response.
- The response is a linear combination of exponential terms of the form $e^{\lambda_i t}$, where λ_i are the eigenvalues. The dynamic behavior depends solely on the roots, rather than on the input.
- Next, determine the unknown constants from the initial conditions. These conditions affect the weights of the natural modes, rather than the form of the solution.
- The form of the solution directly explains the behavior: if the eigenvalues have negative real parts, the terms decay and the system comes to rest.

Exercise 2

Find (a) the zero-input response and (b) the zero-state response of the system described by the LCCDE:

$$\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 15y(t) = 6x(t)$$

with initial conditions $y(0) = 2$ and $y'(0) = 0$, and input $x(t) = e^{-2t}u(t)$.

Answer: (a) **Zero-input response**: The homogeneous equation is:

$$\lambda^2 + 8\lambda + 15 = (\lambda + 3)(\lambda + 5) = 0$$

The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -5$. The zero-input response is:

$$y_{zi}(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} = C_1e^{-3t} + C_2e^{-5t}, \qquad \text{for } t > 0$$

Use the initial conditions to determine the coefficients C_1, C_2:

$$y_{zi}(0) = C_1e^{-3\cdot 0} + C_2e^{-5\cdot 0} = C_1 + C_2 = 2$$

$$y'_{zi}(t) = (-3C_1e^{-3t} - 5c_2e^{-5t})\Big|_{t=0} = -3C_1 - 5C_2 = 0$$

Exercise 2 (continued)

We obtain $C_1 = 5,\ C_2 = -3$. Therefore, the zero-input response is:

$$y_{zi}(t) = (5e^{-3t} - 3e^{-5t})\, u(t)$$

(b) Zero-state response: The input is exponential, $x(t) = e^{-2t}u(t)$. Using the table, choose the particular solution Ce^{-2t}, of the same form as the input. Substituting this solution and its derivatives into the LCCDE gives:

$$(-2)(-2)Ae^{-2t} + 8A(-2)e^{-2t} + 15Ae^{-2t} = 6e^{-2t} \Rightarrow \ldots \Rightarrow A = 2$$

Therefore, the particular solution is $y_p(t) = 2e^{-2t}$.

The zero-state response contains the natural modes and the particular solution:

$$y_{zs}(t) = A_1e^{-3t} + A_2e^{-5t} + 2e^{-2t}$$

Exercise 2 (continued)

Determine the coefficients A_1, A_2 using the initial conditions, which are zero by definition for the zero-state response:

$$y_{zs}(0) = A_1 e^{-3\cdot 0} + A_2 e^{-5\cdot 0} + 2e^{-2\cdot 0} = A_1 + A_2 + 2 = 0$$

$$y_{zs}'(0) = (-3A_1 e^{-3t} - 5A_2 e^{-5t} - 4e^{-2t})\Big|_{t=0} = -3A_1 - 5A_2 - 4 = 0$$

Solving the equations gives $A_1 = -3,\ A_2 = 1$. Therefore, the zero-state response is:

$$y_{zs}(t) = (2e^{-2t} - 3e^{-3t} + e^{-5t})\, u(t)$$

Total response: $y(t) = y_{zi}(t) + y_{zs}(t) = (2e^{-2t} + 2e^{-3t} - 2e^{-5t})\, u(t)$

Exercise 2 (continued)

- This exercise decomposes the **total response** into the zero-input and zero-state responses. This distinction helps explain how the system operates.
- The **zero-input response** depends only on the initial conditions and describes the natural behavior. Solve the homogeneous equation using the characteristic polynomial: the eigenvalues determine the form, and the initial conditions determine the coefficients. The dynamic behavior is therefore encoded in the roots, rather than in the input.
- The **zero-state response** depends only on the input. Choose a trial solution matching the input form. Here the input is exponential, so the trial solution is also exponential.
- The zero-state response still contains the natural modes. Zero initial conditions determine the coefficients needed to satisfy those conditions; they do not eliminate the modes.

Exercise 3

For $t > 0$, an LTI system is described by the LCCDE:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = x(t)$$

The initial conditions are $y(0) = 1, y'(0) = -1$. Find the zero-input response and comment on its form.

Answer: Set $x(t) = 0$ for the zero-input response. The homogeneous differential equation is:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = 0$$

The characteristic equation is:

$$\lambda^2 + 4\lambda + 4 = 0$$

It has a repeated real root of multiplicity two:

$$\lambda_1 = \lambda_2 = -2$$

Exercise 3 (continued)

For a double root, the homogeneous solution cannot take the form $C_1 e^{-2t} + C_2 e^{-2t}$ used in earlier examples: the two terms would be **linear**ly dependent. To obtain **independent** terms, choose:

$$y_{zi}(t) = (C_1 + C_2 t)\, e^{-2t}$$

The second term **must** have the form te^{-2t}.

Differentiate the solution:

$$y'_{zi}(t) = C_2 e^{-2t} - 2(C_1 + C_2 t)\, e^{-2t} = (C_2 - 2C_1 - 2C_2 t)\, e^{-2t}$$

Apply the initial conditions:

- For $y(0) = 1$: $\quad y_{zi}(0) = C_1 = 1$
- For $y'(0) = -1$: $\quad y'_{zi}(0) = C_2 - 2C_1 = -1$

Since $C_1 = 1$, we obtain: $\quad C_2 - 2 = -1 \Rightarrow C_2 = 1$

Therefore, the zero-input response is: $\quad y_{zi}(t) = (1 + t)\, e^{-2t} u(t)$

Exercise 3 (continued)

- The response contains an exponentially decaying term e^{-2t} and a term te^{-2t} arising from the repeated root. Although te^{-2t} initially grows because of the factor t, it eventually decays as exponential damping dominates.
- The zero-input (natural) response follows from the homogeneous equation and depends solely on the initial conditions. Its form depends on both the root values and their multiplicities.
- The double root requires the term te^{-2t} so that the natural modes C_1e^{-2t} and C_2te^{-2t} are **linearly independent**. Omitting te^{-2t} gives dependent solutions and an incorrect response.
- The factor t causes a temporary increase in the output before exponential decay dominates. The response is stable because the double root lies in the left half-plane. Multiplicity changes the form, while the response still tends to zero.

Exercise 4

An LTI system with zero initial conditions and LCCDE:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = e^{-2t}u(t)$$

has input $x(t) = e^{-2t}u(t)$. Find the zero-state response $y_{zs}(t)$.

Answer: With zero initial conditions, only the zero-state response is required. The LCCDE is:

$$\frac{d^2y(t)}{dt} + 4\frac{dy(t)}{dt} + 4y(t) = e^{-2t}u(t)$$

The characteristic equation of the homogeneous equation is: $\lambda^2 + 4\lambda + 4 = 0$

It has a double real root: $\lambda_1 = \lambda_2 = -2$

The natural modes are therefore associated with the term e^{-2t}, as in the earlier examples.

Exercise 4 (continued)

Here, however, the input is $x(t) = e^{-2t}u(t)$.

The input has the same exponential form as the natural mode $\boldsymbol{e^{-2t}}$. Since the corresponding pole is **double**, multiply the trial particular solution by $\boldsymbol{t^2}$.

The exponential excitation coincides with a double pole of the system (**resonance**), so the trial form overlaps the homogeneous solution. We cannot use a particular solution of the form:

$$y_p(t) = Ce^{-2t}$$

or of the form:

$$y_p(t) = (C_1 + C_2 t)\, e^{-2t}$$

because these terms already belong to the homogeneous solution.

The root -2 has multiplicity two, so multiply by t^2. Choose the trial solution:

$$y_p(t) = Ct^2e^{-2t}$$

Exercise 4 (continued)

Its derivatives are:

$$y_p{}'(t) = C(2t - 2t^2)e^{-2t}$$

$$y_p''(t) = C(2 - 8t + 4t^2)e^{-2t}$$

Substitute into the differential equation:

$$y_p''(t) + 4y_p'(t) + 4y_p(t) = C(2 - 8t + 4t^2)\, e^{-2t} + 4C(2t - 2t^2)\, e^{-2t} + 4Ct^2e^{-2t}$$

Collecting terms:

$$y_p''(t) + 4y_p'(t) + 4y_p(t) = C[(2 - 8t + 4t^2) + (8t - 8t^2) + 4t^2]e^{-2t}$$

$$y_p{}''(t) + 4y_p{}'(t) + 4y_p(t) = 2Ce^{-2t}$$

The equation requires $2Ce^{-2t} = e^{-2t}$. Therefore: $C = 1/2$

The zero-state response is: $y_{zs}(t) = \frac{1}{2}t^2e^{-2t}u(t)$

Exercise 4 (continued)

- The term t^2e^{-2t} appears because the input **matches** a natural mode associated with a double root. A simple root would require the term te^{-2t}; the double root requires an additional factor t.
- When the input matches a homogeneous solution, the initial trial form fails and must be modified.
- The root **multiplicity** determines **how many times** the trial solution must be multiplied by $\boldsymbol{t}$. Here -2 is a double root, so the correct trial form is t^2e^{-2t}.
- Similar cases occur when an input has the **same dynamic form as a natural decay mode**, for example in resonance or limiting transient behavior.

Exercise 5

Find the impulse response of the system described by the differential equation:

$$\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 15y(t) = 15x(t) - 2\frac{dx(t)}{dt}$$

Answer: The system is **strictly proper** because the output order N on the left exceeds the input order M on the right. Its impulse response therefore contains only exponential terms, with no delta function or derivatives of it (singular terms).

Assume initial rest and apply the delta function as input, i.e., $x(t) = \delta(t)$. The differential equation becomes:

$$\frac{d^2h(t)}{dt^2} + 8\frac{dh(t)}{dt} + 15h(t) = 15\delta(t) - 2\frac{d\delta(t)}{dt}$$

Exercise 5 (continued)

The homogeneous equation is:

$$\frac{d^2h_o(t)}{dt^2} + 8\frac{dh_o(t)}{dt} + 15h_o(t) = 0$$

The initial conditions are:

$$h_o(0^+) = 0 \text{ and } \frac{dh_o(0^+)}{dt} = \frac{1}{\alpha_2} = 1$$

The function $h_o(t)$ is an **auxiliary homogeneous solution**, rather than the final impulse response. Determine the **new initial conditions** for the homogeneous solution of the normalized system:

$$a_2h_o''(t) + a_1h_o'(t) + a_0h_0(t) = x(t)$$

This system has input $\delta(t)$ on the right-hand side, rather than $15\delta(t) - 2\delta'(t)$. Its new initial conditions at 0^+ are:

$$h_o(0^+) = 0 \quad \text{and} \quad h_o'(0^+) = \frac{1}{\alpha_2} = 1$$

Exercise 5 (continued)

The characteristic polynomial is: $\lambda^2 + 8\lambda + 15 = (\lambda + 3)(\lambda + 5) = 0$.

The roots are $\lambda_1 = -3$ and $\lambda_2 = -5$, and the homogeneous solution is:

$$h_o(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-3t} + C_2 e^{-5t}, \qquad \text{for } t > 0$$

Determine the coefficients C_1, C_2 from the initial conditions:

$$h_o(0^+) = C_1 e^{-3\cdot 0} + C_2 e^{-5\cdot 0} = C_1 + C_2 = 0$$

$$h_o'(0^+) = (-3C_1 e^{-3t} - 5C_2 e^{-5t})\Big|_{t=0} = -3C_1 - 5C_2 = \frac{1}{\alpha_N} = 1$$

where α_N is the coefficient of the highest-order term in the differential equation.

Solving the equations gives $C_1 = 1/2,\ C_2 = -1/2$.

Exercise 5 (continued)

The homogeneous solution for the impulse response is:

$$h_o(t) = \frac{1}{2}(e^{-3t} - e^{-5t})\, u(t)$$

The right-hand side of the LCCDE contains the input terms $15x(t) - 2dx(t)/dt$. Because the system is LTI, it behaves as a cascade of two subsystems:

- A system determined solely by its poles (the left-hand side), with response $h_o(t)$.
- A differentiation and scaling system that applies the operator $15 - 2d(\cdot)/dt$ to the intermediate signal.

The required impulse response is:

$$h(t) = 15h_o(t) - 2\frac{dh_o(t)}{dt}$$

Exercise 5 (continued)

First calculate the derivative $dh_o(t)/dt$ separately:

$$\frac{dh_o(t)}{dt} = \left[\frac{1}{2}(e^{-3t} - e^{-5t})\,u(t)\right]' = \frac{1}{2}[(e^{-3t} - e^{-5t})'u(t) + (e^{-3t} - e^{-5t})u'(t)]$$

$$= \frac{1}{2}[(-3e^{-3t} + 5e^{-5t})\,u(t) + (e^{-3t} - e^{-5t})\delta(t)] = \frac{1}{2}(-3e^{-3t} + 5e^{-5t})\,u(t)$$

Therefore, the impulse response is:

$$h(t) = 15h_o(t) - 2\frac{dh_o(t)}{dt} = \frac{15}{2}(e^{-3t} - e^{-5t})\,u(t) - 2 \cdot \frac{1}{2}(-3e^{-3t} + 5e^{-5t})\,u(t)$$

$$= \frac{1}{2}(21e^{-3t} - 25e^{-5t})\,u(t)$$

The calculation uses the property $x(t)\delta(t) = x(0)\delta(t)$ and the fact that $\delta(t) = du(t)/dt$.

Exercise 5 (continued)

- To find the **impulse response**, apply $\delta(t)$ as the input and assume initial rest. This turns the problem into a differential equation with a specific excitation.
- The LCCDE in this exercise is **strictly proper**. Its impulse response contains no singular terms (the impulse $\delta(t)$ or its derivatives), only exponential terms multiplied by the unit step. This simplifies the solution.
- Solve the homogeneous equation and find the roots (eigenvalues), which determine the form of the impulse response. The initial conditions at $t = 0^+$, caused by the impulse excitation, determine the coefficients. These conditions follow from the effect of $\delta(t)$, rather than being supplied separately.
- Distinguish the auxiliary homogeneous solution from the final impulse response. The homogeneous solution is an intermediate step; the final response includes the **new initial conditions** at $t = 0^+$ caused by $\delta(t)$.

Exercise 6

Find the impulse response of the system described by the LCCDE:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = \frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + x(t)$$

Answer: Since $N = M$, the impulse response also contains a delta function. Apply input $x(t) = \delta(t)$ and assume initial rest. The differential equation becomes:

$$\frac{d^2h(t)}{dt^2} + 4\frac{dh(t)}{dt} + 4h(t) = \frac{d^2\delta(t)}{dt^2} + 3\frac{d\delta(t)}{dt} + \delta(t)$$

The homogeneous equation is:

$$\frac{d^2h_o(t)}{dt^2} + 4\frac{dh_o(t)}{dt} + 4h_o(t) = 0$$

The initial conditions are:

$$h_o(0^+) = 0 \text{ and } \frac{dh_o(0^+)}{dt} = 1$$

Exercise 6 (continued)

The characteristic polynomial is: $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$

The eigenvalues are $\lambda_1 = \lambda_2 = -2$, and the homogeneous solution is:

$$h_o(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_2 t} = (C_1 + C_2 t)e^{-2t}, \qquad \text{for } t > 0$$

Determine the coefficients C_1, C_2 from the initial conditions:

$$h(0^+) = C_1 = 0$$

$$\frac{dh_o(0^+)}{dt} = \left(-2C_1 e^{-2.t} + C_2 e^{-2t}(1 - 2t)\right)\Big|_{t=0} = -2C_1 + C_2 = \frac{1}{\alpha_N} = 1$$

Solving the equations gives $C_1 = 0, C_2 = 1$.

Therefore, the homogeneous solution is:

$$h_o(t) = te^{-2t}u(t)$$

Exercise 6 (continued)

Calculate the impulse response using:

$$h(t) = \frac{d^2 h_o(t)}{dt^2} + 3\frac{dh_o(t)}{dt} + h_o(t) = \frac{d^2}{dt^2}[te^{-2t}u(t)] + 3\frac{d}{dt}[te^{-2t}u(t)] + te^{-2t}u(t)$$

Calculate each derivative separately. The first derivative is:

$$\frac{d}{dt}[te^{-2t}u(t)] = (te^{-2t})'u(t) + (0 \cdot e^{-2\cdot 0})\,\delta(t) = [1 \cdot e^{-2t} + t(-2e^{-2t})]u(t) = (1-2t)e^{-2t}u(t)$$

The second derivative is:

$$\begin{aligned}
&\frac{d^2}{dt^2}[te^{-2t}u(t)] = \frac{d}{dt}[(1-2t)e^{-2t}u(t)] = [(1-2t)e^{-2t}]'u(t) + (1-2t)e^{-2t}u'(t) \\
&= [(1-2t)'e^{-2t} + (1-2t)(e^{-2t})']\,u(t) + (1-2t)e^{-2\mathrm{t}}\delta(t) \\
&= [-2e^{-2t} - 2(1-2t)e^{-2t}]\,u(t) + (1-2\cdot 0)e^{-2\cdot 0}\delta(t)
\end{aligned}$$

Therefore, the impulse response is:

$$h(t) = (4t-4)e^{-2\mathrm{t}}\,u(t) + \delta(t) + 3(1-2t)e^{-2t}u(t) + te^{-2t}u(t) = \delta(t) - (t+1)e^{-2t}u(t)$$

Exercise 6 (continued)

- This system is not strictly proper, so its impulse response contains a singular term involving $\delta(t)$. The form of the LCCDE determines whether the response contains only exponential terms or also singular terms.
- The final impulse response combines the homogeneous solution with terms involving $\delta(t)$ and its derivatives.
- When a system is not strictly proper, part of the input passes directly to the output through an instantaneous component. This behavior occurs in ideal amplifiers or circuits with parallel paths that introduce no delay.