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SIGNALS AND SYSTEMS
Solved Examples – Tutorial Handout 3

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INTRODUCTION TO CONTINUOUS-TIME SYSTEMS

- Types of Systems:
  - Linear and Nonlinear Systems
  - Causal and Noncausal Systems
  - Static and Dynamic Systems
  - Time-Invariant and Time-Varying Systems
  - Stable and Unstable Systems

1. Linear and Nonlinear Systems

📖 Example 1 (*)

Test the system with the following input–output relation for linearity:

$$\frac{dy(t)}{dt} - 3y(t) = x(t)$$

Answer: In this example, we do not know the system's output $y(t)$ for a given input $x(t)$, but we know the differential equation relating the input to the output. We nevertheless use the same method as in the previous examples. The inputs $x_1(t)$ and $x_2(t)$ produce the outputs $y_1(t)$ and $y_2(t)$, satisfying:

$$\frac{dy_1(t)}{dt} - 3y_1(t) = x_1(t)$$

and

$$\frac{dy_2(t)}{dt} - 3y_2(t) = x_2(t)$$

Multiply the first differential equation by a constant α_1 and the second by a constant α_2, giving:

$$\alpha_1 \frac{dy_1(t)}{dt} - 3\alpha_1 y_1(t) = \alpha_1 x_1(t) \Rightarrow \frac{d[\alpha_1 y_1(t)]}{dt} - 3\alpha_1 y_1(t) = \alpha_1 x_1(t)$$

and

$$\alpha_2 \frac{dy_2(t)}{dt} - 3\alpha_2 y_2(t) = \alpha_2 x_2(t) \Rightarrow \frac{d[\alpha_2 y_2(t)]}{dt} - 3\alpha_2 y_2(t) = \alpha_2 x_2(t)$$

Adding the corresponding sides of these equations, we obtain:

$$\frac{d[\alpha_1 y_1(t)]}{dt} + \frac{d[\alpha_2 y_2(t)]}{dt} - 3\alpha_1 y_1(t) - 3\alpha_2 y_2(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

Therefore:

$$\frac{d[\alpha_1 y_1(t) + \alpha_2 y_2(t)]}{dt} - 3\big(\alpha_1 y_1(t) + \alpha_2 y_2(t)\big) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

Let $x(t)$ be the linear combination (combined input) of $x_1(t)$ and $x_2(t)$:

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

and let $y(t)$ be the linear combination (combined output) of $y_1(t)$ and $y_2(t)$:

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

Using these relations, the differential equation becomes:

$$\frac{dy(t)}{dt} - 3y(t) = x(t)$$

The differential equation is therefore satisfied by the combined input $x(t)$ and combined output $y(t)$. Thus, the system is linear.

Comment: This result can readily be extended to a differential equation of order N, showing that:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}$$

This differential equation represents a linear system. The coefficients a_k and b_k may be constants or varying quantities. In this book, we focus on systems described by linear constant-coefficient differential equations (LCCDEs).

Example 2

Determine whether the system with the following input–output relation is linear:

$$y(t) = C + \frac{1}{T}\int_{t-T}^{t} x(\tau)\, d\tau$$

If it is nonlinear, suggest a way to make it linear.

Answer: This system calculates the average value of the input over $[t - T, t]$, then adds a constant C. Such a system is called a biased averager. To test its linearity, we first examine homogeneity. If the input is $\alpha x(t)$, the output is:

$$y'(t) = C + \frac{1}{T}\int_{t-T}^{t} \alpha x(\tau)\, d\tau = C + \frac{\alpha}{T}\int_{t-T}^{t} x(\tau)\, d\tau$$

On the other hand, multiplying the original output $y(t)$ by α gives:

$$\alpha y(t) = \alpha C + \frac{\alpha}{T}\int_{t-T}^{t} x(\tau)\, d\tau$$

In general, we observe that:

$$y'(t) \neq \alpha y(t)$$

The system does not satisfy homogeneity and is therefore nonlinear. Comparing the equations, the difference between the outputs lies in the term C: in the second case, it is multiplied by α, whereas in the first case it is not. Removing C (by setting $C = 0$) in the input–output relation) therefore makes the system linear.

Another approach is to view the system as having two inputs: the signal $x(t)$ and the constant coefficient C. The constant C is the system's zero-input response, $x(t) = 0$. Thus, $y(t)$ can be viewed as the sum of a linear system's response:

$$y_0(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\, d\tau$$

and a zero-input response C:

$$y(t) = y_0(t) + C$$

Comments:

- A spring is a practical example of a system that can depart from linear behavior. While the applied force remains within certain limits, the spring operates in its linear region. When the force is removed, it returns to its original state. If the force exceeds a critical limit, however, the spring leaves its linear operating region, loses its ideal elastic behavior, and does not fully return to its original state after the excitation is removed. It then exhibits permanent deformation.
- Many practical systems behave approximately linearly only for small input amplitudes and become nonlinear at larger amplitudes. Linearity is particularly important because it greatly simplifies mathematical analysis and the study of systems. The analysis of nonlinear systems, which comprise most physical systems, is usually much more demanding. For this reason, theory and practice often use linear models to approximate a system's nonlinear behavior around a particular operating point.

2. Causal and Noncausal Systems

Example 3

Determine whether each of the following systems is causal.

(a) $y(t) = x(t-1)$ (b) $y(t) = x(t) + x(t+1)$

Answer: (a) It is causal because the output depends only on past values of the input.

(b) It is noncausal because the output also depends on future input values. In this case, we can say that the system knows the future input and generates its output using that knowledge.

3. Static and Dynamic Systems

Example 4

Determine whether each of the following systems is static or dynamic.

(a) $y(t) = sin(t)\, cos(t)$ (b) $y(t) = t\, x(t-2)$

(c) $y(t) = x(t)/[1 + x(t-1)]$ (d) $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

Answer: The first system is static because its output at each time t depends only on the input at that same time.

The second, third, and fourth systems are dynamic because their current outputs also depend on previous inputs. Specifically, the third system's output $y(t)$ depends on the

input at $t-1$, whereas the fourth system's output depends on infinitely many past input values. The second system's output $y(t)$ depends on its input at $t-2$.

4. Time-Invariant and Time-Varying Systems

📖 Example 5 (*)

Determine whether the following systems are time-invariant or time-varying.

(a) $y(t) = t\,x(t)$ (b) $y(t) = x(t)\,cos(\Omega_c t)$

Answer: (a) Apply the input $x(t-t_0)$, where t_0 is an arbitrary time shift. The resulting output is $y(t) = t\,x(t-t_0)$. It differs from the original output shifted by t_0, namely $y(t-t_0) = (t-t_0)\,x(t-t_0)$. Therefore, the system is time-varying.

(b) For the input $x(t-t_0)$, where t_0 is an arbitrary time shift, the output is $y(t) = x(t-t_0)\cos(\Omega_c t)$. The system is time-varying because this output differs from the original output shifted by t_0, namely $y(t-t_0) = x(t-t_0)\,cos(\Omega_c(t-t_0))$.

📖 Example 6

The voltage across a capacitor and the current flowing through it are related by:

$$\frac{du_c(t)}{dt} = \frac{1}{C}i(t)$$

with initial condition $u_c(t) = 0$. Determine the conditions under which this capacitor system is time-invariant.

Answer: Solving the differential equation for the voltage $u_c(t)$ and using the initial condition $u_c(t) = 0$, we obtain:

$$u_c(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

Delay the input current $i(t)$ by λ. Using the substitution $\xi = \tau - \lambda$, the output voltage $u_c'(t)$ across the capacitor for the shifted input $i(\tau-\lambda)$ is:

$$u_c'(t) = \frac{1}{C}\int_0^t i(\tau-\lambda)d\tau = \frac{1}{C}\int_{-\lambda}^0 i(\xi)d\xi + \frac{1}{C}\int_0^{t-\lambda} i(\xi)d\xi \qquad (1)$$

The output shifted by λ, namely $u_c(t-\lambda)$, is given by:

$$u_c(t-\lambda) = \frac{1}{C}\int_0^{t-\lambda} i(\xi)d\xi \qquad (2)$$

For the system to be time-invariant, we require:

$$u_c'(t) = u_c(t-\lambda) \qquad (3)$$

Comparing (1) and (2), condition (3) is satisfied if the first integral in (1) is zero:

$$\frac{1}{C}\int_{-\lambda}^0 i(\xi)d\xi = 0$$

Thus, the current must be zero for negative time, namely $i(t) = 0$ for $t < 0$. Together with the initial condition $u_c(0) = 0$, this means that, for the capacitor to be a time-invariant system, it must have no stored energy at 0.

5. Stable and Unstable Systems

📖 Example 7

Determine whether the system described by the following input–output relation is BIBO stable.

$$y(t) = \alpha_1 x(t-\tau_1) + \alpha_2 x(t-\tau_2)$$

Answer: Assume a bounded input signal $x(t)$, namely $|x(t)| \leq M < \infty$. All values of $x(t)$ and its time-shifted versions $x(t-\tau_1)$ and $x(t-\tau_2)$ therefore lie within $[-M, M]$. Consequently, the signal $y(t)$ satisfies:

$$|y(t)| = |\alpha_1|\,|x(t-\tau_1)| + |\alpha_2||x(t-\tau_2)| \leq [|\alpha_1| + |\alpha_2|]M$$

The output is also bounded, so the system is BIBO stable. Alternatively, we can determine its BIBO stability by calculating its impulse response and testing absolute integrability using $\int_{-\infty}^{\infty}|h(t)|\,dt < \infty$. The impulse response is obtained by setting $x(t) = \delta(t)$:

$$h(t) = \alpha_1\delta(t-\tau_1) + \alpha_2\delta(t-\tau_2)$$

The relation $\int_{-\infty}^{\infty}|h(t)|\,dt < \infty$ gives:

$$\int_{-\infty}^{\infty}|h(t)|\,dt = \int_{-\infty}^{\infty}|\alpha_1\delta(t-\tau_1) + \alpha_2\delta(t-\tau_2)|\,dt = |\alpha_1|\int_{-\infty}^{\infty}\delta(t-\tau_1)\,dt + |\alpha_2|\int_{-\infty}^{\infty}\delta(t-\tau_2)\,dt = |\alpha_1| + |\alpha_2| < \infty$$

The impulse response is therefore absolutely integrable, so the system is BIBO stable.

📖 Example 8

Test the following system for asymptotic stability:

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} - 2x(t)$$

Answer: Setting the input $x(t)$ and its derivatives to zero gives the homogeneous differential equation:

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 4y(t) = 0$$

This yields the characteristic polynomial $p(\lambda)$, which we factor as follows:

$$p(\lambda) = \lambda^3 + 4\lambda^2 + 6\lambda + 4 \Rightarrow (\lambda+2)(\lambda^2+2\lambda+2) = 0$$

The roots of the characteristic polynomial are:

$$\lambda_1 = -2, \qquad \lambda_2 = -1 + j, \qquad \lambda_3 = -1 - j$$

All roots have negative real parts. Therefore, the system is asymptotically stable.

Note: Examples marked with an asterisk (*) also appear in the lecture slides.