UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

# Signals and Systems

## Lecture 3: Introduction to Continuous-Time Systems

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# Lecture Contents

1. Definition and Categories of Systems
   - Continuous-Time Systems
   - Discrete-Time Systems
2. Interconnections of Continuous-Time Systems
3. Types of Systems
   - Linear and Nonlinear Systems
   - Causal and Noncausal Systems
   - Static and Dynamic Systems
   - Time-Invariant and Time-Varying Systems
   - Stable and Unstable Systems

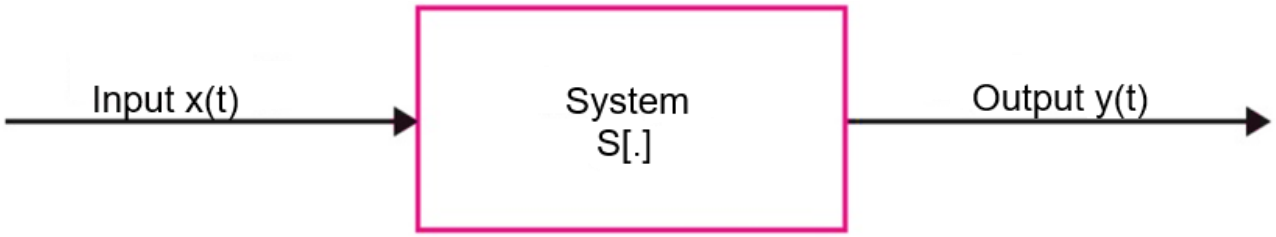
# 1. Definition and Categories of Systems

# Definition of a System

A system is any entity that acts on a signal $x(t)$ and produces a (new) signal $y(t)$.

Mathematically, a system is a transformation $\boldsymbol{S}[.]$ that maps a signal $x(t)$ into another signal $y(t)$, according to $y(t) = \boldsymbol{S}[x(t)]$.

This approach is called the **input–output relation**. It describes a system in the time domain through its input and output, without considering its internal structure or description.




Block diagram of a system

The signal $x(t)$ that excites the system is called its **input**. The result of this excitation, the signal $y(t)$, is called its **output**.

# Categories of Systems (1/3)

(A) According to the **number of inputs and outputs**:

- **Single-input, single-output** systems (SISO). Simple examples are the scalar multiplier $y(t) = \alpha\, x(t)$ and the delay system $y(t) = x(t - t_0)$..
- **Multiple-input, single-output** systems (MISO). Examples include the adder of two or more signals
  $y(t) = x_1(t) + x_2(t)$ and the multiplier $y(t) = x_1(t)\, x_2(t)$.
- **Single-input**, **multiple-output** systems (SIMO).
- **Multiple-input, multiple-output** systems (MIMO).

# Categories of Systems (2/3)

(A) According to the **nature of the inputs and outputs**:

- **Continuous-time** or **analog systems** have analog input and output signals.

  When the input and output signals are **discrete-time** signals, the systems are called discrete-time systems.

- **Deterministic systems** have deterministic input and output signals.

  When the input and output signals are stochastic, the systems are called **stochastic systems**.

# Categories of Systems (3/3)

- A **continuous-time system** is any entity that acts on a **continuous-time** input signal $x(t)$ and transforms it into the **continuous-time** output signal:

$$y(t) = S[x(t)]$$

- A **discrete-time system** is any entity that acts on a discrete-time input signal $\{x(n)\}$ and transforms it into the **discrete-time** output signal $\{y(n)\}$:

$$\{y(n)\} = T[\{x(n)\}]$$

# 2. Interconnections of Continuous-Time Systems

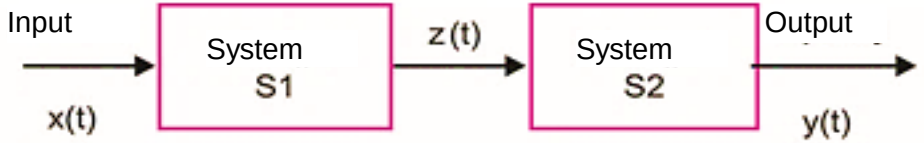
# System Interconnections (1/2)

A system can be decomposed into simpler systems interconnected in various ways.

The main system interconnections are **cascade**, **parallel**, **mixed**, and **feedback** connections.

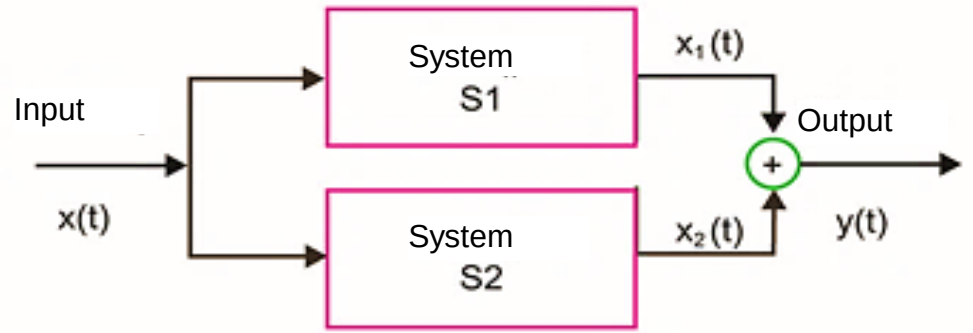
- **Cascade connection:**

  The output is: $y(t) = S2[\,S1[x(t)]\,]$



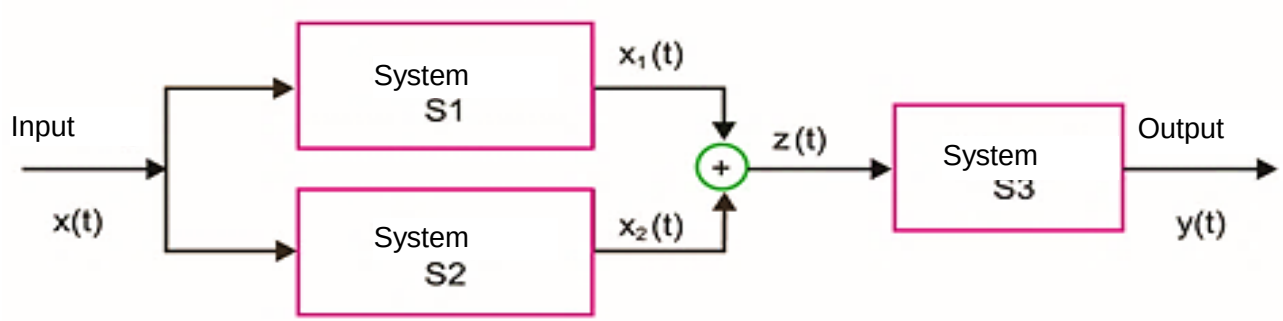

- **Parallel connection:**

  The output is: $y(t) = S1[x(t)] + S2[x(t)]$



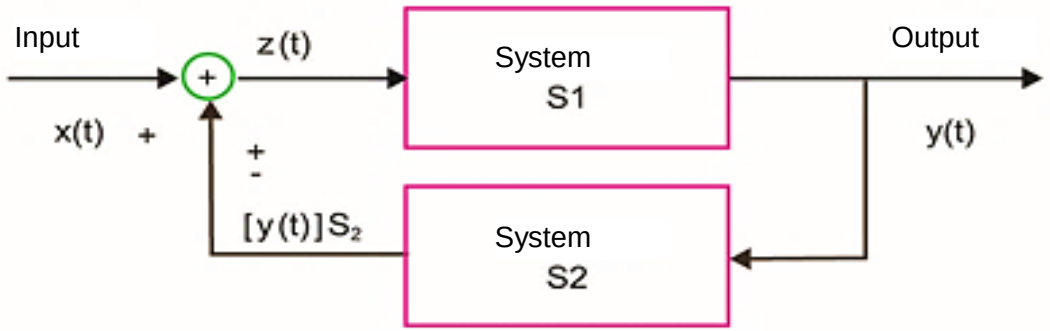
# System Interconnections (2/2)

- **Mixed connection:**




The output is: $y(t) = S3\big[\, S1[x(t)] + S2[x(t)]\big]$

- **Feedback connection:**




The output is: $y(t) = S1[x(t) \pm S2[y(t)]]$

# 3. Types of Continuous-Time Systems

# Types of Systems

A system is **at rest** at time $t_0$ if no signal has excited it for $t < t_0$. This means that the system has no stored energy at time $t = t_0$.

Types of systems under initial rest conditions:

- Linear and Nonlinear Systems
- Causal and Noncausal Systems
- Static and Dynamic Systems
- Time-Invariant and Time-Varying Systems
- Stable and Unstable Systems

# Linear and Nonlinear Systems (1/2)

A system initially at rest is **linear** if and only if, for any two input signals $x_1(t)$ and $x_2(t)$, the following relation holds:

$$S[a_1x_1(t) + a_2x_2(t)] = a_1S[x_1(t)] + a_2S[x_2(t)]$$

The response to a linear combination of two input signals equals the corresponding linear combination of the responses to each signal.

This is the **principle of superposition**. It extends to any finite linear combination of signals:

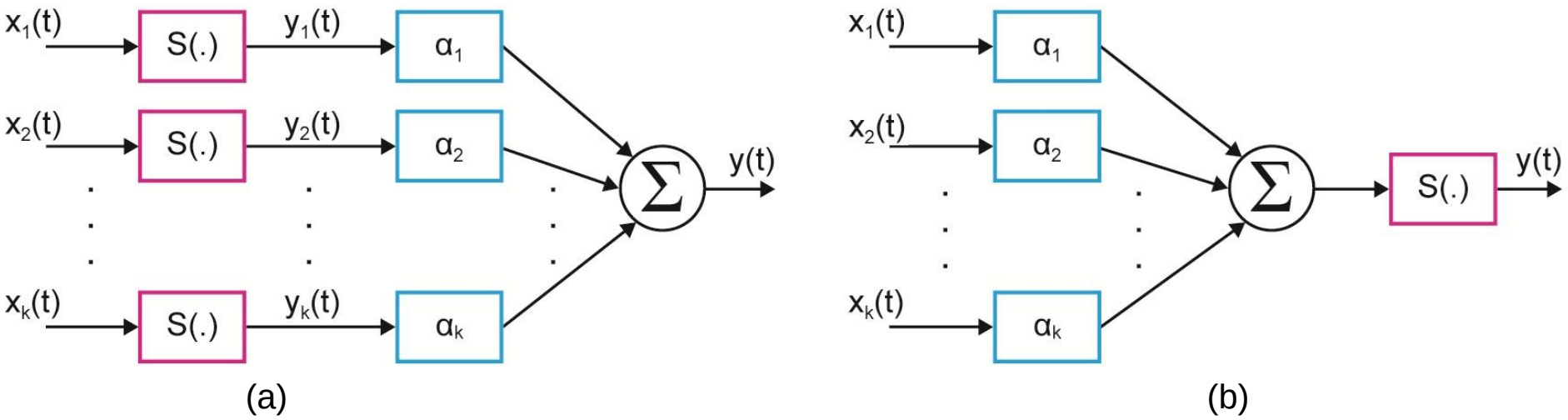
$$x_1(t) \xrightarrow{S} y_k(t), \qquad k = 1,2, \dots, M$$

then, for scalars $a_k, k = 1,2, \dots, M$, we have:

$$x(t) = \sum_{k=1}^{M} a_k x_k(t) \xrightarrow{S} y(t) = \sum_{k=1}^{M} a_k\, y_k(t)$$

# Linear and Nonlinear Systems (2/2)

- As the diagram shows, a system is linear when its output for a linear combination of inputs (combined input) equals the corresponding linear combination of the individual outputs (combined output).



# Causal and Noncausal Systems (1/2)

- A system is **causal** when its **present output does not depend** on **future input values.**
- For every input signal $x(t)$, the corresponding output $y(t)$ depends only on present and/or past input values.
- Conversely, if the output $y(t_0)$ at time $t_0$ **depends** on later values of the input signal $x(t)$, i.e., for $t \geq t_0$, the system is **noncausal.**
- In other words, a system is causal if changes in its output (effect) **never precede** changes in its input (cause).
- All physical passive systems have the property of **causality.**
- Causal systems satisfy the fundamental **cause-and-effect relation**: the cause (input) comes first and the effect (output) follows. All practical systems obey this relation.

# Causal and Noncausal Systems (2/2)

- A noncausal system must know future input values to produce its output. This is impossible in real-time systems, so real-time systems are always causal.
- If the input is recorded in **memory**, all its values, including future values, are available at any time. Memory therefore allows noncausal processing when real-time operation is unnecessary. Audio and image compression use this approach: algorithms such as JPEG and MP3 achieve high compression ratios because they "know" future input values.
- Noncausal systems are easier to implement in discrete time because digital memory is convenient and inexpensive.
- An ideal filter, which introduces no signal distortion, is **always noncausal** and cannot operate in real time. A practical filter is **causal** but always introduces some distortion into the input signal.
- Based on this observation, noncausal systems can be regarded as an **upper performance limit** for causal systems.

# Static and Dynamic Systems (1/2)

- A system is **static**, or **memoryless**, if its output at any time depends only on the input at that **same** time, rather than on past or future input values.
- A system that is not static is **dynamic**, or **has memory**.
- The output of a dynamic system depends on the input at the same time and on at least some past input values. It may also depend on future input values.
- A system whose output at time $t$ is completely determined by input values from $t - T$ to $t$ (where $T \geq 0$) has a **memory** of **length** T.
- If $T$ is finite, the system has **finite memory**. If $T \to \infty$, the system has **infinite memory**.
- A static system has **zero memory** ($T = 0$).

# Static and Dynamic Systems (2/2)

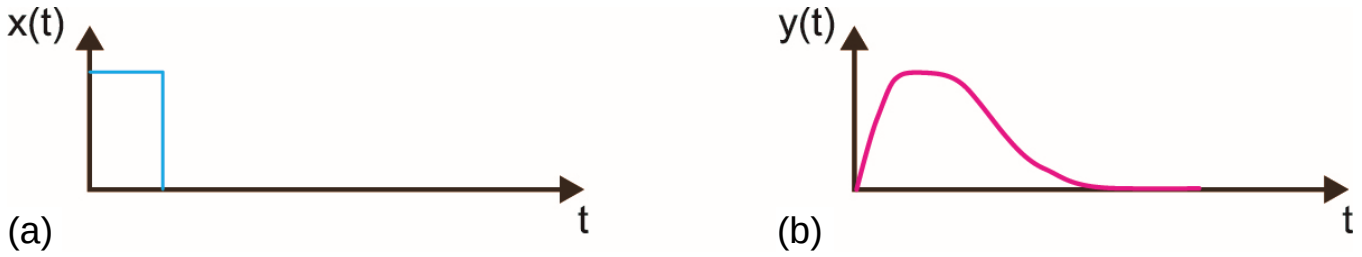
A dynamic system can store energy. Examples of static and dynamic systems:

- A resistor is **memoryless**: the voltage $u_R(t)$ across it (output) depends on the current $i(t)$ through it (input) at the same instant:

$$u_R(t) = R\, i(t)$$

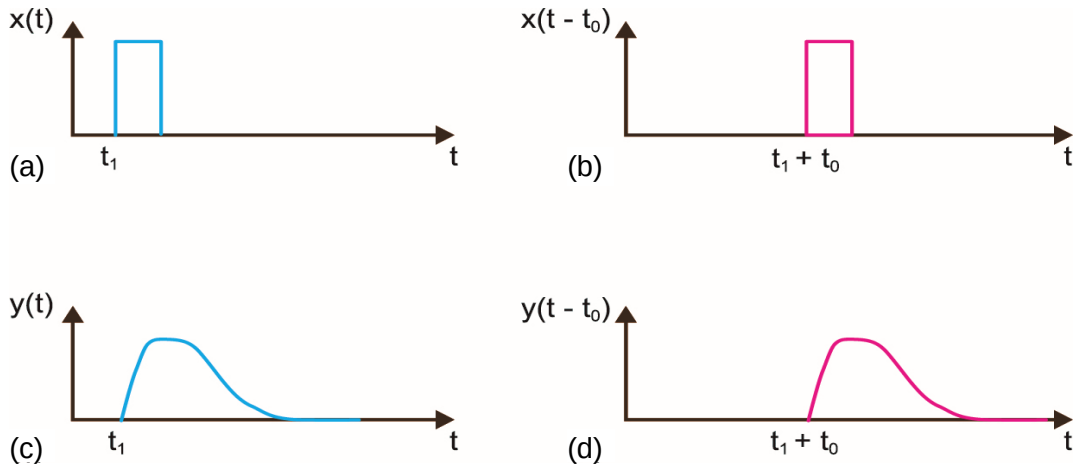
- A capacitor **has memory**: the voltage $u_c(t)$ across it at any time depends on the entire history of the current $i(t)$ through it:

$$u_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau$$



# Time-Invariant and Time-Varying Systems (1/2)

- A system is **time-invariant** if and only if a time shift of the input produces the same time shift of the output.
- In a time-invariant system, if an input signal $x(t)$ produces output $y(t)$, then the input $x(t - t_0)$ produces output $y(t - t_0)$.




- To test time invariance, calculate the output $y(t)$ for input $x(t)$. If $y(t) = y(t - t_0)$ for every input $x(t)$ and every time shift $t_0$, the system is time-invariant.

# Time-Invariant and Time-Varying Systems (2/2)

- Although systems are often time-varying, they can be approximated as time-invariant over a **short interval** in which their behavior is effectively constant.
- For example, in **speech synthesis**, the human vocal system continually changes the shape of its components (larynx, glottis, tongue, lips, nasal cavity, nose, etc.). Nevertheless, a time-invariant model is a good approximation over short intervals of about 20 msec.
- Synthetic speech systems use a low-order linear, time-invariant system with different inputs for **unvoiced and voiced sounds**, corresponding to **consonants** and **vowels**, respectively. A pulse train excites the system for a voiced sound; a noise signal excites it for an unvoiced sound.
- The **low system order** was a major advantage for real-time speech processing, especially in earlier decades.

# Stable and Unstable Systems

- A system has **bounded-input, bounded-output** stability, or BIBO stability, if and only if every bounded input produces a bounded output.
- A system is BIBO stable if, for every positive number $M_1 < \infty$ such that $|x(t)| \leq M_1$, there exists a positive number $M_2 < \infty$ such that $|y(t)| \leq M_2$.
- This definition of stability requires input and output signals to have finite (bounded) amplitudes.

# Asymptotic Stability (1/2)

The stability of a linear system can be studied when its internal structure is known. As we will see in the next lecture, this structure is described by a linear differential equation with constant coefficients:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}$$

The **locations of the roots** $\lambda_i, i = 1,2, \dots N$ of the **characteristic polynomial** in the complex frequency plane determine the stability of the system.

Each root $\lambda_i$ of the characteristic polynomial describes the behavior of the **term** $e^{\lambda_i t}$ as $t \to \infty$..

- The term tends to **zero** when the real part of the root is negative, i.e., $Re\{\lambda_i\} < 0$.
- The term grows **without bound** when the real part of the root is positive, i.e., $Re\{\lambda_i\} > 0$.

# Asymptotic Stability (2/2)

Studying stability through the roots of the characteristic polynomial is called **asymptotic** or **zero-input stability**. It examines the natural response: the response with no external input, determined entirely by initial conditions. The cases are:

- Asymptotically stable: all roots $\lambda_i$ of the characteristic polynomial have negative real parts, i.e., $Re\{\lambda_i\} < 0,\ i = 1,2, \dots, N$. The free response decays to zero as $t \to \infty$.
- Asymptotically unstable: at least one root has a positive real part, i.e., $Re\{\lambda_i\} > 0$ for some $i$. The corresponding response component grows exponentially with time.
- Asymptotically unstable due to repeated roots on the imaginary axis: the characteristic polynomial has roots of multiplicity $p > 1$ on the imaginary axis. The free response contains terms that become unbounded as time increases.
- **Marginally stable**: all roots lie in the closed left half-plane, any roots on the imaginary axis are simple, and all other roots have negative real parts. The free response remains bounded but does not decay to zero.

# Routh Algorithm

For a system with characteristic polynomial $\alpha_n\,\lambda^n + \alpha_{n-1}\,\lambda^{n-1} + \cdots + \alpha_1\,\lambda + \alpha_0$ (usually with $\alpha_n = 1$), construct the Routh array:

| | | | | | | |
|---|---|---|---|---|---|---|
| $\boldsymbol{\lambda^N}$ | $\alpha_n$ | $\alpha_{n-2}$ | $\alpha_{n-4}$ | … | $\alpha_2$ | $\alpha_0$ |
| $\boldsymbol{\lambda^{N-1}}$ | $\alpha_{n-1}$ | $\alpha_{n-3}$ | $\alpha_{n-5}$ | … | $\alpha_1$ | 0 |
| $\boldsymbol{\lambda^{N-2}}$ | $b_1$ | $b_2$ | $b_3$ | … | | |
| $\boldsymbol{\lambda^{N-3}}$ | $c_1$ | $c_2$ | | | | |
| … | … | … | … | | | |
| $\boldsymbol{\lambda^2}$ | $e_1$ | $a_0$ | | | | |
| $\boldsymbol{\lambda^1}$ | $f_1$ | | | | | |
| $\boldsymbol{\lambda^0 = 1}$ | $a_0$ | | | | | |

where: $b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}}$, $b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}}$, …, $c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1}$, $c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1}$, …

According to the Routh criterion, the system is stable when **all** coefficients in the first column $\alpha_n, \alpha_{n-1}, b_1, c_1$ … have the **same sign** as the coefficient $\alpha_n$.

Each **sign change** corresponds to a root (eigenvalue) in the right half-plane of complex frequency, which causes instability.

# Conclusions on System Stability

- Stability is especially important in practical systems. Unstable systems are avoided in practical applications because their outputs grow without bound.
- BIBO stability requires input and output signals to remain finite (bounded) throughout their duration.
- If the impulse response is **absolutely integrable**, the system is BIBO stable. We will study this in a later section.
- Although BIBO stability is defined through input and output, stability is an **intrinsic property** of a system, **independent** of its input and output. Asymptotic or **zero-input stability** concerns the **internal structure** and the response to nonzero initial conditions, with no applied input.
- An asymptotically stable system is also BIBO stable. A marginally stable or unstable system is BIBO unstable.
- The Laplace transform provides a convenient way to study system stability.

# Exercises

# Exercise 1

Test the linearity of the system with input–output relation:

$y(t) = sin\{x(t)\}$.

Answer: For input $x_1(t)$, the output is
$y_1(t) = sin\{x_1(t)\}$. For input $x_2(t)$, the output is $y_2(t) = sin\{x_2(t)\}$.

For the combined input $x(t) = a_1x_1(t) + a_2x_2(t)$, the output $y(t)$ is:

$$y(t) = sin\big(a_1x_1(t) + a_2x_2(t)\big) \neq sin\big(a_1x_1(t)\big) + sin\big(a_2x_2(t)\big) = y'(t)$$

Here, $y'(t)$ is the linear combination of the outputs $y_1(t)$ and $y_2(t)$:

$$y'(t) = \alpha_1y_1(t) + \alpha_2y_2(t)$$

Since $y(t) \neq y'(t)$, the system is nonlinear.

# Exercise 2

Test the linearity of the system with input–output relation:

$$y(t) = \frac{x(t)}{1 + x(t-1)}$$

Answer: For inputs $x_1(t)$ and $x_2(t)$, the outputs $y_1(t)$ and $y_2(t)$ are:

$$y_1(t) = \frac{x_1(t)}{1+x_1(t-1)} \text{ and } y_2(t) = \frac{x_2(t)}{1+x_2(t-1)}$$

For the linear combination $x(t) = a_1 x_1(t) + \alpha_2 x_2(t)$ as input, the output $y(t)$ is:

$$y(t) = \frac{x(t)}{1 + x(t-1)} = \frac{\alpha_1 x_1(t) + \alpha_2 x_2(t)}{1 + \alpha_1 x_1(t-1) + \alpha_2 x_2(t-1)}$$

The linear combination $y'(t)$ of the outputs $y_1(t)$ and $y_2(t)$ is:

$$y'(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 \frac{x_1(t)}{1 + x_1(t-1)} + \alpha_2 \frac{x_2(t)}{1 + x_2(t-1)}$$

We observe that $y(t) \neq y'(t)$. Therefore, the system is nonlinear.

# Exercise 3

Determine whether each of the following systems is causal:

(a) $y(t) = x^3(t+3)$ (b) $y(t) = e^{-\alpha t}u(t)$

Answer:

(a) Noncausal, because the output depends on future input values.

(b) Causal, because the output depends only on the present input value.

# Exercise 4

Which of the systems described by the following input–output relations are static, and which are dynamic?

(a) $y(t) = \alpha x(t)$

(b) $y(t) = x^2(t)$

(c) $y(t) = x(t) + 2x(t-1)$

(d) $y(t) = y(t-1) + x(t)$

Answer:

(a) Static: the output at time $t$ depends only on the input at the same time.

(b) Static, for the same reason.

(c) Dynamic: the output at time $t$ depends on the input at an earlier time.

(d) Dynamic: the output at time $t$ depends on an earlier output and therefore on an earlier input.

# Exercise 5

Determine whether the system with input–output relation $y(t) = t\ x(t)$ is time-invariant or time-varying.

Answer: Apply the input $x(t - t_0)$, where $t_0$ is an arbitrary time shift. The output is $y(t) = t\ x(t - t_0)$.

This output differs from the output shifted by $t_0$, namely:

$$y(t - t_0) = (t - t_0)\ x(t - t_0).$$

Therefore, the system is time-varying.

# Exercise 6

Determine whether the system with input–output relation $y(t) = cos[\, x(t) \,]$ is time-invariant or time-varying.

Answer: Apply the input $x(t - t_0)$, where $t_0$ is an arbitrary time shift. The output is $y(t) = cos[\, x(t - t_0) \,]$.

This output equals the output shifted by $t_0$, namely:

$$y(t - t_0) = cos[\, x(t - t_0) \,].$$

Therefore, the system is time-invariant.

# Exercise 7

Determine whether the system with input–output relation $y(t) = x(t)\ cos(\Omega_c t)$ is time-invariant or time-varying.

Answer: For input $x(t - t_0)$, where $t_0$ is an arbitrary time shift, the output is $y(t) = x(t - t_0)\ \cos(\Omega_c t)$.

This output differs from the output shifted by $t_0$:

$$y(t - t_0) = x(t - t_0)\ cos(\Omega_c(t - t_0)).$$

Therefore, the system is time-varying.

# Exercise 8

Test the BIBO stability of the systems described by these input–output relations:

$$\text{(a) } y(t) = x(t)\, sin(\Omega t), \qquad \text{(b) } y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Answer: (a) Using the system definition and the fact that $|\sin \Omega t| \leq 1$, we have:

$$|y(t)| = |x(t)\, sin(\Omega t)| = |x(t)\,||sin(\Omega t)| \leq |x(t)|$$

If the input $x(t)$ is bounded, this relation shows that the output $y(t)$ is also bounded. Therefore, the system is **BIBO stable**.

(b) Apply the bounded function $|x(t)| \leq a$ as the input $x(t)$. The output $y(t)$ is:

$$|y(t)| = \int_{-\infty}^{t} x(\tau)\, d\tau \leq \int_{-\infty}^{t} \alpha\, d\tau = \alpha \int_{-\infty}^{t} d\tau = \infty$$

The output is unbounded. Therefore, the system is **BIBO unstable**.

# Exercise 9

Test the asymptotic stability of the system described by the differential equation:

$$\frac{d^4y(t)}{dt^4} + 4\frac{d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 6y(t) = x(t)$$

Answer: Set the input $x(t)$ and its derivatives to zero to obtain the homogeneous differential equation:

$$\frac{d^4y(t)}{dt^4} + 4\frac{d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 6y(t) = 0$$

The characteristic polynomial is:

$$\lambda^4 + 4\lambda^3 + 3\lambda^2 - 2\lambda - 6 = (\lambda^2 + 2\lambda - 3)(\lambda^2 + 2\lambda + 2)$$

The roots of the characteristic polynomial are:

$$\lambda_1 = -3, \qquad \lambda_2 = 1, \qquad \lambda_3 = -1 + j, \qquad \lambda_4 = -1 - j$$

The root $\lambda_1$ has a **positive real part**, so the system is asymptotically unstable.

# Exercise 10

Solve the previous exercise using the Routh algorithm.

Answer: The characteristic polynomial is fourth-order: $\lambda^4 + 4\lambda^3 + 3\lambda^2 - 2\lambda - 6$. Therefore: $\alpha_n = \alpha_4 = 1$,

$$\alpha_{n-1} = \alpha_3 = 4, \quad \alpha_{n-2} = \alpha_2 = 3, \quad \alpha_{n-3} = \alpha_1 = -2, \quad \alpha_{n-4} = \alpha_0 = -6$$

Calculate the coefficients $b_1, b_2, c_1, c_2$ using:

$$b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}} = \frac{a_2 a_3 - a_4 a_1}{a_3} = \frac{3.4 - 1 \cdot (-2)}{4} = 3.5$$

$$b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}} = \frac{a_0 a_3 - a_4 0}{a_3} = \frac{(-6) \cdot 4}{4} = -6$$

$$c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1} = \frac{a_1 b_1 - a_3 b_2}{b_1} = \frac{(-2) \cdot 3.5 - 4 \cdot (-6)}{3.5} = \frac{34}{7}$$

$$c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1} = \frac{0 b_1 - a_3 0}{b_1} = 0$$

# Exercise 10 (continued)

Construct the Routh array:

| | | | |
|---|---|---|---|
| $\lambda^4$ | $\boldsymbol{\alpha_n = \alpha_4 = 1}$ | $\alpha_{n-2} = \alpha_2 = 3$ | $\alpha_{n-4} = \alpha_0 = -6$ |
| $\lambda^3$ | $\boldsymbol{\alpha_{n-1} = \alpha_3 = 4}$ | $\alpha_{n-3} = \alpha_1 = -2$ | 0 |
| $\lambda^2$ | $\boldsymbol{b_1 = 3.5}$ | $b_2 = -6$ | 0 |
| $\lambda^1$ | $\boldsymbol{c_1 = 34/7}$ | $c_2 = 0$ | |
| $\lambda^0 = 1$ | $\boldsymbol{a_0 = -6}$ | | |

The first-column coefficients $\alpha_4, \alpha_3, b_1, c_1, a_0$ do not all have the same sign. There is one sign change, corresponding to one root in the right half-plane of complex frequency $\lambda$. The system is therefore asymptotically unstable.

- The Routh criterion does not calculate the roots; it provides information about their locations.
- Each sign change in the first column corresponds to one root in the right half-plane. Even one sign change therefore proves that the system is unstable.
- The Routh method is particularly efficient for high-order polynomials.

# Review Exercises

# Exercise 11

Determine whether the system described below, called an integrator, is linear, time-invariant, and causal.

$$y(t) = \boldsymbol{S}[\, x(t)] = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Answer: (a) Linearity: For integrator input $x_1(t)$, the output is $\int_{-\infty}^{t} x_1(\tau)\, d\tau$. For input $x_2(t)$, the output is $\int_{-\infty}^{t} x_2(\tau)\, d\tau$.

For the combined input $\alpha_1 x_1(t) + \alpha_2 x_2(t)$, the output is:

$$\int_{-\infty}^{t} [\alpha_1\, x_1(t) + \alpha_2\, x_2(t)] d\tau = \alpha_1 \int_{-\infty}^{t} x_1(t)\, d\tau + \alpha_2 \int_{-\infty}^{t} x_2(t)\, d\tau$$

Therefore, the system is **linear**.

# Exercise 11 (continued)

(b) Time invariance: For integrator input $x(t)$, the output is:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Using the change of variable $\tau = \lambda - t_0$, the response to the signal $x(t - t_0)$ is:

$$y_1(t) = \int_{-\infty}^{t} x(\xi - t_0)\, d\xi = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau = y(t - t_0)$$

Therefore, the system is **time-invariant**.

(c) **Causal**ity: The integrator is causal because its output depends only on present and past input values.

# Exercise 12

Test the linearity of the system with input–output relation:

$$\frac{dy(t)}{dt} - 3y(t) = x(t)$$

Answer: Inputs $x_1(t)$ and $x_2(t)$ produce outputs $y_1(t)$ and $y_2(t)$, satisfying the differential equations:

$$\frac{dy_1(t)}{dt} - 3y_1(t) = x_1(t) \quad \text{and} \quad \frac{dy_2(t)}{dt} - 3y_2(t) = x_2(t)$$

Multiply the first differential equation by a constant $\alpha_1$ and the second by a constant $\alpha_2$:

$$\alpha_1 \frac{dy_1(t)}{dt} - 3\alpha_1 y_1(t) = \alpha_1 x_1(t) \Rightarrow \frac{d[\alpha_1 y_1(t)]}{dt} - 3\alpha_1 y_1(t) = \alpha_1 x_1(t)$$

and

$$\alpha_2 \frac{dy_2(t)}{dt} - 3\alpha_2 y_2(t) = \alpha_2 x_2(t) \Rightarrow \frac{d[\alpha_2 y_2(t)]}{dt} - 3\alpha_2 y_2(t) = \alpha_2 x_2(t)$$

# Exercise 12 (continued)

Adding the two equations term by term gives:

$$\frac{d[\alpha_1 y_1(t)]}{dt} + \frac{d[\alpha_2 y_2(t)]}{dt} - 3\alpha_1 y_1(t) - 3\alpha_2 y_2(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

Therefore:

$$\frac{d[\alpha_1 y_1(t) + \alpha_2 y_2(t)]}{dt} - 3\big(\alpha_1 y_1(t) + \alpha_2 y_2(t)\big) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

Combined input $x(t)$ of $x_1(t)$ and $x_2(t)$: $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$

Combined output $y(t)$ of $y_1(t)$ and $y_2(t)$: $y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$

The differential equation becomes:

$$\frac{dy(t)}{dt} - 3y(t) = x(t)$$

The differential equation is satisfied by the combined input $x(t)$ and combined output $y(t)$.
Therefore, the system is **linear.**

# Exercise 12 (continued)

This result extends to an Nth-order differential equation. In other words, a system described by the following differential equation is linear:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}$$

The coefficients $a_k$ and $b_k$ may be constants or time-varying quantities.

A similar proof establishes time invariance for systems described by linear differential equations with constant coefficients.

Therefore, systems described by linear differential equations with constant coefficients (LCCDEs) are **linear and time-invariant** (LTI).