UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
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SIGNALS AND SYSTEMS
Solved Examples – Tutorial Handout 2

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CONTINUOUS-TIME SIGNALS

- Elementary Continuous-Time Signals

1. Elementary Continuous-Time Signals

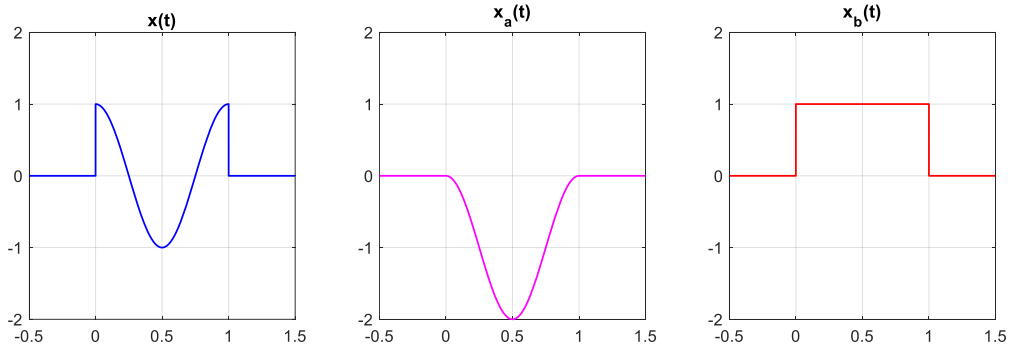
📖 Example 1 (*)
Given the discontinuous signal $x(t) = cos(2\pi t)[u(t) - u(t-1)]$. (a) Express it as a sum of continuous signals and unit step functions. (b) Find its derivative.

Answer: (a) The given signal is one period of $\cos(2\pi t)$ from 0 to 1, and zero everywhere else. It has discontinuities at $t = 0$ and $t = 1$.

Removing the pulse $u(t) - u(t-1)$ from the signal $x(t)$ gives a continuous signal. To compensate for its removal, however, we must introduce a unit pulse between $t = 0$ and $t = 1$. Thus:

$$x(t) = (cos(2\pi t) - 1)[u(t) - u(t-1)] + [u(t) - u(t-1)] = x_a(t) + x_b(t) \qquad (1)$$

where $x_a(t) = (cos(2\pi t) - 1)[u(t) - u(t-1)]$ and $x_b(t) = u(t) - u(t-1)$. The term $x_a(t)$ is a continuous function, whereas $x_b(t)$ is discontinuous. The signal waveforms are shown in the figure.



Signal waveforms: (a) $x_a(t)$, *(b)* $x_b(t)$, (c) $x(t) = x_a(t) + x_b(t)$

(b) Since $du(t)/dt = \delta(t)$, the derivative of the given signal (1) is:

$$\frac{dx(t)}{dt} = -2\pi \sin(2\pi t)[u(t) - u(t-1)] + (cos(2\pi t) - 1)[\delta(t) - \delta(t-1)] + \delta(t) - \delta(t-1)$$
$$= -2\pi \sin(2\pi t)[u(t) - u(t-1)] + \delta(t) - \delta(t-1)$$

📖 Example 2 (*)

Given the signal

$$x(t) = \begin{cases} 0, & t < -1 \\ t+1, & -1 \le t < 0 \\ 1, & 0 \le t < 2 \\ 3-t, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

(a) Express it using unit step functions.
(b) Calculate its first derivative in the sense of distributions.

Answer: (a) We write the signal as a sum of segments. To do this, we multiply the expression for each segment by the difference $u(t-t_1) - u(t-t_2)$, so that the function starts at t_1 and ends at t_2.

On the interval $[-1,0)$, we have $x(t) = t+1$, so:

$$x_1(t) = (t+1)\,[u(t+1) - u(t)]$$

On the interval $[0,2)$, we have $x(t) = 1$, so:

$$x_2(t) = 1\,[u(t) - u(t-2)]$$

On the interval $[2,3)$, we have $x(t) = 3-t$, so:

$$x_3(t) = (3-t)\,[u(t-2) - u(t-3)]$$

Thus

$$x(t) = x_1(t) + x_2(t) + x_3(t)$$

Expanding and collecting terms, we obtain:

$$x(t) = (t+1)u(t+1) - t\,u(t) + (2-t)u(t-2) + (t-3)u(t-3)$$

(b) To calculate the derivative, we use:

$$\frac{d}{dt}[f(t)u(t-\alpha)] = f'(t)u(t-\alpha) + f(t)\delta(t-\alpha) = f'(t)u(t-\alpha) + f(\alpha)\delta(t-\alpha)$$

Derivative of the first term:

$$\frac{d}{dt}[(t+1)u(t+1)] = u(t+1)$$

since $(t+1)\delta(t+1) = 0$.

Derivative of the second term:

$$\frac{d}{dt}[-t\,u(t)] = -u(t)$$

since $t\delta(t) = 0$.

Derivative of the third term:

$$\frac{d}{dt}[(2-t)u(t-2)] = -u(t-2)$$

since at $t = 2$, we have $2 - t = 0$.

Derivative of the fourth term:

$$\frac{d}{dt}[(t-3)u(t-3)] = u(t-3)$$

since $(t-3)\delta(t-3) = 0$.

Therefore:

$$x'(t) = u(t+1) - u(t) - u(t-2) + u(t-3)$$

No impulse terms appear because the signal is continuous at -1, 0, 2, and 3, although it is not differentiable at these points.

Comments:

- This example distinguishes two concepts that are often confused: continuity and differentiability. A signal may be continuous without having a smooth derivative at the points where its slope changes.
- Dirac terms appear only when there is an actual jump in the signal's amplitude. If the signal is continuous, only the ordinary derivative changes, and no impulse terms appear.
- The unit step representation is more suitable than the piecewise form when calculating derivatives, integrals, or Laplace/Fourier transforms.

Example 3

Evaluate each of the following integrals:

(a) $\int_{-\infty}^{\infty}(t-1)^2\,\delta(t-1)\,dt$ (b) $\int_{-\infty}^{\infty}t^2\,\delta'(t-1)\,dt$

(c) $\int_{-\infty}^{\infty}\cos(2\pi t)\,\delta(t-1)dt$ (d) $\int_{-\infty}^{\infty}(e^{-2t}+\sin(10\pi t))\,\delta'(t)\,dt$

Answer: Using the properties of the delta function, we obtain:

(a) $$\int_{-\infty}^{\infty}(t-1)^2\,\delta(t-1)\,dt=(t-1)^2|_{t=1}=0$$

(b) $$\int_{-\infty}^{\infty}t^2\,\delta'(t-1)\,dt=(-1)^1\frac{d(t^2)}{dt}\bigg|_{t=1}=-2t|_{t=1}=-2$$

(c) $$\int_{-\infty}^{\infty}\cos(2\pi t)\,\delta(t-2)dt=\cos(2\pi t)\,|_{t=2}=1$$

(d) $$\int_{-\infty}^{\infty}(e^{-2t}+\sin(2\pi t))\,\delta'^{(t)}dt$$

$$=\frac{d(e^{-2t}+\sin(2\pi t))}{dt}\bigg|_{t=0}$$

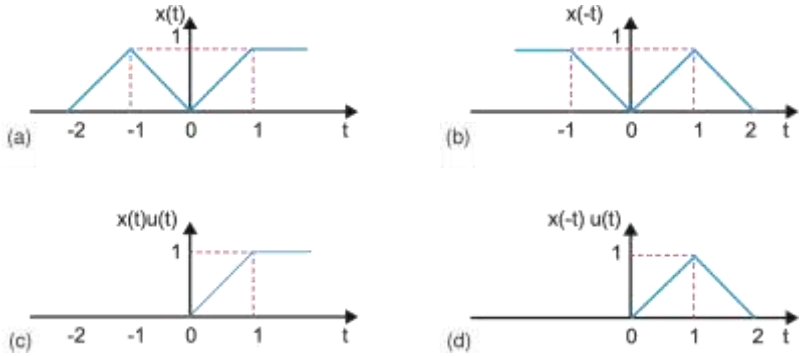
$$=2e^{-2t}-2\pi\cos(2\pi t)\,|_{t=0}$$

$$=2-2\pi$$

📖 Example 4

Sketch the signals $x_1(t) = x(t)u(t)$ and $x_2(t) = x(-t)u(t)$, where $x(t)$ is given by $x(t) = r(t+2) - 2r(t+1) + 2r(t) - r(t-1)$.

Answer:



(a) Original signal $x(t)$, (b) Time reversal $x(-t)$, (c) Signal $x_1(t)$, (d) Signal $x_2(t)$

Figure (a) shows the signal $x(t)$, and (b) shows its time reversal $x(-t)$. The signal $x_1(t) = x(t)u(t)$ is the causal part of $x(t)$, shown in (c), whereas $x_2(t) = x(-t)u(t)$ is the causal part of $x(-t)$, shown in (d). These figures show that, although $x(t)$ is noncausal, the signals $x_1(t)$and $x_2(t)$ are causal. This means that a noncausal signal $x(t)$ can be expressed as a sum of two causal signals using $x(t) = x_1(t) + x_2(-t)$. In this case, $x(t)$ equals the sum of $x_1(t)$ and the time reversal $x_2(-t)$ of $x_2(t)$.

📖 Example 5 (*)

Given the signals $x_1(t) = rect\left(\frac{t}{2}\right)$, $x_2(t) = rect\left(\frac{t}{4}\right)$ and $x_3(t) = tri(t)$. Calculate and plot the following signals:

(a) $y(t) = x_1(t) + x_2(t)$
(b) $w(t) = x_1(t) + x_3(t)$
(c) $z(t) = x_2(t) - x_1(t) - x_3(t)$

Answer: (a) We first calculate the signals $x(t), y(t)$and $z(t)$ analytically, and then use Octave.

The function $rect(\square)$ is defined by equation (1.88), and tri() by equation (1.92). Therefore, $x_1(t)$ is a rectangular pulse with unit amplitude, centered at zero, and two time units wide:

$$x_1(t) = rect\left(\frac{t}{2}\right) = \begin{cases} 1, & -1 < t < 1 \\ 0, & t < -1 \ or \ t > 1 \end{cases}$$

The signal $x_2(t)$ is a rectangular pulse with unit amplitude, centered at zero, and four time units wide:

$$x_2(t) = rect\left(\frac{t}{4}\right) = \begin{cases} 1, & -2 < t < 2 \\ 0, & t < -2 \ \ or \ \ t > 2 \end{cases}$$

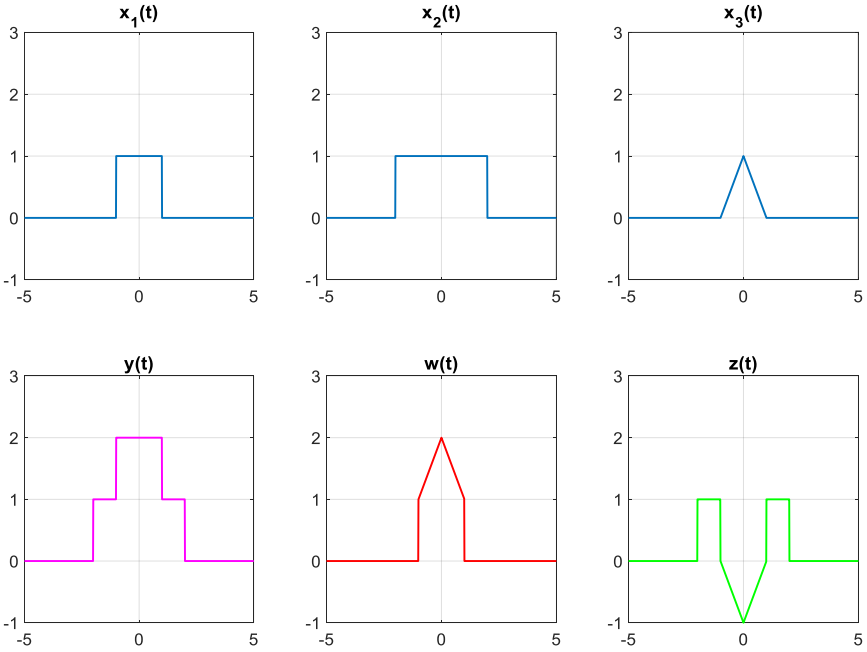
The signal $x_3(t)$ is a triangular pulse with unit amplitude, centered at zero, and two time

units wide:

$$x_3(t) = tri(t) = \begin{cases} 1 - \frac{|t|}{1}, & |t| < 1 \\ 0, & t < -1 \; or \; t > 1 \end{cases} = \begin{cases} 1 + t, & -1 < t < 0 \\ 1 - t, & 0 < t < 1 \\ 0, & t < -1 \; or \; t > 1 \end{cases}$$

To calculate the signals $y(t), w(t)$and $z(t)$, we construct the following table using the boundary points of the inequalities defining $x_1(t)$, $x_2(t)$, and $x_3(t)$. The last three rows give the signals $y(t), w(t)$ and $z(t)$.

	-∞	-2	-1	0	1	2	∞
$x_1(t)$	0	0	1	1	0	0	
$x_2(t)$	0	1	1	1	1	0	
$x_3(t)$	0	0	1+t	1-t	0	0	
$y(t) = x_1(t) + x_2(t)$	0	1	2	2	1	0	
$w(t) = x_1(t) + x_3(t)$	0	0	2+t	2-t	0	0	
$z(t) = x_2(t) - x_1(t) - x_3(t)$	0	1	-1-t	-1+t	1	0	



Signal waveforms: (a) $x_1(t)$, *(b)* $x_2(t)$, (c) $x_3(t)$, (d) $y(t)$, (e) $w(t)$, and (f) $z(t)$.

📖 Example 6

Calculate and sketch the signal:

$$x(t) = rect\left(\frac{t}{10}\right) + 3rect\left(\frac{t+1}{2}\right) + 2tri(t)$$

Answer: For the given problem, we have:

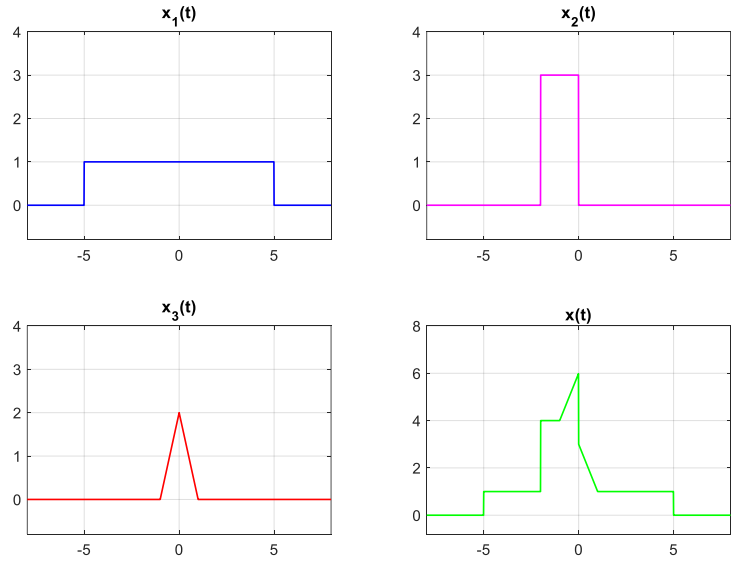
$$rect\left(\frac{t}{10}\right) = \begin{cases} 1, & -5 < t < 5 \\ 0, & t < -5 \text{ or } t > 5 \end{cases}$$

$$3\,rect\left(\frac{t+1}{2}\right) = \begin{cases} 3, & -1 < t+1 < 1 \\ 0, & t+1 < -1 \text{ or } t+1 > 1 \end{cases} = \begin{cases} 3, & -2 < t < 0 \\ 0, & t < -2 \text{ or } t > 0 \end{cases}$$

$$2\,tri(t) = \begin{cases} 2\left(1 - \frac{|t|}{1}\right), & |t| < 1 \\ 0, & t < -1 \text{ or } t > 1 \end{cases} = \begin{cases} 2(1+t), & -1 < t < 0 \\ 2(1-t), & 0 < t < 1 \\ 0, & t < -1 \text{ or } t > 1 \end{cases}$$

To calculate $x(t)$, we construct the following table using the boundary points of the inequalities. The last row gives the sum of all the rows above, namely the signal $x(t)$.

	-∞	-5	-2	-1	0	1	2	5	∞
$rect\left(\frac{t}{10}\right)$	0	1	1	1	1	1	1	0	
$3\,rect\left(\frac{t+1}{2}\right)$	0	0	3	3	0	0	0	0	
$2\,tri(t)$	0	0	0	2+2t	2-2t	0	0	0	
Total	0	0	4	6+2t	3-2t	1	1	0	



Signals: (a) $rect\left(\frac{t}{10}\right)$, *(b)* $3rect\left(\frac{t+1}{2}\right)$, (c) $2tri(t)$, (d) $x(t)$

📖 Example 7

Show that any sufficiently smooth signal $x(t)$ can be represented as:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

and interpret this relation.

Answer: Consider the known property of the time-shifted delta function:

$$\int_{-\infty}^{\infty} f(\tau)\, \delta(\tau-\tau_0)\, d\tau = f(\tau_0)$$

Setting $f(\tau) = x(\tau)$ and $\tau_0 = t$, we obtain:

$$\int_{-\infty}^{\infty} x(\tau)\, \delta(\tau-t)\, d\tau = x(t)$$

Since the delta function is even, $\delta(\tau - t) = \delta(t - \tau)$, and therefore:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

This relation states that the signal $x(t)$ can be viewed as a continuous sum of infinitely many shifted impulse functions $\delta(t - \tau)$, each weighted by the coefficient $x(\tau)$.

Note: Examples marked with an asterisk (*) also appear in the lecture slides.