

UNIVERSITY OF THE PELOPONNESE – SCHOOL OF ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

Signals and Systems

Lecture 2: Elementary Continuous-Time Signals

Michalis Paraskevas
Professor

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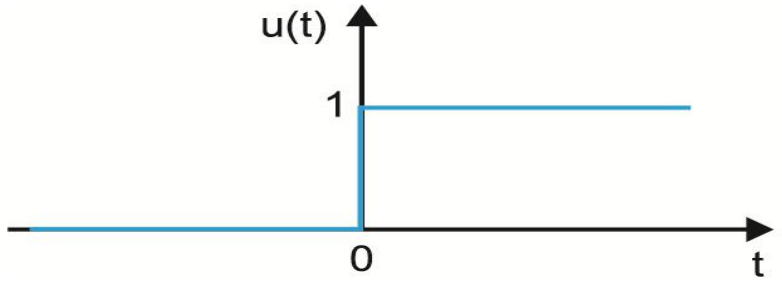
1. Unit Step Function

Unit Step Function (1/2)

The continuous-time **unit step function**, also called the Heaviside function, is defined by:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

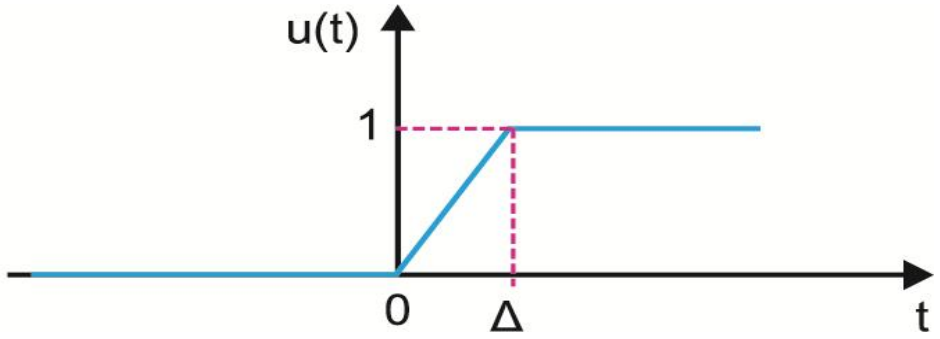
The function $u(t)$ is discontinuous, since it is undefined at $t = 0$.



Unit Step Function (2/2)

Alternatively, the unit step function can be defined as the limit of a sequence of functions $u_n(t)$, i.e. as $u(t) = \lim_{\Delta \to 0} u_\Delta(t)$, where:

$$u_\Delta(t) = \begin{cases} 0, & t < 0 \\ t/\Delta, & 0 \le t \le \Delta \\ 1, & t \ge \Delta \end{cases}$$



Continuous approximation of the unit step function

Properties of the Unit Step Function

The most important properties of the unit step function are:

(a) Scaling:

$$\mathrm{A}\,u(t) = \begin{cases} 0, & t < 0 \\ \mathrm{A}, & t > 0 \end{cases}$$

(b) Time shifting by a time interval t_0:

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

Properties of the Unit Step Function

(c) Change of variable:

$$u(-t) = \begin{cases} 0, & t > 0 \\ 1, & t < 0 \end{cases}$$

(d) Subtracting $u(t)$ and $u(-t)$, we obtain:

$$u(t) - u(-t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases} = sgn(t)$$

This is the **sign function**: it returns $+1$, 0 and -1 when t is positive, zero or negative, respectively.

Properties of the Unit Step Function

- Subtracting two step functions, one shifted in time relative to the other, produces a **rectangular pulse.** The first step determines the start of the pulse and the second determines its end.
- The pulse duration is the difference between the time shifts of the two functions. Specifically, subtracting two shifted step functions, $u(t-a)-u(t-b),\ \ a<b$, produces a rectangular pulse of amplitude 1.
  It starts at $t=a$ and ends at $t=b$. Its duration is $b-a$.

Notes on the Unit Step Function

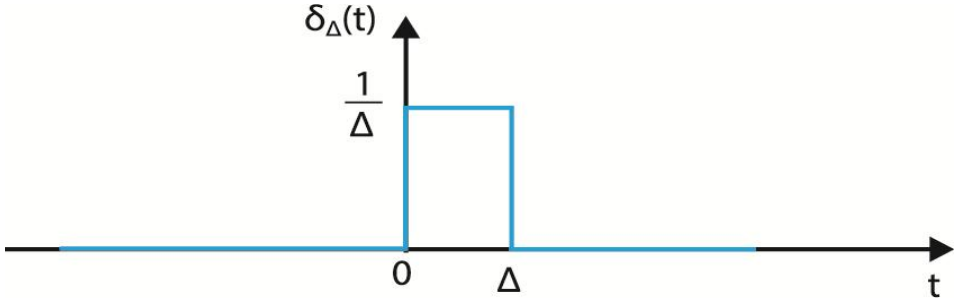
- Signals with discontinuities can be represented as the sum of a continuous signal and unit step signals at the points of discontinuity. This makes it easier to calculate their derivatives.
- The unit step function is useful in signal processing because multiplying it by any signal removes (sets to zero) the part at negative times. It leaves the part at positive times unchanged, since it multiplies it by 1.
- The result is a signal defined only for $t > 0$, called a **causal signal** (as discussed in a later section). Causal signals are widely used in practical applications.

2. Impulse Function or Delta Function (Dirac)

Impulse Function (1/3)

The derivative of the function $u_{\Delta}(t)$ is:

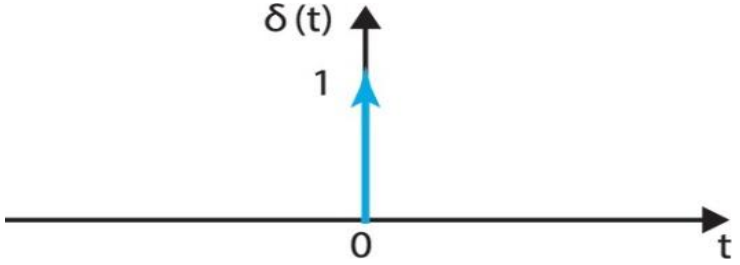
$$\delta_{\Delta}(t) = \frac{du_{\Delta}(t)}{dt} = \begin{cases} 0, & t < 0 \\ 1/\Delta, & 0 \le t \le \Delta \\ 0, & t \ge \Delta \end{cases}$$



As $\Delta \to 0$, the pulse duration decreases and its height increases, but its area remains constant and equal to one.

The limit of $\delta_{\Delta}(t)$ as $\Delta \to 0$ defines the **delta function**, also called the Dirac function or **impulse function** $\delta(t)$, by:

$$\delta(t) = \lim_{\Delta \to 0} \delta_{\Delta}(t)$$



Impulse Function (2/3)

- The delta function $\delta(t)$ can be viewed as the limit of a family of pulses $p_\varepsilon(t)$ with unit area:

$$\int_{-\infty}^{\infty} p_\varepsilon(t)\, dt = 1$$

  as the pulse duration tends to zero. The specific pulse shape is not critical, provided its area remains equal to one. As the pulse narrows, its amplitude increases so that the total area stays constant.

- This definition is not mathematically rigorous, because $\delta(t)$ is not a classical function that can be defined by finite values. Instead, it is interpreted as a **generalized function** (distribution).

- Relations involving $\delta(t)$ are interpreted through their action on suitable smooth test functions $\varphi(t)$, or within integrals, rather than as pointwise equalities. Thus $\delta(t)$ can be regarded as an operator acting on test functions $\varphi(t)$ through the property:

$$\int_{-\infty}^{\infty} \delta(t)\varphi(t)\, dt = \varphi(0)$$

Impulse Function (3/3)

- This property shows that $\delta(t)$ selects the value of the function at the point where the impulse is located. More generally, the shifted impulse $\delta(t - t_0)$ selects the function value at $t = t_0$.
- The previous definition can be generalized to describe the time-shifted function $\delta(t - t_0)$:

$$\int_{-\infty}^{+\infty} \delta(t - t_0)\, \varphi(t)\, dt = \int_{-\infty}^{+\infty} \delta(t - t_0)\, \varphi(t_0)\, dt = \varphi(t_0) \int_{-\infty}^{+\infty} \delta(t - t_0)\, dt = \varphi(t_0)$$

This relation describes the mathematical model of **sampling**, the first of the three stages in converting a signal from analog to digital form.

For $\varphi(t) = 1$, we obtain:

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = \int_{0^-}^{0^+} \delta(t)\, dt = 1$$

- The function $\delta(t)$ is very important in signal processing, especially for describing impulsive excitations and analyzing linear systems.

Properties of the Impulse Function (1/3)

- $\delta(t) = \delta(-t)$
- $\delta(t) = \frac{du(t)}{dt}$
- $u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$
- $\varphi(t)\, \delta(t - t_0) = \varphi(t_0)\, \delta(t - t_0)$
- $\delta(\alpha t) = \frac{1}{|a|}\, \delta(t)$
- $\int_{-\infty}^{+\infty} \delta(at)\, \varphi(t)\, dt = \frac{1}{|a|} \varphi(0)$
- $\int_{-\infty}^{+\infty} \delta'(t)\, \varphi(t)\, dt = -\varphi'(0)$
- $\int_{-\infty}^{+\infty} \delta'(t)\, dt = 0$

The delta function can be produced not only from the derivative of a rectangular pulse, but also from the derivative of a pulse of any shape, provided the pulse area remains constant and equal to one as its duration tends to zero.

Properties of the Impulse Function (2/3)

- $\int_{-\infty}^{+\infty} \delta(t)\, e^{-j\Omega t}\, dt = 1$ (Fourier transform)
- $\varphi(t)\, \delta'(t - t_0) = \varphi(t_0)\, \delta'(t - t_0) - \varphi'(t_0)\, \delta(t - t_0)$
- $\int_{-\infty}^{+\infty} A\, \delta(t - t_0)\, dt = A$
- $\int_{t_1}^{t_2} \delta(t - t_0)\, \varphi(t)\, dt = \begin{cases} \varphi(t_0), & t_1 < t_0 < t_2 \\ 0, & t_0 < t_1, \quad t_0 > t_2 \end{cases}$
- $\int_{t_1}^{t_2} \delta(\tau - t)\delta(t - t_0)\, dt = \delta(t - t_0),\ t_1 < t_0 < t_2$

Properties of the Impulse Function (3/3)

The derivative of the delta function is also a generalized function (distribution), with the following properties:

- A $\delta'(t - t_0) = 0, \quad t \neq t_0$
- $t\ \delta'(t) = -\delta(t)$
- $\delta'(-t) = -\delta'(t)$
- $\delta'(t) * x(t) = x'(t)$ (* denotes convolution, discussed in a later section)
- $x(t) * \delta'(t - t_0) = x'(t - t_0)$
- $\int_{-\infty}^{+\infty} A\ \delta'(t - t_0)\, dt = 0$
- $\int_{-\infty}^{+\infty} x(t) A\ \delta'(t - t_0)\, dt = -A\ x'(t_0)$

Notes on the Impulse Function (1/2)

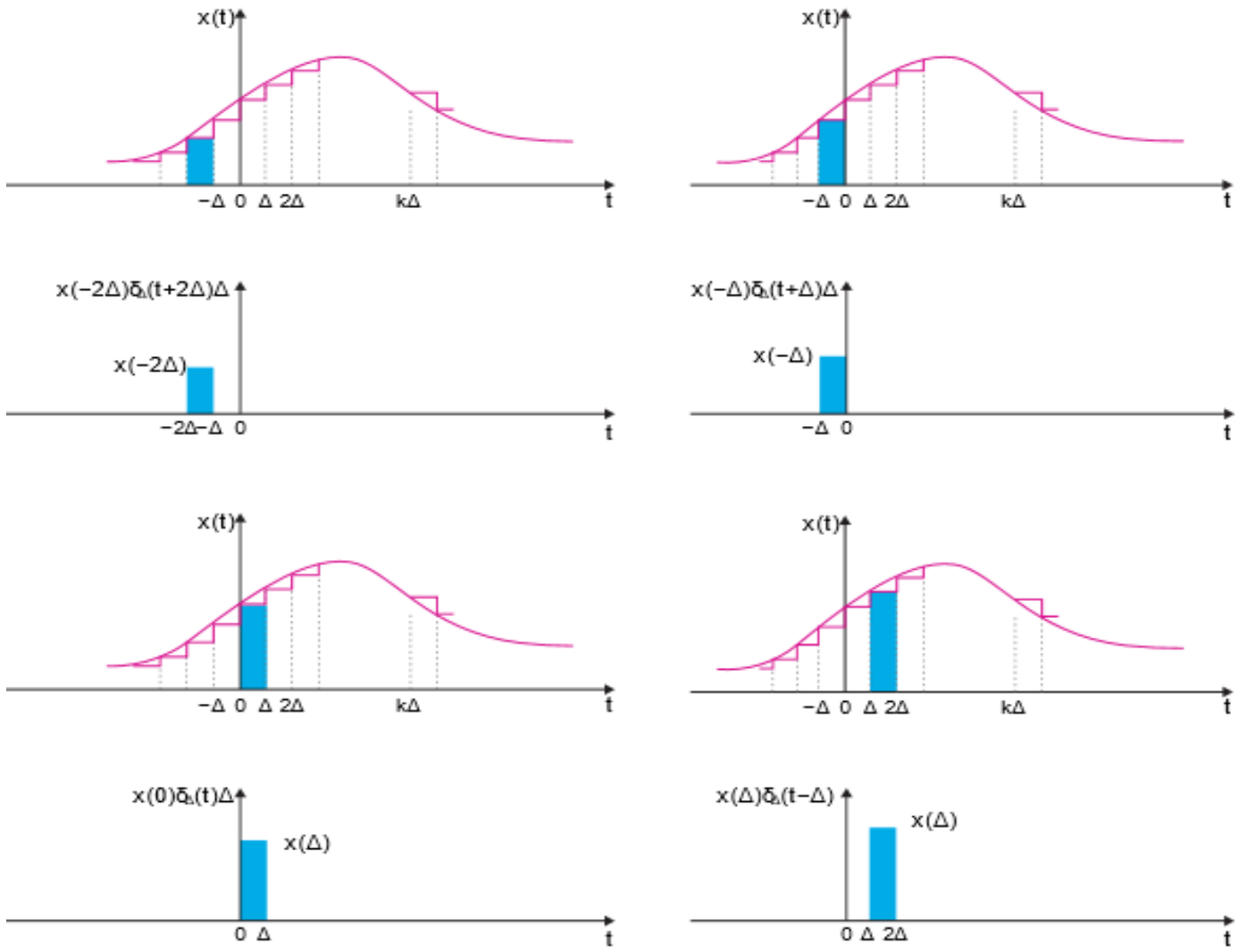
- Multiplying a function $\varphi(t)$, continuous at $t = 0$, by the delta function gives the delta function weighted by the function value $\varphi(0)$.
- The property $\varphi(t)\,\delta(t - t_0) = \varphi(t_0)\,\delta(t - t_0)$ shows that multiplying a delta function shifted by t_0 by $\varphi(t)$ gives a delta function weighted by $\varphi(t_0)$. Thus, multiplying a continuous function $\varphi(t)$ by the delta function and its time-shifted versions gives samples of $\varphi(t)$.
- Although the duration of the impulse $\delta(t)$ tends to zero, its area always remains equal to one. This is critical: it distinguishes the delta function from any other pulse, rectangular or otherwise, however short its duration.

Notes on the Impulse Function (2/2)

- A pulse with very short duration and constant amplitude has a small area, so its effect when applied to a system input may be negligible. The delta function, by contrast, always has unit area, so its effect on a system is always nonnegligible.
- The delta function contains infinitely many frequency components, all with unit amplitude. This helps us study continuous-time systems in the frequency domain.
- The delta function is a key mathematical tool for studying continuous-time signals and systems. Its greater mathematical complexity is offset by the advantages it offers.

3. Signal Representation Using the Delta Function

Signal Representation Using the Delta Function



Signal Representation Using the Delta Function

Every continuous-time signal $x(t)$ can be written as a linear combination of time-shifted functions $\delta_\Delta(t)$.

Proof: We know that $\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$, where $\delta_\Delta(t)$ is the derivative of the rectangular pulse $u_\Delta(t)$. Thus:

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt} = \begin{cases} 1/\Delta, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

Since $y(t) = x(t)\ \delta_\Delta(t)$, the signal $y(t)$ is zero everywhere except within the time interval $0 \le t < \Delta$. If the pulse width Δ is very small, the value of $y(t)$ over $0 \le t < \Delta$ is approximately constant and equal to $y(0)$.. We therefore write:

$$x(t)\ \delta_\Delta(t) \approx x(0)\ \delta_\Delta(t)$$

In the limit as $\Delta \to 0$, this gives the familiar property $x(t)\delta(t) = x(0)\delta(t)$.

In general form:

$$x(t)\ \delta(t - t_0) = x(t_0)\ \delta(t - t_0)$$

Signal Representation Using the Delta Function

Multiplying a signal $x(t)$ by a time-shifted function $\delta_{\Delta}(t)$ therefore isolates a segment of width Δ. We regard the full signal $x(t)$ as the combination of infinitely many such elementary signals, written mathematically as:

$$\tilde{x}(t) = \sum_{n=-\infty}^{+\infty} x(n\Delta)\, \delta_{\Delta}(t - n\Delta)\Delta$$

As the duration $\Delta \to 0$, $\tilde{x}(t)$ approaches $x(t)$ according to:

$$x(t) = \lim_{\Delta \to 0} \tilde{x}(t) = \lim_{\Delta \to 0} \sum_{n=-\infty}^{+\infty} x(n\Delta)\, \delta_{\Delta}(t - n\Delta)\Delta$$

Setting $\tau = n\Delta$,, the sum becomes an integral:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\, d\tau$$

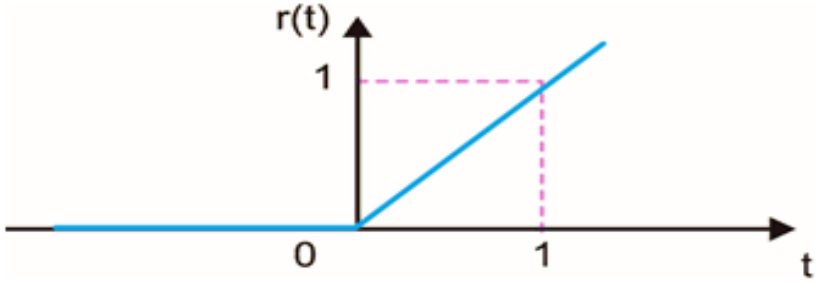
This relation is particularly important in the study of continuous-time signals and systems: it represents any continuous-time signal $x(t)$ using the delta function $\delta(t - \tau)$. The operation between $x(t)$ and $\delta(t)$ is called **convolution**.

4. Unit Ramp Function

Unit Ramp Function

The continuous-time unit ramp function is defined by:

$$r(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases} \quad \text{or} \quad r(t) = t\,u(t)$$



The following relations hold:

$$\frac{dr(t)}{dt} = \frac{d}{dt}[t\,u(t)] = u(t)$$

$$r(t) = \int_{-\infty}^{\infty} u(\tau)d(\tau)$$

$$\delta(t) = \frac{d''r(t)}{dt''}$$

5. Exponential Signals

Exponential Signals

General definition of an exponential signal:

$$x(t) = A\beta^{st}$$

where:

- A is the signal **amplitude.**
- β is a real number, most commonly:

$$\beta = e = 2.171828$$

- s can take real (positive or negative) or complex values and largely determines the signal's behavior.

We next analyze the signal for different values of s.

Real Exponential Signals (1/2)

(a) Negative s: exponential decay

If $s = -1/T$, the signal is written as $x(t) = Ae^{-t/T}$.

T is called the **time constant** and describes how quickly the signal decreases.

- For $t = T$, we obtain: $x(T) = A\,e^{-1} = 0.368\,A$
- For $t = 5T$, we obtain: $x(T) = A\,e^{-5} = 0.0067\,A$

After five (5) time constants, the exponential signal has a negligible value relative to its initial amplitude.

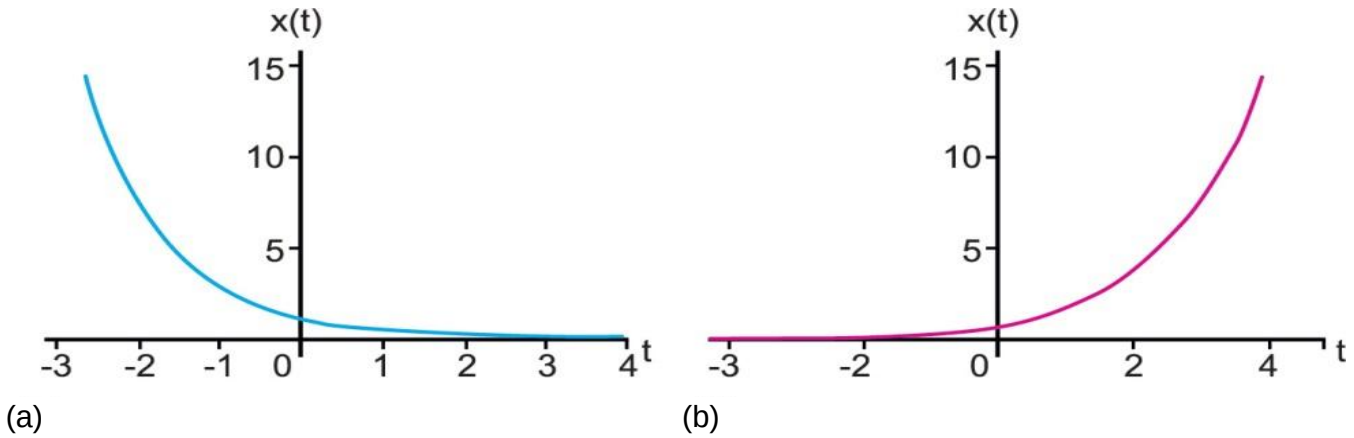
Examples of exponential decay include capacitor discharge, radioactive decay, decreasing drug concentration in the blood, declining demand for a product, etc.

Real Exponential Signals (2/2)

(b) Positive s: exponential growth

If $s = 1/T$, the signal is $x(t) = Ae^{t/T}$

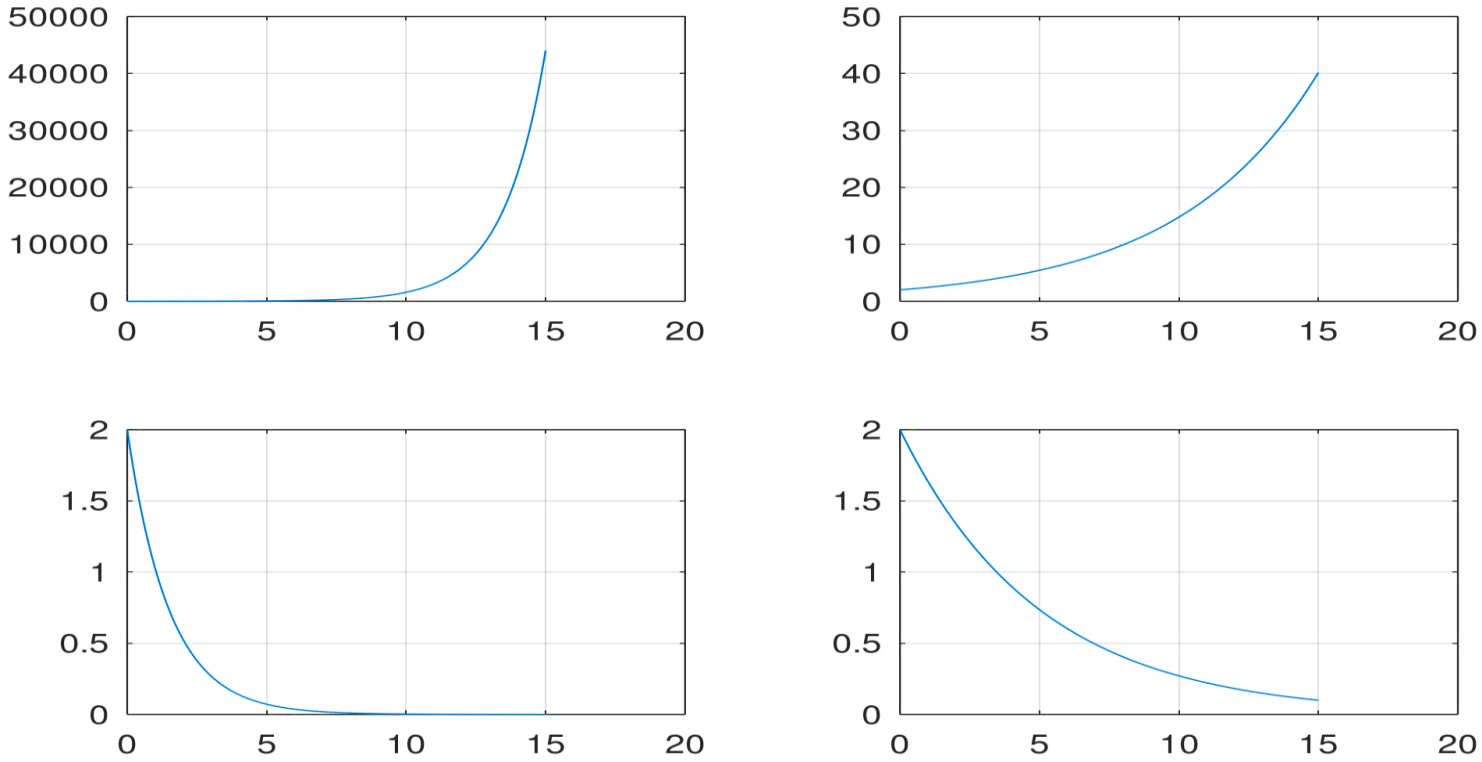
The conclusions are analogous to those in the previous case.



Exponential signal for $s = -1/T$ Exponential signal for $s = 1/T$

Examples of exponential growth include a nuclear chain reaction, the spread of rumors, increasing computing power, viral spread in a host without immunity, human population growth, etc.

Example of an Exponential Signal



The signal $x(t) = 2\,e^{-t/T}$ Top: positive exponent (T = −1.5 and T = −5)

Bottom: negative exponent (T = 1.5 and T = 5)

Euler's Formulas

Euler's formula is frequently used in the analysis of complex functions:

$$e^{j\theta} = \cos(\theta) + jsin(\theta)$$

Setting $\varphi = -\varphi$ in the relation above, we obtain:

$$e^{-j\theta} = \cos(\theta) - jsin(\theta)$$

Adding and subtracting these equations gives the inverse Euler relations, which express cosine and sine in equivalent complex forms:

$$cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$$

$$j\ sin\theta = \frac{1}{2}\left[e^{j\theta} - e^{-j\theta}\right]$$

Complex Exponential Signals

If $s = j\Omega_0$, the signal is written as $x(t) = A\, e^{j\Omega_0 t}$, where $j = \sqrt{-1}$ is the imaginary unit.

Complex exponential signals have no direct physical realization, but are used to study signals because they simplify the algebra.

Using Euler's formulas:

$$A\, cos(\Omega_0 t + \varphi) = A\, Re\{e^{j(\Omega_0 t + \varphi)}\}$$

$$A\, sin(\Omega_0 t + \varphi) = A\, Im\{e^{j(\Omega_0 t + \varphi)}\}$$

where $Re\{.\}$ and $Im\{.\}$ are the real and imaginary parts of a complex number, respectively.

A cosine signal can therefore be regarded as the real part of a complex exponential signal, and a sine signal as its imaginary part.

Complex Exponential Signals

A complex exponential signal can also be represented as:

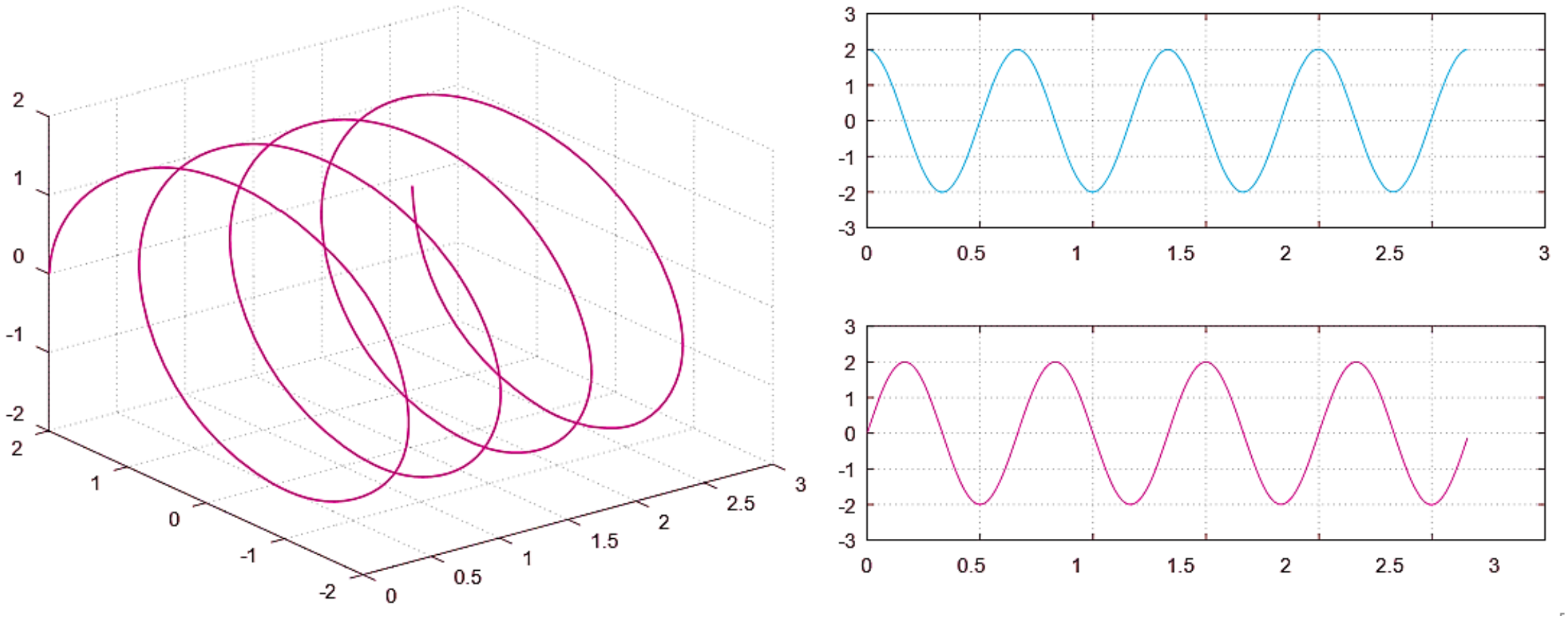
$$x(t) = A\, e^{j(\Omega_0 t+\varphi)} = Xe^{j\Omega_0 t}$$

where $X = A\, e^{j\varphi}$ is called the **complex amplitude** of the signal.

Since the signal can also be written as:

$$x(t) = A\, e^{j(\Omega_0 t+\varphi)} = A\, cos\Omega_0 t + j\, A\, sin\, \Omega_0 t$$

- We conclude that the complex exponential signal $x(t)$ is periodic, with **fundamental period** $T_0 = 2\pi/\Omega_0(sec)$.
- The signal **frequency** is given by $f_0 = \Omega_0/2\pi\ (Hertz)$, where $\Omega_0\ (rad/\text{sec})$ is the angular frequency.
- The quantity φ is called the **phase shift**, or simply the **phase.** It determines the time shift of the signal within one period.

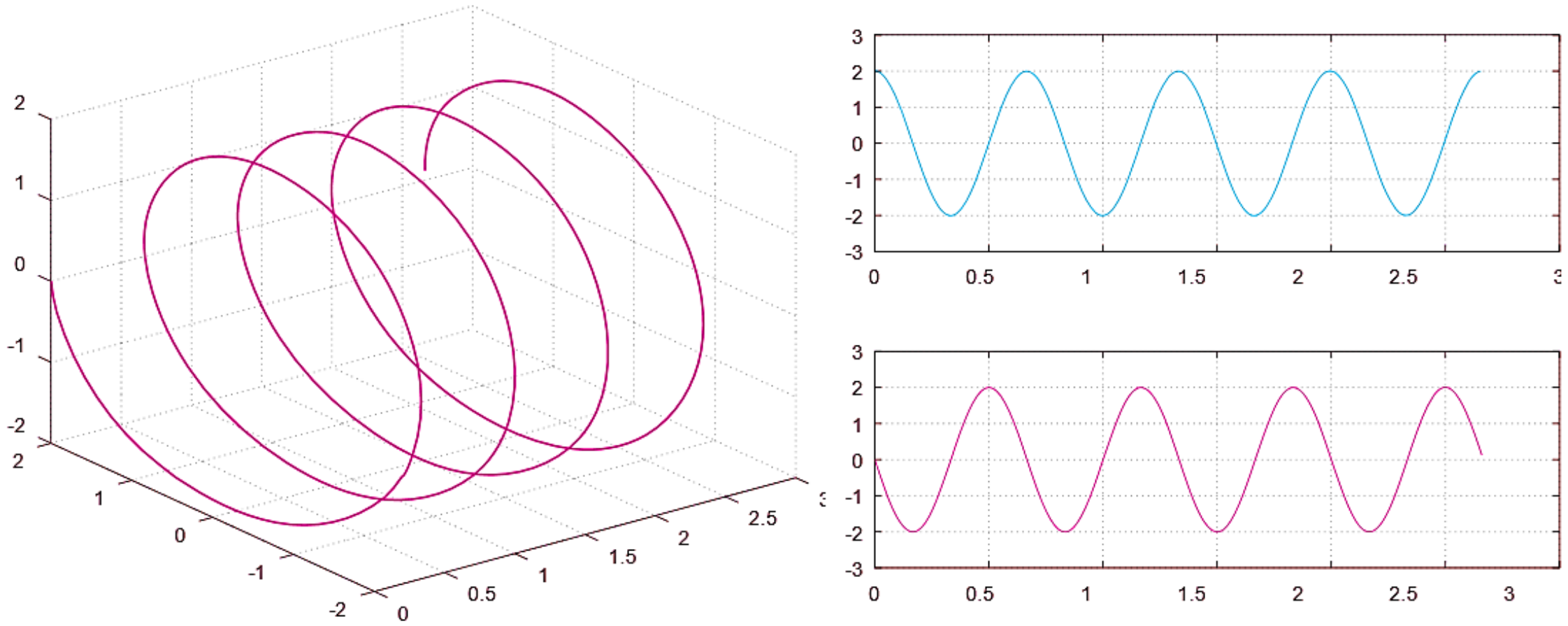


(a) Complex exponential signal $x(t) = 2\, e^{j3\pi t}$ over 4 periods
(b) Real part (cosine), (c) imaginary part (sine)

Observations

The diagram above shows that:

- The three-dimensional graph (a) represents the complex exponential signal in three-dimensional space. The horizontal axis corresponds to time.
- The 4 turns of the helix in graph (a) correspond to a time interval of four periods.
- The direction of rotation in graph (a) is clockwise.
- Projecting graph (a) onto individual axes produces graphs (b) and (c).
- Graph (b) shows the real component (cosine function).
- Graph (c) shows the imaginary component (sine function).
- Both components are periodic with the same period. Their sum, the complex exponential signal, is therefore also periodic with that period.



(d) Complex exponential signal $x(t) = 2\,e^{-j3\pi t}$ over 4 periods
(e) Real part (cosine), (f) imaginary part (sine)

Observations

The diagram above shows that:

- The three-dimensional graph (d) is identical to (a), except for the direction of rotation. Here the signal rotates counterclockwise. This change is due to the negative sign in the complex exponent.
- Graph (e) shows the real component (cosine function), which is unchanged from graph (b).
- Graph (f) shows the imaginary component (sine function), with a phase difference of π (180°) relative to graph (c). This change is also due to the negative sign in the complex exponent.

6. Sinusoidal Signals

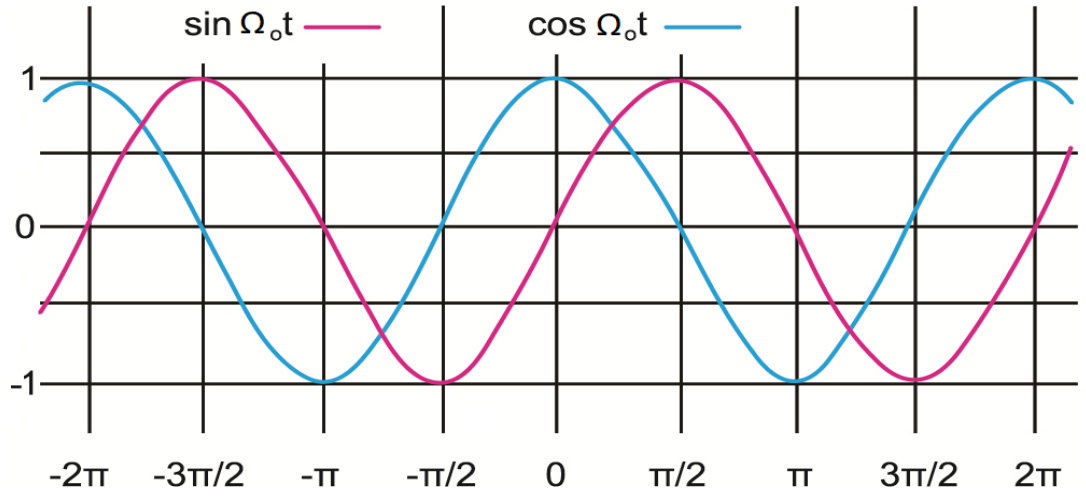
Sinusoidal Signals

The general expression for a continuous-time sinusoidal signal is:

$$x(t) = A\cos(\Omega_0 t + \theta) = A\sin(\Omega_0 t + \theta + \pi/2)$$

The sine and cosine functions have a constant phase difference of $\pi/2$ (90°).

The sinusoidal signal is periodic, with **fundamental period** $T_0 = 2\pi/\Omega_0$ (sec) and **frequency** $f_0 = \Omega_0/2\pi = 1/T_0$ (Hz)..
The quantity φ is the **phase** angle, or simply the **phase**.



Sinusoidal signals are a particularly useful class of periodic signals because they describe many real-world signals, such as sound waves and electric currents.

Furthermore, nonsinusoidal signals can be expressed as sums of sinusoidal signals using Fourier analysis.

Modulation

A sinusoidal function can be expressed as:

$$\boldsymbol{x(t)} \;=\; \boldsymbol{A(t)}\,[\,\boldsymbol{cos}(\boldsymbol{\Omega(t)}\,\boldsymbol{t} \;+\; \boldsymbol{\theta(t)})\,]$$

where A(t), Ω(t) and θ(t) are determined by an information signal m(t) carrying useful information (a message).

The following types of modulation are used to transmit the message through a communication channel:

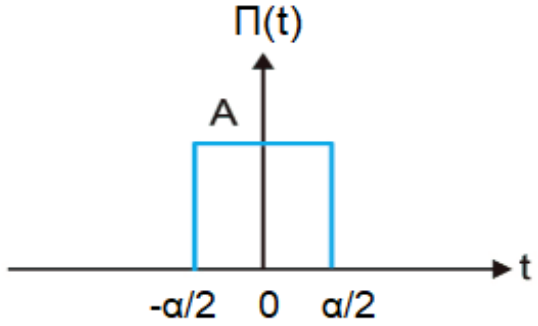
- **Amplitude Modulation** (AM): the amplitude A(t) varies with the message m(t), while frequency and phase remain unchanged.
- **Frequency Modulation** (FM): the frequency Ω(t) varies with the message m(t), while amplitude and phase remain unchanged.
- **Phase Modulation** (PM): the phase θ(t) varies with the message m(t), while amplitude and frequency remain unchanged.

7. Rectangular Pulse

Rectangular Pulse

Rectangular pulse of duration α and amplitude A:

$$\Pi_\alpha(t-t_0) = rect\left(\frac{t-t_0}{a}\right) = \begin{cases} A, & |t-t_0| < \alpha/2 \\ 0, & |t-t_0| > \alpha/2 \end{cases}$$



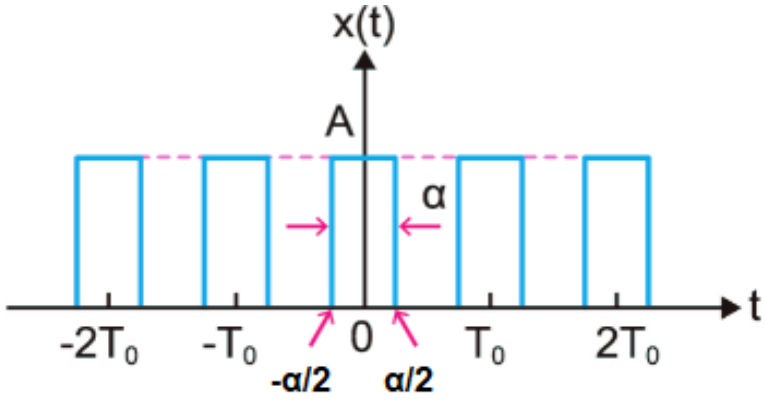
- A rectangular pulse can be produced by subtracting two unit step functions:

$$\Pi_\alpha(t-t_0) = rect\left(\frac{t-t_0}{a}\right) = A\left[u\left(t-\left(t_0-\frac{\alpha}{2}\right)\right) - u\left(t-\left(t_0+\frac{\alpha}{2}\right)\right)\right]$$

- The rectangular pulse has **even symmetry.**
- Multiplying a signal by a rectangular pulse isolates a specific segment of the signal. This allows us to extract a segment for further processing.

Rectangular Pulse Train

Repeating pulses with period T_0 produces a periodic pulse train with period T_0 and pulse duration α..



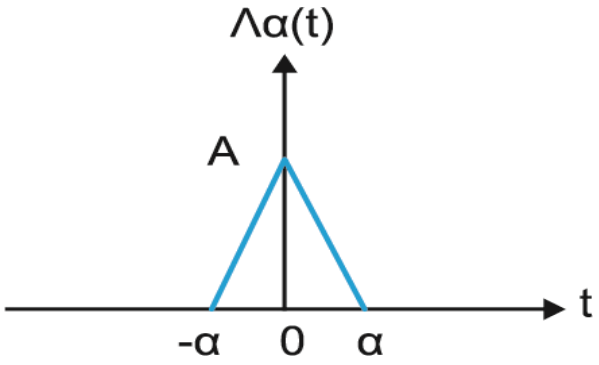
Rectangular pulse trains are of particular interest in digital communications: they describe transmitted sequences of binary digits that represent the samples of a digital signal.

8. Triangular Pulse

Triangular Pulse

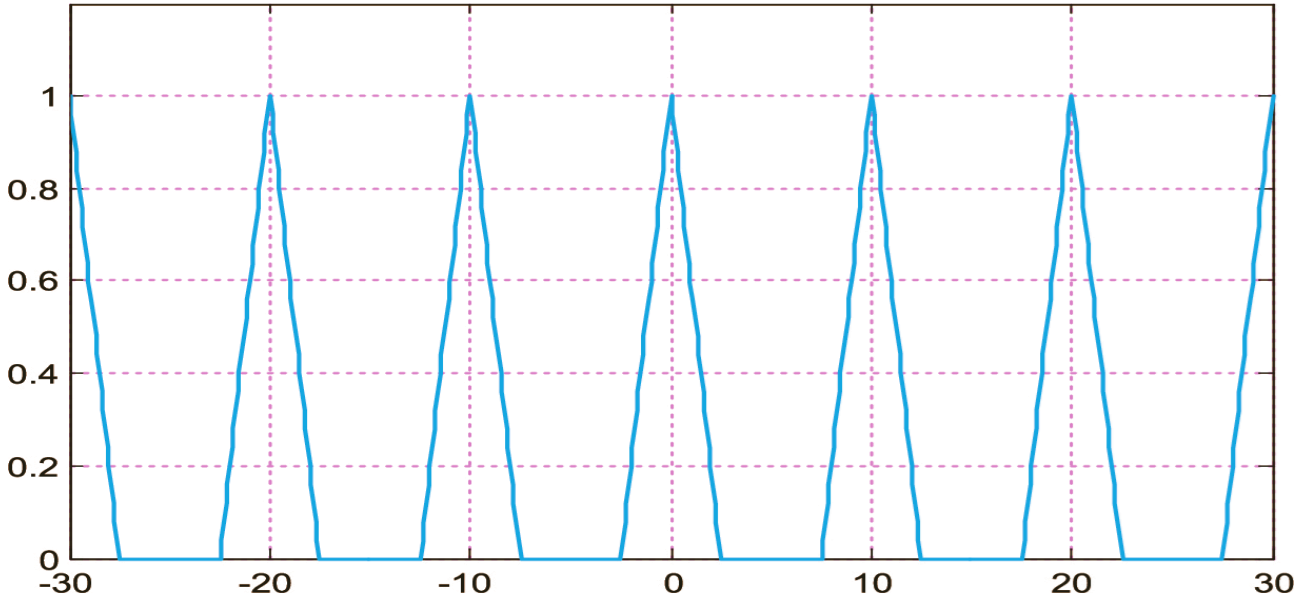
Triangular pulse of duration 2α and amplitude A:

$$tri\left(\frac{t-t_0}{a}\right) = \Lambda_\alpha(t-t_0) = \begin{cases} A\left(1-\dfrac{|t-t_0|}{a}\right), & |t-t_0| < a \\ 0, & |t-t_0| > \alpha \end{cases}$$



- The triangular pulse has **even symmetry.**
- Note the duration (width) of the triangular pulse $\Lambda_\alpha(t)$: it is 2α, rather than α as for the rectangular pulse $\Pi_\alpha(t)$.
- Multiplying a signal by a triangular pulse isolates a specific segment while giving greater weight to signal values near the center of the pulse.

Triangular Pulse Train



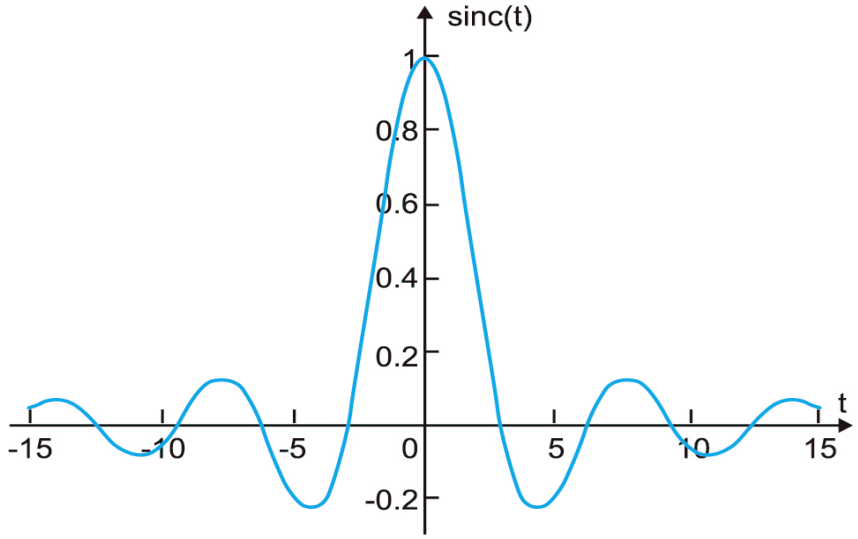
Repeating a triangular pulse produces a periodic triangular pulse train with period T_0 and individual pulse duration 2α.

9. Sampling Function

Sampling Function

Sampling function:

$$sinc(t) = \begin{cases} \sin(t)/t & t \neq 0 \\ 1 & t = 0 \end{cases}$$



Observations:

- The function $sinc(t)$ has **even symmetry.**
- It crosses zero periodically at $t = \pm n\pi, \quad n = 1,2 \ldots$
- The heights of the secondary lobes decrease asymptotically to zero.
- Its maxima and minima occur approximately halfway between successive zero crossings, i.e. approximately at:

$$t = \pm\left(n + \frac{1}{2}\right)\pi, \qquad \text{where } |sint| = 1$$

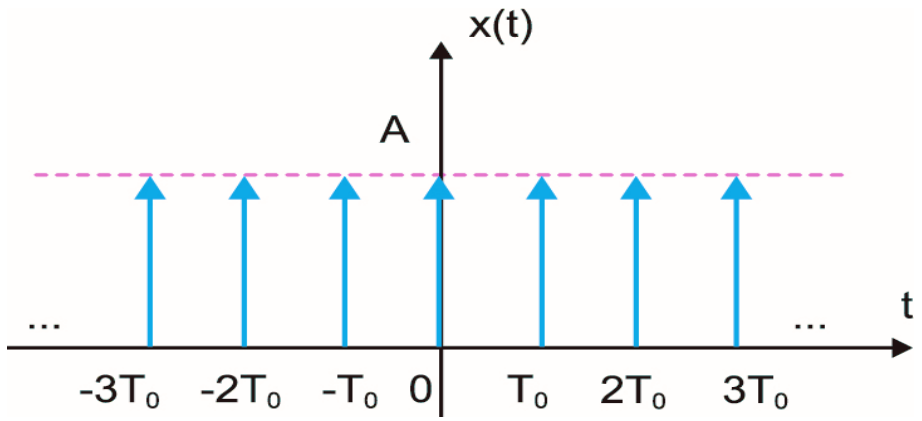
10. Impulse Train (Comb Signal)

Impulse Train (Comb)

Repeating the function $\delta(t)$ with period T_0 produces an impulse train (the *comb* function):

$$comb(t) = A \sum_{k=-\infty}^{+\infty} \delta(t - kT_0)$$

The series on the right-hand side can be shown to converge, so the function $comb(t)$ can be defined.



Sampling a Continuous-Time Signal (1/3)

The function $comb(t)$ samples a continuous-time signal $x(t)$ with sampling period T_s, equal to the period of $comb(t)$.

The continuous-valued, discrete-time signal $x_s(t)$ obtained by sampling $x(t)$ with sampling period T_s is:

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s)$$

The impulses $\delta(t - nT_s)$ select the values of $x(t)$ at the time instants nT_s όπου $n = -\infty, \dots, -1,0,1, \dots, \infty$, thereby sampling the signal $x(t)$.

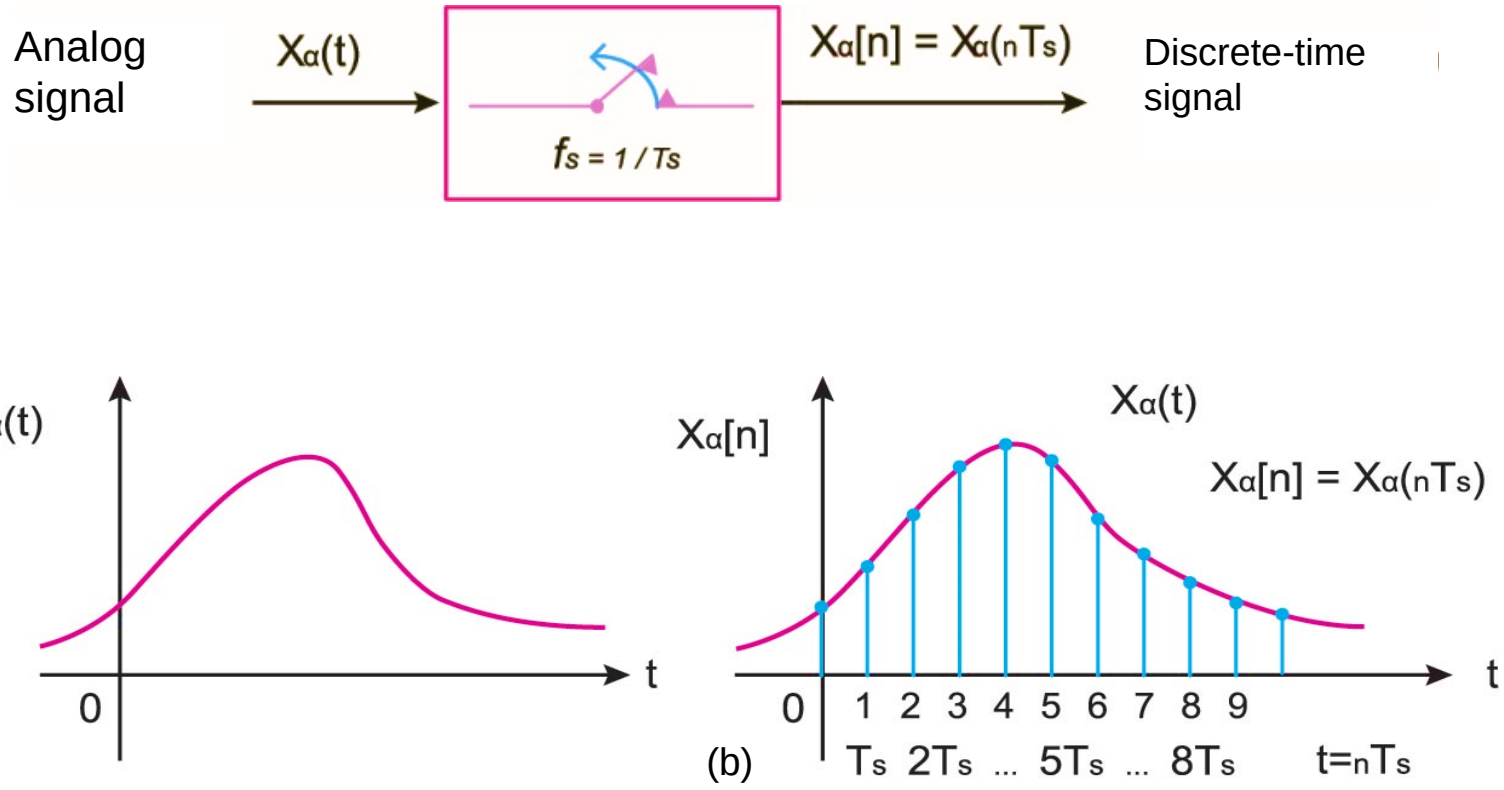
Sampling a Continuous-Time Signal (2/3)

- The resulting sampled signal $x_s(t)$ is a sequence of impulses with amplitudes equal to the samples $x(nT_s)$..
- The spectrum of the sampled signal can be shown to be:

$$X_s(f) = \frac{1}{T_S}\sum_{k=-\infty}^{\infty} X(f - kf_S)\,, \qquad f_S = \frac{1}{T_S}$$

  Thus, sampling in time produces periodic repetitions of the spectrum in frequency, with period f_S.
- In a later lecture we will see the converse: a periodic signal in time always has a discrete spectrum in frequency.
- The periodic spectral repetitions produced by ideal sampling are the basic mechanism that allows exact reconstruction of the original signal from its samples, under suitable conditions.
- If the sampling frequency is not sufficiently high, the spectral repetitions overlap, producing aliased frequencies that were absent from the original signal.

Sampling a Continuous-Time Signal (3/3)



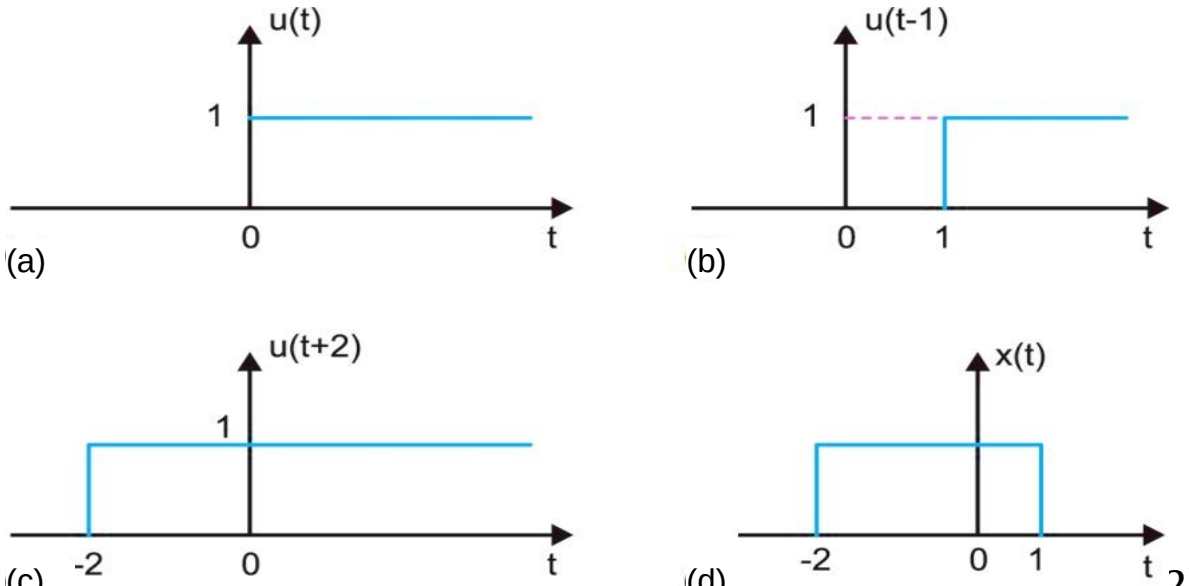
Analog signal sampling process

Exercises

Exercise 1

Plot the signal $x(t) = u(t+2) - u(t-1)$

Answer: The figures below show step by step how the signal $x(t)$ is constructed (graph d). The rectangular pulse starts at $t = -2$, where the first step function begins, and ends at $t = 1$, where the second step function begins.



Signals: (a) $u(t)$, (b) $u(t-1)$, (c) $u(t+2)$ and (d) $x(t) = u(t+2) - u(t-1)$

Exercise 2

Given the discontinuous signal $x(t) = cos(2\pi t)[u(t) - u(t-1)]$: (a) represent it as a sum of continuous signals and unit step functions; (b) find its derivative.

Answer: (a) The signal $x(t)$ is one period of $\cos(2\pi t)$ from 0 to 1, and zero elsewhere. It is discontinuous at $t = 0$ and $t = 1$. Removing the pulse $u(t) - u(t-1)$ from $x(t)$ gives a continuous signal. To compensate, we must add a unit pulse between $t = 0$ and $t = 1$. Thus:

$$x(t) = (cos(2\pi t) - 1)[u(t) - u(t-1)] + [u(t) - u(t-1)] = x_a(t) + x_b(t) \qquad (1)$$

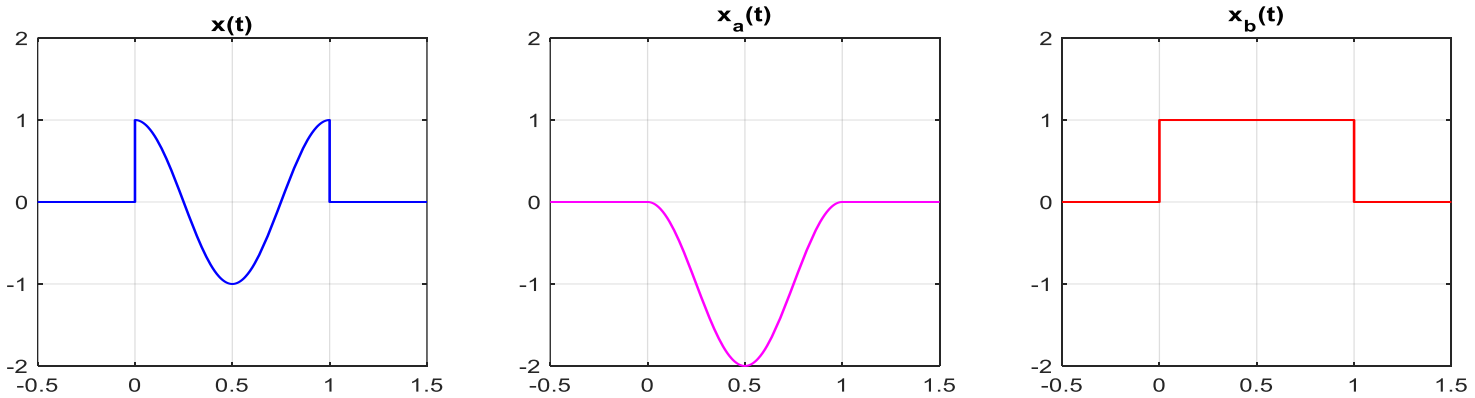
where:

$x_a(t) = (cos(2\pi t) - 1)[u(t) - u(t-1)]$ and $x_b(t) = u(t) - u(t-1)$

The term $x_a(t)$ is continuous, while the term $x_b(t)$ is discontinuous.

The waveforms are plotted in the next figure.

Exercise 2



Signal waveforms: (a) $x_a(t)$, (b) $x_b(t)$, (c) $x(t) = x_a(t) + x_b(t)$

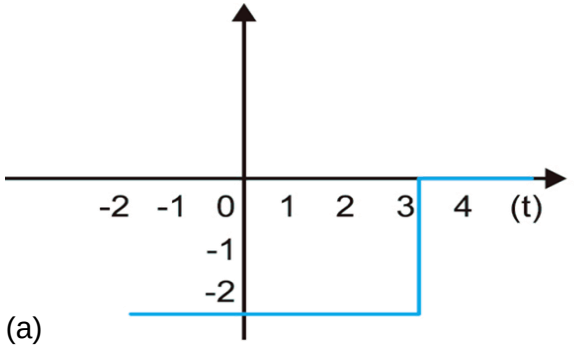
(b) Since $du(t)/dt = \delta(t)$, , the derivative of the given signal (1) is:

$$\frac{dx(t)}{dt} = -2\pi \sin(2\pi t)[u(t) - u(t-1)] + (cos(2\pi t) - 1)[\delta(t) - \delta(t-1)] + \delta(t) - \delta(t-1)$$

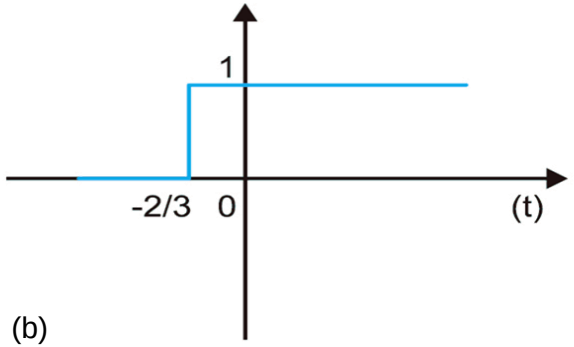
$$= -2\pi \sin(2\pi t)[u(t) - u(t-1)] + \delta(t) - \delta(t-1)$$

Exercise 3

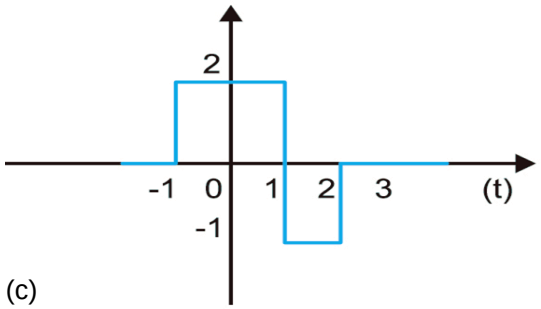
Plot the signals:



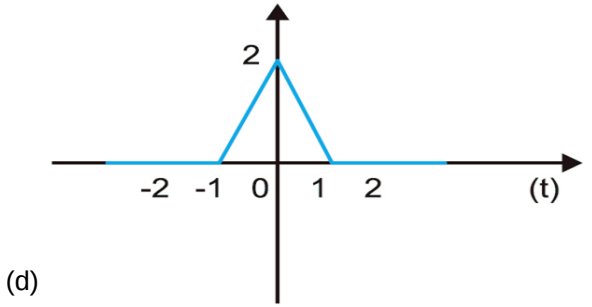
(a) $-2u(3-t)$



(b) $u(3t+2)$



(c) $2u(t+1) - 3u(t-1) + u(t-2)$



(d) $2r(t+1) - 4r(t) + 2r(t-1)$

Exercise 4

Evaluate each of the following integrals:

$$1)\ \int_0^{+\infty}(t^2+3t-1)\,\delta(t-1)\,dt \qquad 2)\ \int_0^{+\infty}(t^2+3t-1)\,\delta'(t-1)\,dt$$

$$3)\ \int_0^{+\infty}(t^2+3t-1)\,\delta''(t-1)\,dt \qquad 4)\ \int_0^{+\infty}(t^2+3t-1)\delta^{(3)}(t-1)\,dt$$

Answer: Using the property $\int_{t_1}^{t_2}\delta(\tau-t)\delta(t-t_0)\,dt=\delta(t-t_0),\ t_1<t_0<t_2$, we obtain:

$$\int_0^{+\infty}(t^2+3t-1)\delta(t-1)dt=[\,t^2+3t-1\,]_{t=1}=3$$

$$\int_0^{+\infty}(t^2+3t-1)\,\delta'(t-1)\,dt=-[(t^2+3t-1)'\,]_{t=1}=-5$$

$$\int_0^{+\infty}(t^2+3t-1)\,\delta''(t-1)\,dt=(-1)^2\,[(t^2+3t-1)''\,]_{t=1}=2$$

$$\int_0^{+\infty}(t^2+3t-1)\,\delta^{(3)}(t-1)\,dt=(-1)^3\left[(t^2+3t-1)^{(3)}\right]\ =0$$

Exercise 5

Evaluate each of the following integrals:

$$\text{(a) } \int_0^\infty t\, u(2-t)\, u(t)\, dt \quad \text{(b) } \int_0^\infty \delta'(t-1)e^{-t}\, u(t)\, dt$$

Answer:

$$\int_0^\infty t\, u(2-t)\, u(t)\, dt = \int_0^2 t\, dt = 2$$

$$\int_0^\infty \delta'(t-1)e^{-t}\, u(t)\, dt = [(e^{-t})']_{t=1} = e^{-1}$$

Exercise 6

Given the signal $x(t) = e^{-t}\cos(t)\,u(t).$, determine whether it is an energy signal.

Answer: The signal energy is calculated as:

$$Ex = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{0}^{\infty} e^{-2t}\cos^2(t)\,dt \leq \int_{0}^{\infty} e^{-2t}\,dt = -\frac{1}{2}e^{-2t}\Big|_{0}^{\infty} = -\frac{1}{2}(\lim_{t\to\infty} e^{-2t} - e^{0})$$

$$= -\frac{1}{2}(0 - 1) = \frac{1}{2}$$

Since its energy is finite, it is an energy signal.

The solution uses the trigonometric inequality $\cos^2(t) \leq 1$.

Exercise 7

Show that the complex exponential signal $x(t) = e^{j\Omega_0 t}, -\infty < t < \infty$ is periodic with fundamental period $T_0 = 2\pi/\Omega_0$.

Answer: A periodic signal must have infinite duration (guaranteed by the problem statement) and satisfy $x(t+T) = x(t), \forall t$, where T is positive. From the definition of the signal:

$$e^{j\Omega_0(t+T)} = e^{j\Omega_0 t} \Rightarrow e^{j\Omega_0 t}\, e^{j\Omega_0 T} = e^{j\Omega_0 t} \Rightarrow \boldsymbol{e^{j\Omega_0 T} = 1}$$

The condition for periodicity is therefore $e^{j\Omega_0 T} = 1$. We distinguish two cases:

- For $\boldsymbol{\Omega_0 = 0}$, we have the constant signal $x(t) = 1$, which is periodic with an unspecified period.
- For $\boldsymbol{\Omega_0 \neq 0}$, the complex exponential signal is periodic for values of T satisfying $\Omega_0 T = 2\pi m$. Thus:

$$T = \frac{2\pi}{\Omega_0} m, \qquad m = 1, 2, \ldots$$

The fundamental period corresponds to $m = 1$ and is $T_0 = 2\pi/\Omega_0$.

Exercise 8

Given the signals $x_1(t) = rect(t/2),\ x_2(t) = rect(t/4)$ and $x_3(t) = tri(t)$, calculate and plot:

$$\textbf{(a) } \boldsymbol{y(t) = x_1(t) + x_2(t)} \qquad \textbf{(b) } \boldsymbol{w(t) = x_1(t) + x_3(t)} \qquad \textbf{(c) } \boldsymbol{z(t) = x_2(t) - x_1(t) - x_3(t)}$$

Answer: (a) The signal $x_1(t)$ is a rectangular pulse of unit amplitude, centered at zero, with a width of two time units:

$$x_1(t) = rect\left(\frac{t}{2}\right) = \begin{cases} 1, & -1 < t < 1 \\ 0, & t < -1, t > 1 \end{cases}$$

The signal $x_2(t)$ is a rectangular pulse of unit amplitude, centered at zero, with a width of four time units:

$$x_2(t) = rect\left(\frac{t}{4}\right) = \begin{cases} 1, & -2 < t < 2 \\ 0, & t < -2\ , t > 2 \end{cases}$$

Exercise 8 (continued)

The signal $x_3(t)$ is a triangular pulse of unit amplitude, centered at zero, with a width of two time units:

$$x_3(t) = tri(t) = \begin{cases} 1 - \dfrac{|t|}{1}, & |t| < 1 \\ 0, & t < -1, t > 1 \end{cases} = \begin{cases} 1 + t, & -1 < t < 0 \\ 1 - t, & 0 < t < 1 \\ 0, & t < -1, t > 1 \end{cases}$$

To calculate $y(t), w(t)$and $z(t)$, we construct the table on the next slide using the discontinuity points in the inequalities defining $x_1(t)$, $x_2(t)$ and $x_3(t)$.

The last three rows give the values of $y(t), w(t)$ and $z(t)$.

Exercise 8 (continued)

	-∞	-2	-1	0	1	2	∞
$x_1(t)$	0	0	1	1	0	0	
$x_2(t)$	0	1	1	1	1	0	
$x_3(t)$	0	0	$1+t$	$1-t$	0	0	
$y(t) = x_1(t) + x_2(t)$	0	1	2	2	1	0	
$w(t) = x_1(t) + x_3(t)$	0	0	$2+t$	$2-t$	0	0	
$z(t) = x_2(t) - x_1(t) - x_3(t)$	0	1	$-1-t$	$-1+t$	1	0	

