Ηλεκτρομαγνητισμός

Διάλεξη 15

Α. Δροσόπουλος

07-12-2022

Outline

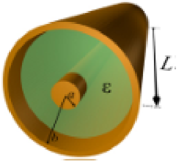
Topic

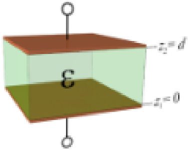
1. **Παράδειγμα χωρητικότητας**

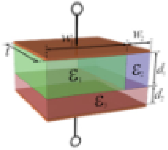
2. Μαγνητοστατικά πεδία

Παράδειγμα χωρητικότητας

- Parallel plate capacitor
- How big is a Farad?
- Coaxial capacitor
- RG-59 coax
- Inhomogeneous capacitor



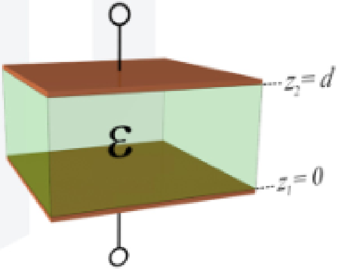




Παράδειγμα χωρητικότητας 1

Step 1 – Choose a convenient coordinate system.

Cartesian

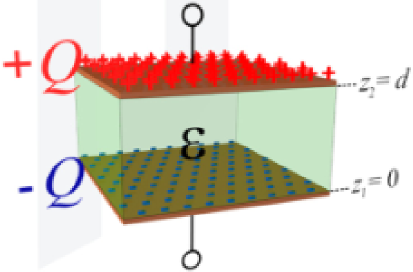


Παράδειγμα χωρητικότητας 1

Step 1 – Choose a convenient coordinate system.

Cartesian

Step 2 – Let the plates carry charges $+Q$ and $-Q$.



Παράδειγμα χωρητικότητας 1

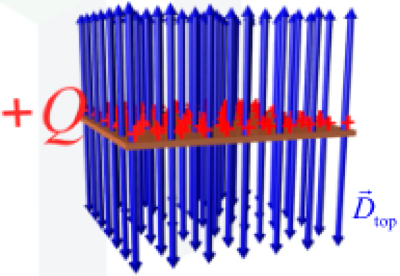
Step 3 – Calculate $\vec{D}$ using Gauss' law.

Recall the field around an infinite plate.

$$\vec{D} = \frac{\rho_S}{2}\hat{a}_n$$

Field below the top plate,

$$\vec{D}_{top} = -\frac{\rho_S}{2}\hat{a}_z$$



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Field above the bottom plate,

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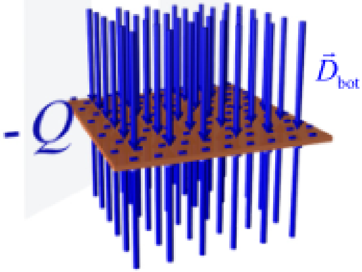

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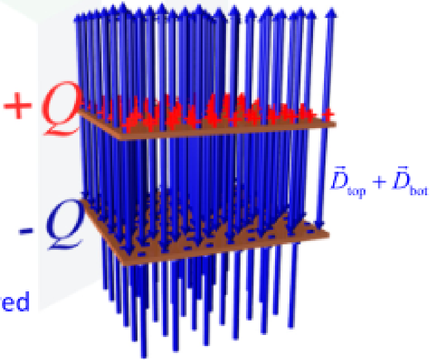
Field above the bottom plate,

$$\vec{D}_{bot} = -\frac{\rho_S}{2}\hat{a}_n$$

When both plates are considered

$$\vec{D} = \vec{D}_{top} + \vec{D}_{bot} = -\rho_S\hat{a}_z$$



Παράδειγμα χωρητικότητας 1

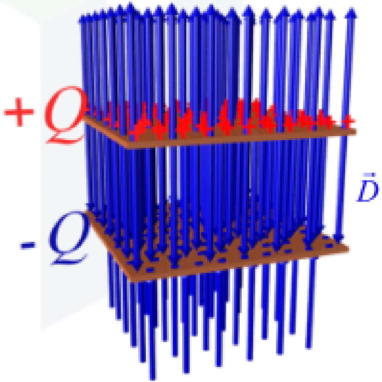
Step 3 – Calculate $\vec{D}$ using Gauss' law.

The surface charge density is

$$\rho_S = \frac{Q}{S}$$

The final express for $\vec{D}$ is

$$\vec{D} = -\rho_S \hat{a}_z = -\frac{Q}{S}\hat{a}_z$$

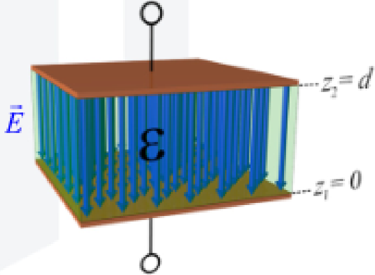


Παράδειγμα χωρητικότητας 1

Step 4 – Calculate $\vec{E}$.

Calculate $\vec{E}$ from the constitutive relation.

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = -\frac{Q}{\varepsilon S}\hat{a}_z$$



Παράδειγμα χωρητικότητας 1

Step 5 – Calculate V_0.

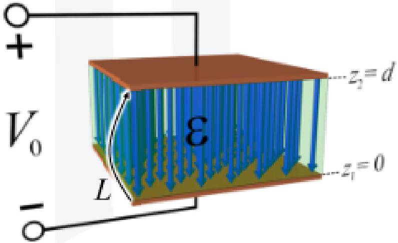
Given $\vec{E}$, calculate V_0 by integrating from the bottom plate to the top plate.

$$V_0 = -\int_L \vec{E} \bullet d\vec{\ell}$$

$$V_0 = -\int_0^d \left(-\frac{Q}{\varepsilon S}\hat{a}_z\right) \bullet dz\hat{a}_z$$

$$V_0 = \frac{Q}{\varepsilon S}\int_0^d dz$$

$$V_0 = \frac{Qd}{\varepsilon S}$$



Παράδειγμα χωρητικότητας 1

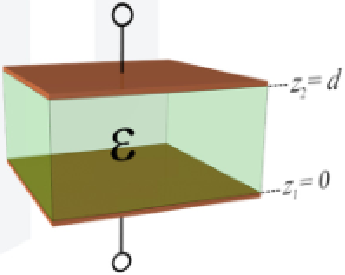
Step 6 – Calculate capacitance C.

$$C = \frac{|Q|}{|V_0|} = \frac{|Q|}{\left|\frac{Qd}{\varepsilon S}\right|} = \frac{Q}{\frac{Qd}{\varepsilon S}} = \frac{\varepsilon S}{d}$$

The final answer is

$$\boxed{C = \frac{\varepsilon S}{d}}$$

Self-check – C should not be a function of Q or V_0.



Παράδειγμα χωρητικότητας

Suppose the plates of a capacitor are 10 m by 20 m and the gap between the plates is 1 mm.

$$C = \frac{\varepsilon S}{d} = \frac{\varepsilon_0 \varepsilon_r L W}{d} = \frac{\left(8.854\times10^{-12}\ \mathrm{F/m}\right)\left(1.0\right)\left(10\ \mathrm{m}\right)\left(20\ \mathrm{m}\right)}{\left(0.001\ \mathrm{m}\right)}$$

$$= 1.78\times10^{-6}\ \mathrm{F}$$

$$= 1.78\ \mu\mathrm{F}$$

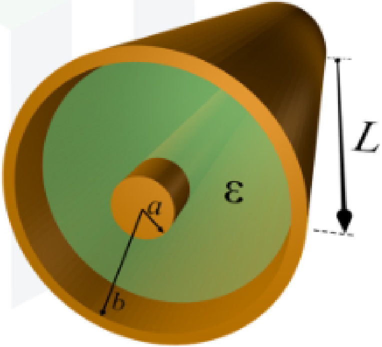
The capacitor is physically very large, yet the capacitance is very small.

The Farad is a HUGE unit!!!

Παράδειγμα χωρητικότητας 2

Step 1 – Choose a convenient coordinate system.

Cylindrical (ρ,ϕ,z)

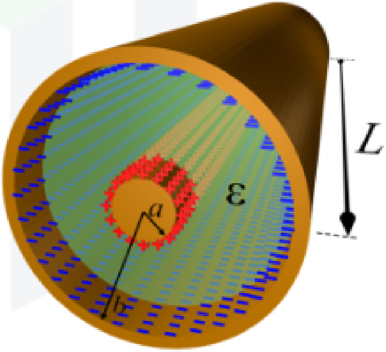


Παράδειγμα χωρητικότητας 2

Step 1 – Choose a convenient coordinate system.

Cylindrical (ρ, ϕ, z)

Step 2 – Let the plates carry charges $+Q$ and $-Q$.



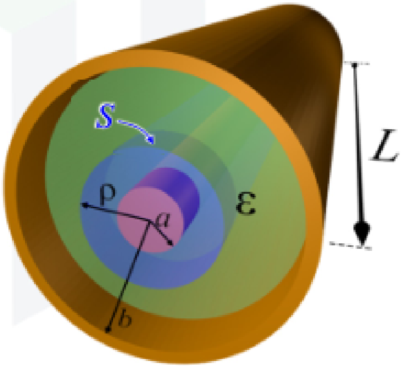
Παράδειγμα χωρητικότητας 2

Step 3 – Calculate $\vec{D}$ using Gauss' law.

Define a Gaussian surface with radius ρ to be inside of the dielectric.

$$Q = \oiint_S \vec{D} \bullet d\vec{s}$$

From the boundary conditions, it is known that the electric field will be normal at the interfaces to the metal.

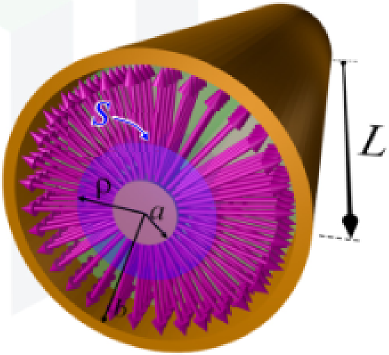


Παράδειγμα χωρητικότητας 2

Step 3 – Calculate $\vec{D}$ using Gauss' law.

The only field configuration that makes sense considering the boundary conditions is when the field is purely radially directed.

$$\vec{D} = D_\rho(\rho,\phi,z)\hat{a}_\rho$$



Παράδειγμα χωρητικότητας 2

Step 3 – Calculate $\vec{D}$ using Gauss' law.

Gauss' law becomes

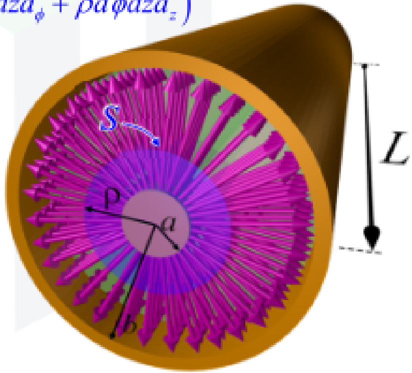
$$Q=\int_0^L\int_0^{2\pi}\left(D_\rho\hat{a}_\rho\right)\bullet\left(\rho d\phi dz\hat{a}_\rho+d\rho dz\hat{a}_\phi+\rho d\phi dz\hat{a}_z\right)$$

$$=\int_0^L\int_0^{2\pi}D_\rho\rho d\phi dz$$

$$=D_\rho\rho\int_0^L\left(\int_0^{2\pi}d\phi\right)dz$$

$$=D_\rho\rho\int_0^L\left(2\pi\right)dz$$

$$=2\pi D_\rho\rho\left(\int_0^L dz\right)=2\pi D_\rho\rho L$$



Παράδειγμα χωρητικότητας 2

Step 3 – Calculate $\vec{D}$ using Gauss' law.

Solving for $\vec{D}$ gives

$$D_\rho(\rho,\phi,z) = \frac{Q}{2\pi\rho L}$$

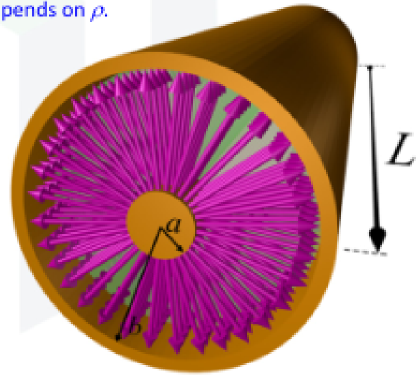
$\vec{D}$ only depends on ρ.

$$\vec{D}(\rho) = \frac{Q}{2\pi\rho L}\hat{a}_\rho$$

Step 4 – Calculate $\vec{E}$.

Calculate $\vec{E}$ from the constitutive relation.

$$\vec{E}(\rho) = \frac{\vec{D}}{\varepsilon} = \frac{Q}{2\pi\varepsilon\rho L}\hat{a}_\rho$$



Step 5 – Calculate V_0.

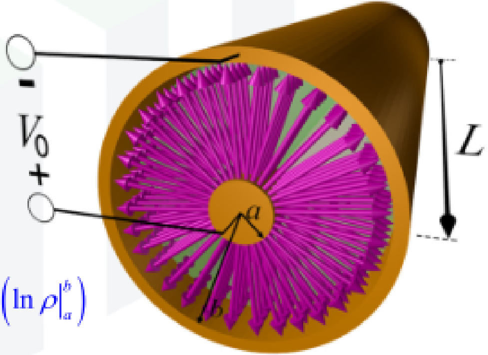
Given $\vec{E}$, calculate V_0 by integrating from the inner conductor to the outer conductor.

$$V_0 = -\int_L \vec{E} \bullet d\vec{\ell}$$

$$= -\int_a^b \left(\frac{Q}{2\pi\varepsilon\rho L} \hat{a}_\rho \right) \bullet \left(d\rho \hat{a}_\rho \right)$$

$$= -\int_a^b \frac{Q}{2\pi\varepsilon\rho L} d\rho$$

$$= -\frac{Q}{2\pi\varepsilon L} \int_a^b \frac{1}{\rho} d\rho = -\frac{Q}{2\pi\varepsilon L} \left(\ln \rho \Big|_a^b \right)$$



Παράδειγμα χωρητικότητας 2

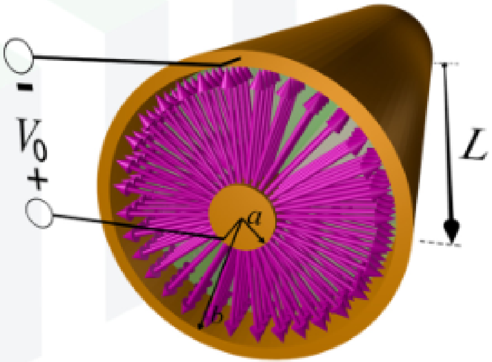
Step 5 – Calculate V_0.

Continued…

$$V_0 = -\frac{Q}{2\pi\varepsilon L}\left(\ln \rho\Big|_a^b\right)$$

$$= -\frac{Q}{2\pi\varepsilon L}\left(\ln b - \ln a\right)$$

$$= -\frac{Q}{2\pi\varepsilon L}\ln\left(\frac{b}{a}\right)$$



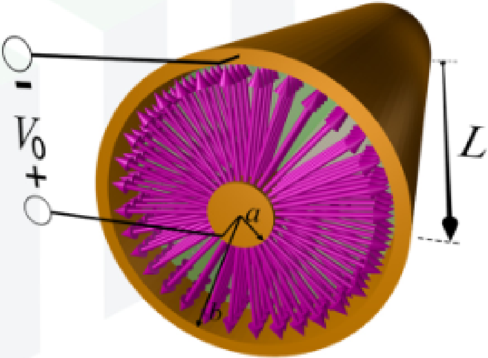
Παράδειγμα χωρητικότητας 2

Step 6 – Calculate capacitance C.

$$C = \frac{|Q|}{|V_0|}$$

$$= \frac{Q}{\frac{Q}{2\pi\varepsilon L}\ln\left(\frac{b}{a}\right)}$$

$$\boxed{C = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}}$$



* Self Check – C is not a function of Q or V_0.

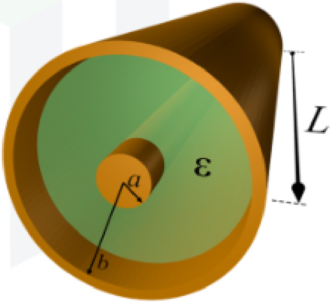
Παράδειγμα χωρητικότητας

Distributed Capacitance

Sometimes it is desired to specify the capacitance without knowledge of L.

This is done using the distributed capacitance, which is defined as capacitance per unit length.

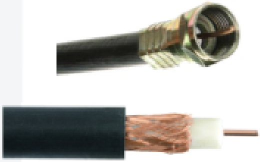
$$\frac{C}{L} = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}$$



Παράδειγμα χωρητικότητας

A standard RG-59 coax has

Inner conductor diameter:	0.81 mm (20 AWG)
Outer conductor diameter:	3.66 mm
Dielectric constant:	2.1
Specified capacitance:	86.9 pF/m

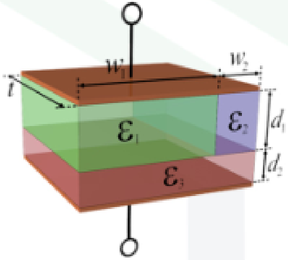


$$\frac{C}{L} = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln(b/a)}$$

$$= \frac{2\pi\left(8.854\times10^{-12}\ \text{F/m}\right)(2.1)}{\ln\left(3.66\ \text{mm}/0.81\ \text{mm}\right)}$$

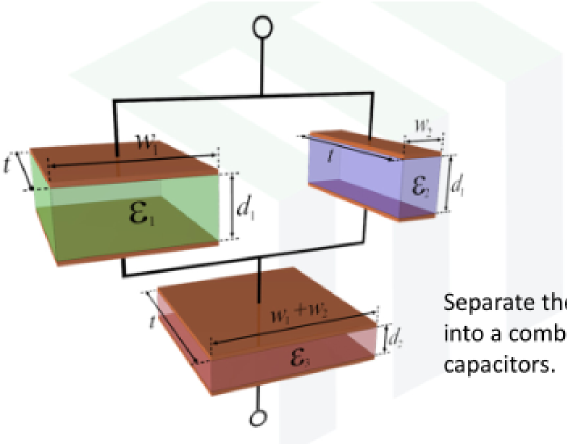
$$= 7.746\times10^{-11}\ \text{F/m} = \boxed{77.46\ \text{pF/m}}$$

Παράδειγμα χωρητικότητας 3



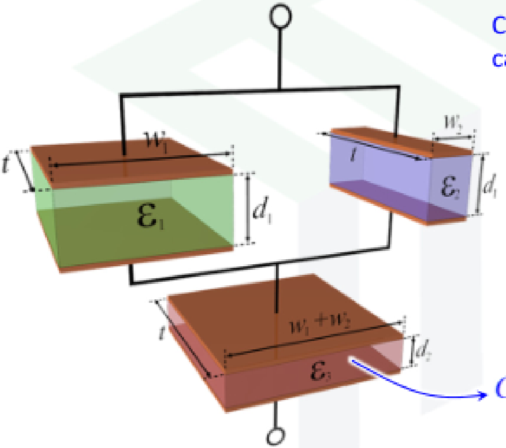
Suppose there exists an inhomogeneous capacitor.

Παράδειγμα χωρητικότητας 3



Separate the inhomogeneous capacitor into a combination of homogeneous capacitors.

Παράδειγμα χωρητικότητας 3



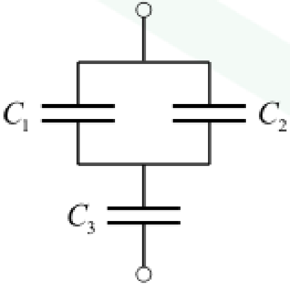
Calculate each homogeneous capacitor independently.

$$C_1 = \frac{\varepsilon_1 S_1}{d_1} = \frac{\varepsilon_1 t w_1}{d_1}$$

$$C_2 = \frac{\varepsilon_2 S_2}{d_1} = \frac{\varepsilon_2 t w_2}{d_1}$$

$$C_3 = \frac{\varepsilon_3 S_3}{d_2} = \frac{\varepsilon_3 t \left(w_1 + w_2\right)}{d_2}$$

Παράδειγμα χωρητικότητας 3



View the capacitor as a series/parallel combination of capacitors.

The equivalent capacitance is

$$C_{eq} = (C_1 + C_2) \| C_3$$
$$= \left(\frac{\varepsilon_1 t w_1}{d_1} + \frac{\varepsilon_2 t w_2}{d_1} \right) \Bigg\| \frac{\varepsilon_3 t (w_1 + w_2)}{d_2}$$

$$C_{eq} = \frac{t\varepsilon_3 (\varepsilon_1 w_1 + \varepsilon_2 w_2)(w_1 + w_2)}{d_2 (\varepsilon_1 w_1 + \varepsilon_2 w_2) + \varepsilon_3 d_1 (w_1 + w_2)}$$

Topic

1. Παράδειγμα χωρητικότητας
2. Μαγνητοστατικά πεδία

Από εξισώσεις Maxwell

Maxwell's Equations

$$\nabla \times \vec{E} = \cancel{-\frac{\partial \vec{B}}{\partial t}}$$

$$\nabla \times \vec{H} = \vec{J} \cancel{- \frac{\partial \vec{D}}{\partial t}}$$

$$\nabla \bullet \vec{D} = \rho_v$$

$$\nabla \bullet \vec{B} = 0$$

Constitutive Relations

$$\vec{D} = [\varepsilon]\vec{E}$$

$$\vec{B} = [\mu]\vec{H}$$

Electrostatic Equations

$$\nabla \times \vec{E} = 0$$

$$\nabla \bullet \vec{D} = \rho_v$$

$$\vec{D} = [\varepsilon]\vec{E}$$

Magnetostatic Equations

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \bullet \vec{B} = 0$$

$$\vec{B} = [\mu]\vec{H}$$

Εξισώσεις Maxwell για μαγνητοστατική

Νόμος Ampere

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Έχουμε μαγνητικό πεδίο μόνο αν έχουμε ρεύμα. Τα μαγνητικά πεδία κυκλοφορούν γύρω από ρεύματα.

Νόμος Gauss για μαγνητικά πεδία

$$\nabla \cdot \mathbf{B} = 0$$

Δεν έχουμε απόκλιση. Δεν υπάρχουν μαγνητικά μονόπολα. Οι μαγνητικές δυναμικές γραμμές δεν έχουν αρχή ή τέλος αλλά υπάρχουν. Άρα εμφανίζονται μόνο σε βρόχους.

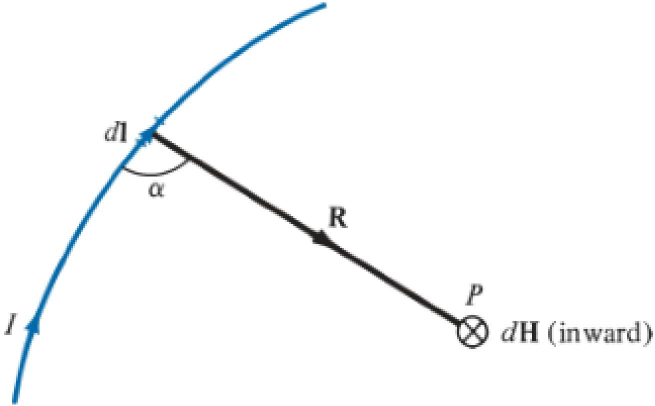
Καταστατικές εξισώσεις

$$\mathbf{B} = [\mu]\mathbf{H}$$

Το μαγνητικό πεδίο συνδέεται με την πυκνότητα μαγνητικής ροής μέσω της μαγνητικής διαπερατότητας. Το μαγνητικό πεδίο δεν βλέπει ηλεκτρική διαπερατότητα.

Νόμος Biot-Savart

Το στοιχειώδες μαγνητικό πεδίο dH σε σημείο P που παράγεται από στοιχειώδες ρεύμα $Id\ell$ είναι ανάλογο με το γινόμενο του $Id\ell$ με το ημίτονο της γωνίας που σχηματίζεται μεταξύ του στοιχειώδους ρεύματος και της γραμμής που συνδέει το ρεύμα με το P και αντιστρόφως ανάλογο με το τετράγωνο της μεταξύ τους απόστασης.



Νόμος Biot-Savart (συνέχεια 1)

Το μέτρο:

$$dH \sim \frac{I d\ell \sin\alpha}{R^2}$$

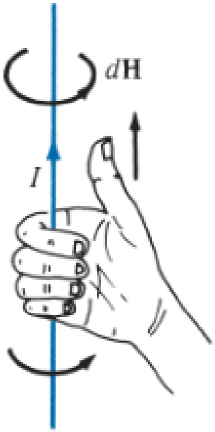
$$dH = \frac{I d\ell \sin\alpha}{4\pi R^2}$$

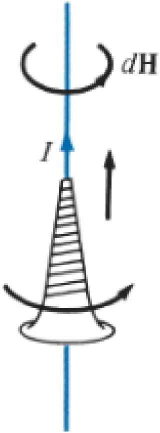
Και η πλήρης μορφή:

$$d\mathbf{H} = \frac{I d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{I d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

Νόμος Biot-Savart (συνέχεια 2)

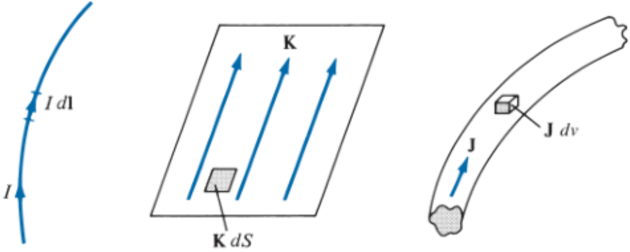
Κανόνας δεξιού χεριού ή δεξιόστροφου κοχλία





Νόμος Biot-Savart (συνέχεια 3)

Κατανομές ρεύματος: Γραμμική, επιφανειακή και χώρου



Νόμος Biot-Savart (συνέχεια 4)

Ολικό μαγνητικό πεδίο γραμμικού ρεύματος

$$\mathbf{H} = \int_L \frac{I d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2}$$

Ολικό μαγνητικό πεδίο επιφανειακού ρεύματος

$$\mathbf{H} = \int_S \frac{\mathbf{K}\, dS \times \mathbf{a}_R}{4\pi R^2}$$

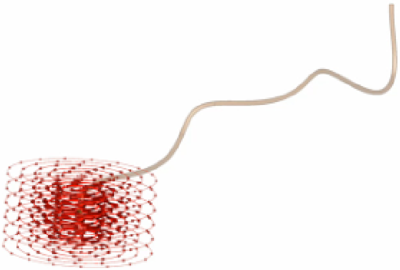
Ολικό μαγνητικό πεδίο ρεύματος χώρου

$$\mathbf{H} = \int_v \frac{\mathbf{J}\, dv \times \mathbf{a}_R}{4\pi R^2}$$

Total Magnetic Field

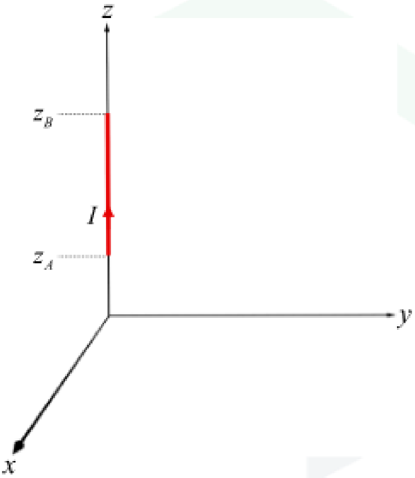
The total magnetic field $\vec{H}$ due to a long wire is obtained by integrating the Biot-Savart law over the length of the wire.

$$\vec{H} = \int_L d\vec{H} = \int_L \frac{I d\vec{\ell} \times \hat{a}_R}{4\pi R^2}$$



Παράδειγμα 1

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$

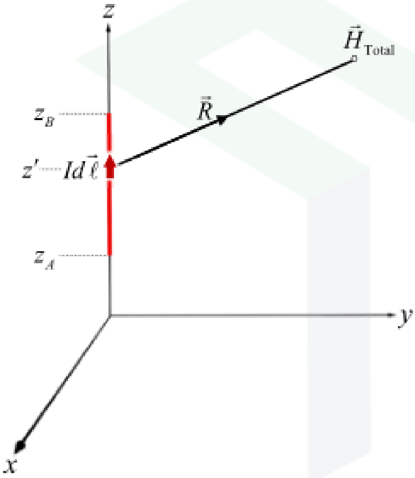
The total magnetic field is obtained by integrating the Biot-Savart law.

$$\vec{H} = \int_L d\vec{H}$$

$$= \int_{z_A}^{z_B} \frac{I d\vec{\ell} \times \hat{a}_R}{4\pi R^2}$$

$$= \int_{z_A}^{z_B} \frac{I d\vec{\ell} \times \vec{R}}{4\pi R^3}$$

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$?

For this problem

$$d\vec{\ell} = dz\hat{a}_z \qquad \vec{R} = \rho\hat{a}_\rho + (z - z')\hat{a}_z$$

Thus, our cross product becomes

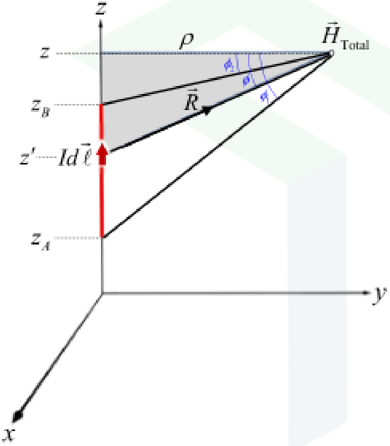
$$d\vec{\ell}\times\vec{R} = (dz\hat{a}_z)\times\left[\rho\hat{a}_\rho + (z - z')\hat{a}_z\right]$$
$$= \rho dz\hat{a}_\phi$$

Putting this back into the integral gives

$$\vec{H} = \int_{z_A}^{z_B} \frac{I\rho dz\hat{a}_\phi}{4\pi R^3} = \frac{\rho I}{4\pi}\hat{a}_\phi \int_{z_A}^{z_B} \frac{dz}{R^3}$$

Παράδειγμα 1 (συνέχεια 2)

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$?

Instead of integrating over z, integrate over angle ϕ.

$$z_A \to \phi_1$$
$$z_B \to \phi_2$$
$$dz' \to ?$$
$$\vec{R}/\left|\vec{R}\right|^3 \to ?$$

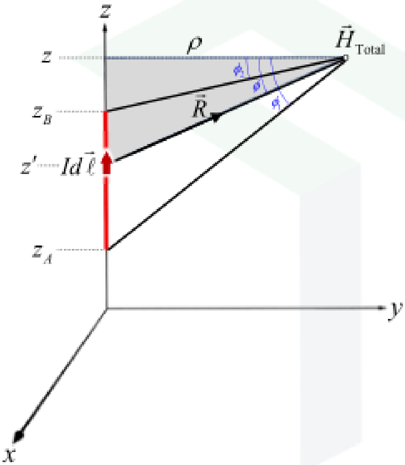
From the figure, it can be seen that

$$\tan\phi = \frac{z - z'}{\rho} \quad \to \quad z' = z - \rho\tan\phi$$

Differentiate this expression to get

$$dz' = -\rho\sec^2\phi d\phi$$

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$?

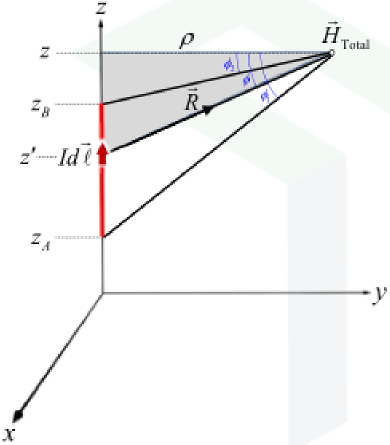
The vector $\vec{R}$ is

$$\vec{R} = \rho\hat{a}_\rho + (z - z')\hat{a}_z$$
$$= \rho\hat{a}_\rho + \rho\tan\phi\hat{a}_z$$
$$= \frac{\rho}{\cos\phi}(\cos\phi\hat{a}_\rho + \sin\phi\hat{a}_z)$$
$$= \rho\sec\phi\underbrace{(\cos\phi\hat{a}_\rho + \sin\phi\hat{a}_z)}_{\text{Magnitude is 1}}$$

An expression can now be written for R^3.

$$R^3 = |\vec{R}|^3$$
$$= |\rho\sec\phi(\cos\phi\hat{a}_\rho + \sin\phi\hat{a}_z)|^3$$
$$= (\rho\sec\phi)^3$$

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$?

The integral in terms of ϕ becomes

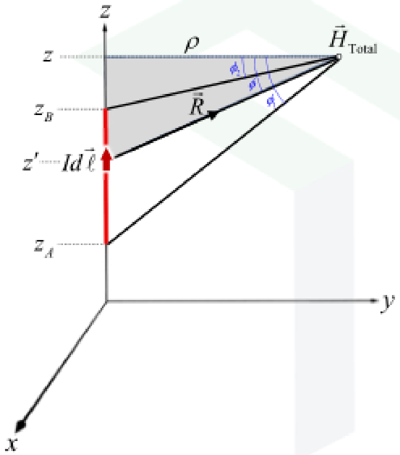
$$\vec{H} = \frac{\rho I}{4\pi}\hat{a}_\phi \int_{z_A}^{z_B} \frac{dz}{(\rho \sec\phi)^3}$$

$$= \frac{\rho I}{4\pi}\hat{a}_\phi \int_{\phi_1}^{\phi_2} \frac{(-\rho \sec^2\phi\, d\phi)}{(\rho\sec\phi)^3}$$

$$= -\frac{I}{4\pi\rho}\hat{a}_\phi \int_{\phi_1}^{\phi_2} \frac{d\phi}{\sec\phi}$$

$$= -\frac{I}{4\pi\rho}\hat{a}_\phi \int_{\phi_1}^{\phi_2} \cos\phi\, d\phi$$

Example #1 – Finite Length Wire



What is the magnetic field $\vec{H}$?

The integral can now be evaluated as

$$\vec{H} = -\frac{I}{4\pi\rho}\hat{a}_\phi \int_{\phi_1}^{\phi_2} \cos\phi \, d\phi$$

$$= -\frac{I}{4\pi\rho}\hat{a}_\phi \left(\sin\phi \Big|_{\phi_1}^{\phi_2} \right)$$

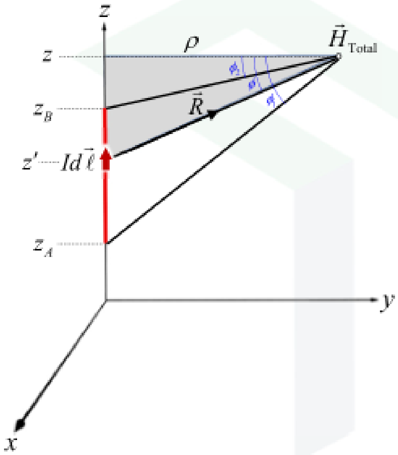
$$= -\frac{I}{4\pi\rho}\hat{a}_\phi \left(\sin\phi_2 - \sin\phi_1 \right)$$

$$= \frac{I}{4\pi\rho}\left(\sin\phi_1 - \sin\phi_2 \right)\hat{a}_\phi$$

$$\boxed{\vec{H} = \frac{I}{4\pi\rho}\left(\sin\phi_1 - \sin\phi_2 \right)\hat{a}_\phi}$$

Παράδειγμα 1 (συνέχεια 6)

Example #1 – Finite Length Wire



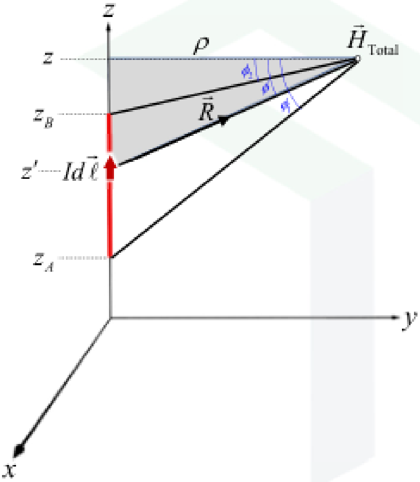
Observations about the solution:

$$\vec{H} = \frac{I}{4\pi\rho}\left(\sin\phi_1 - \sin\phi_2\right)\hat{a}_\phi$$

1. This solution is applicable to any straight wire with uniform current.
2. Magnitude of $\vec{H}$ decays as $1/\rho$.
3. Magnetic has only an $\hat{a}_\phi$ component. This means the magnetic field forms loops around the wire.

Παράδειγμα 1 (συνέχεια 7)

Example #2 – Infinite Length Wire



What is the magnetic field $\vec{H}$?

For the infinite length wire,

$\phi_1 = 90°$ $\phi_2 = -90°$

The expression for $\vec{H}$ reduces to

$$\vec{H} = \frac{I}{4\pi\rho}(\sin\phi_1 - \sin\phi_2)\hat{a}_\phi$$

$$= \frac{I}{4\pi\rho}[\sin(90°) - \sin(-90°)]\hat{a}_\phi$$

$$= \frac{I}{4\pi\rho}[\ 1 \ - \ (-1)\ ]\hat{a}_\phi$$

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$