6. COMPLEX NUMBERS

Objectives

To understand that complex numbers consist of a real part and an imaginary part and that they are used to investigate what might happen if they were in fact completely real.

To be able to cary out algebraic manipulations with complex numbers.

To be able to illustrate complex numbers on an argand diagram in both cartesian and polar form.

To understand that a locus is a point, z, moving through a set of points and not a curve or line.

To know and be able to use de Moivre's theorem.

6.1 Introduction to Complex Numbers

We know that some equations such as $x^2 + 1 = 0$ cannot be solved using the usual methods, because the square root of a negative number does not exist. Now consider this,

$$x^2 = -1$$

$$x = \sqrt{(-1)} \quad \text{let } i^2 = -1$$

then $$x = \sqrt{i^2}$$

$$x = \pm i$$

This is an **imaginary** solution but it is useful to use in mathematics because it allows us to see the results as if they were in fact possible.

Usually a complex number z is made up of a **real** part, x, and an imaginary part, y.

$$z = x + iy$$

Re(z) = x denotes the real part of z and Im(z) = y denotes the imaginary part. In mathematics, i represents *imaginary* but in engineering j is used instead of i. So in an engineering context, all the letters i may be regarded as j.

In quadratic equations, when $\Delta = b^2 - 4ac < 0$ we can have imaginary solutions.

eg $x^2 + 2x + 2 = 0$

using the formula we have $x = \dfrac{-2 \pm \sqrt{(4 - 4.1.2)}}{2}$

$= -1 \pm \dfrac{\sqrt{-4}}{2}$

$= -1 \pm \dfrac{2i}{2}$

$= -1 \pm i$

The most important point to remember with complex numbers is that

$$i^2 = -1$$

Find the following values in their lowest terms.

$i^3 =$ .... $i^4 =$ .... $i^{-1} =$ ....

$i^{-2} =$ .... $i^{-3} =$ ....

[Solutions: -i, 1, -i, -1, i]

6.2 Operations with Complex Numbers

Consider two complex numbers, $z_1 = a + ib$ and $z_2 = c + id$.

$$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$$
$$z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i(b - d)$$

These two results are themselves complex numbers because they are composed of a real part and an imaginary part. The same is true when two complex numbers are multiplied or divided.

$$\begin{aligned} z_1 z_2 &= (a + ib)(c + id) \\ &= ac + iad + ibc + i^2 bd \\ &= ac - bd + i(ad + bc) \end{aligned}$$

$$\begin{aligned} z_1/z_2 = \frac{(a + ib)}{(c + id)} &= \frac{(a + ib)(c - id)}{(c + id)(c - id)} \\ &= \frac{(ac + bd) + i(ad + bc)}{c^2 + d^2} \\ &= \frac{(ac + bd)}{c^2 + d^2} + \frac{i(ad + bc)}{c^2 + d^2} \end{aligned}$$

Notice that we multiplied the numerator and the denominator by (c - id). This is called the **complex conjugate** of the denominator. The reason for doing so is so that we eliminate the complex number in the denominator. It is a similar method to the one used in surds.

$z = x + iy$ $\qquad$ $\bar{z} = x - iy$ is the complex conjugate of z

Note that

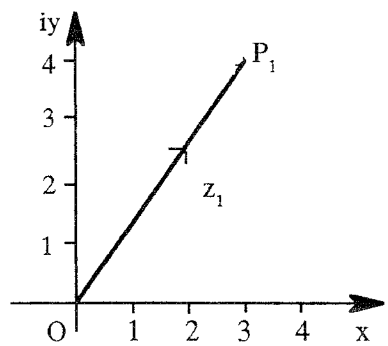
$$z + \bar{z} = 2x \qquad z - \bar{z} = 2iy$$

The Argand Diagram

This is the way we represent complex numbers in two dimensions. The complex number $z = x + iy$ is represented by a vector $OP = (x\ y)'$ where P is the point with co-ordinates (x,y). The x axis is the real axis and the y axis is the imaginary one.

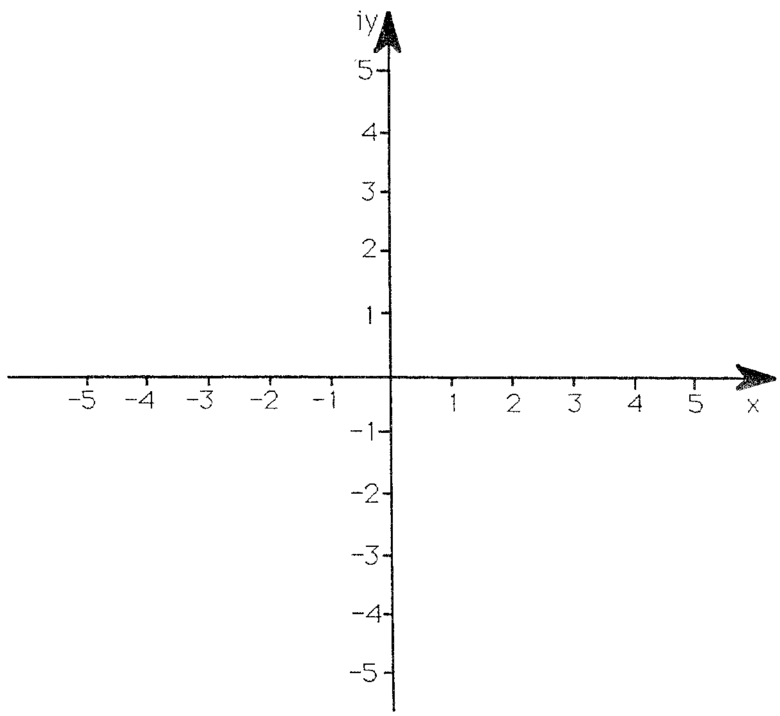
eg $z_1 = 3 + 4i$ is represented as

The Argand Diagram for z=3 + 4i



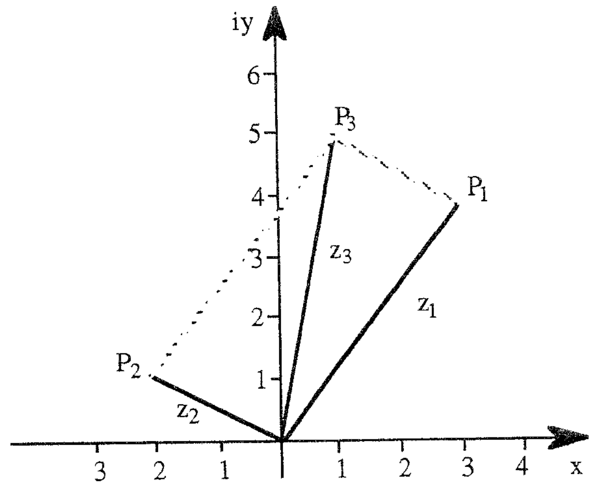
Represent the following complex numbers on the argand diagram below.

$w_1 = -3-i$ $w_2 = 2 - 2i$ $w_3 = 3$ $w_4 = -3 + 3i$ $w_5 = 2 + 3i$

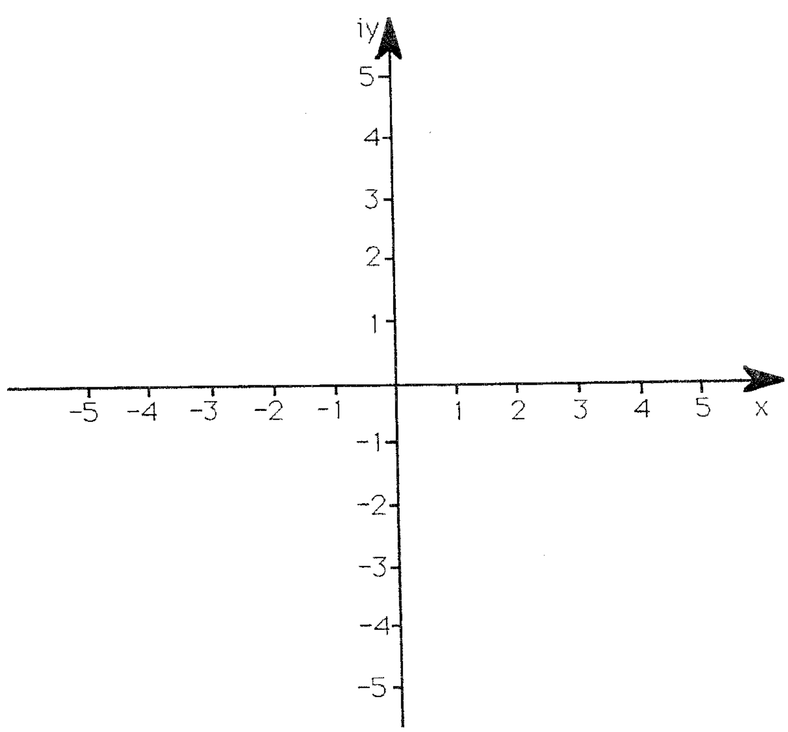


Consider the point P_2 (-2, 1) which represents the point z_2= -2 + i together with z_1 as in the previous example. Then $z_1 + z_2 = z_3 = 1 + 5i$ is represented in the diagram below. The point P_3 can be found geometrically by extending lines parallel to z_1 and z_2 from P_1 and P_2 respectively.

Addition of two complex numbers



What are $z_4 = z_1 - z_2$ and $z_5 = z_2 - z_1$? Illustrate them below.



Notice that -z goes in the opposite direction to z and that the two new *vectors* z_4 and z_5 are reflections of each other.

6.3 Modulus and Argument

So for we have seen how to represent $z = x + iy$ on a diagram using *Cartesian* co-ordinates. Now we will use **polar** co-ordinates to illustrate complex numbers.

Cartesian co-ordinates use x and y values.
Polar co-ordinates use angles and distances from the origin O.

In polar form $z = r(\cos\vartheta + i \sin\vartheta)$ or in short-hand, $z = (r, \vartheta)$

r is called the **modulus** of z ie $r = |z|$
ϑ is called the **argument** of z

We can find values for r and ϑ from the Cartesian co-ordinates using the formulae below.

$r = |z| = \sqrt{(x^2 + y^2)}$

$\arg z = \vartheta$ where $\quad x = r\cos\vartheta \quad -\pi < \vartheta < \pi$

$y = r\sin\vartheta$

r is the length of the vector z and ϑ is the (anticlockwise) angle it makes with the x axis at the origin.

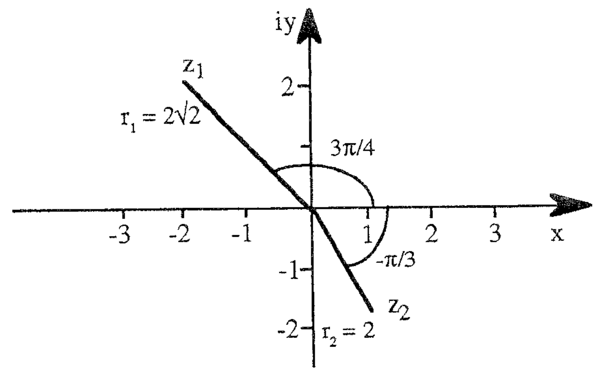
eg $\quad z_1 = -2 + 2i \quad\quad z_2 = 1 - i\sqrt{3}$

$r_1 = \sqrt{\{(-2)^2 + 2^2\}} \quad\quad r_2 = \sqrt{(1 + 3)}$

$= 2\sqrt{2} \quad\quad = 2$

$\cos\vartheta_1 = -2/2\sqrt{2} \quad\quad \cos\vartheta_2 = 1/2$

$\vartheta_1 = 3\pi/4 \quad\quad \vartheta_2 = -\pi/3$



The Argand Diagram illustrating modulus and argument

The complex numbers z and w are given in polar form as $z = (3, \pi/2)$ and $w = (2, \pi/6)$. Calculate zw and express it in polar form.

....

....

....

....

....

[Solution: $zw = (6, 2\pi/3)$]

From the result above we can see that in general

$$|zw| = |z||w| \qquad \arg(zw) = \arg z + \arg w$$

This is a result which shows that whilst it is easy to add and subtract complex numbers in Cartesian form, it is easier to multiply and divide them in polar form.

Another general result is the triangle inequality below.

$$|z + w| \leq |z| + |w| \qquad \big||z| + |w|\big| \leq |z - w|$$

6.4 Loci in the Complex Plane

Using moduli and arguments we can begin to describe lines and circles in the complex plane.

eg $|z_1| = 2$

This means that the number z_1 is two units away from the origin but at what angle to the x axis? None is specified so we take all possible angles, ie z_1 traces a circular path of radius 2 around the origin.

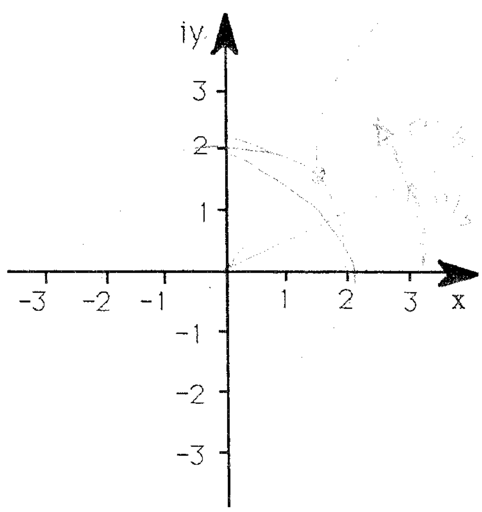
eg $\arg z_2 = \pi/3$

This is a similar situation, no distance from the origin is specified so we have a line extending from the origin into infinity at an angle of $\pi/3$ to the x axis.

The paths traced by the set of points P given by these two complex numbers are called **loci**.

Where do the two paths z_1 and z_2 cross? Clearly it is at $z_3 = (2, \pi/3) = 1 + i\sqrt{3}$.

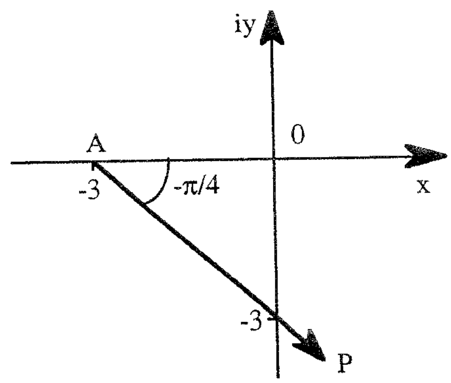
Sketch z_1, z_2 and z_3.



So far we have talked about z in relation to the origin O. Consider $\arg(z + 3) = -\pi/4$. Where is the set of points P which satisfy this?

Let the complex numbers z and -3 be represented by P and A. Then AP = OP - OA = z - (-3) = z + 3 using vector algebra. Therefore the locus of z given by arg (z + 3) = -π/4 is the set of points P such that AP makes an angle of -π/4 with the x axis.
Note that A and O are fixed points, P is a moving point.

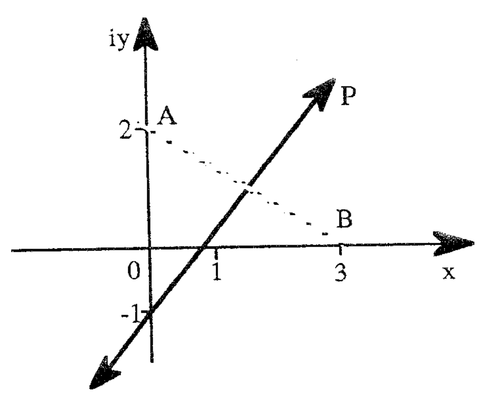
The Locus of the point P



What is the locus of | z - 3 | = | 2i - z | ?

As before, let A = 2i, B= 3 and P = z so that |z - 3 | is the distance BP and |2i - z | is the distance AP. So we have BP = AP and so the equation above is the set of points P equal distances from fixed points A and B.

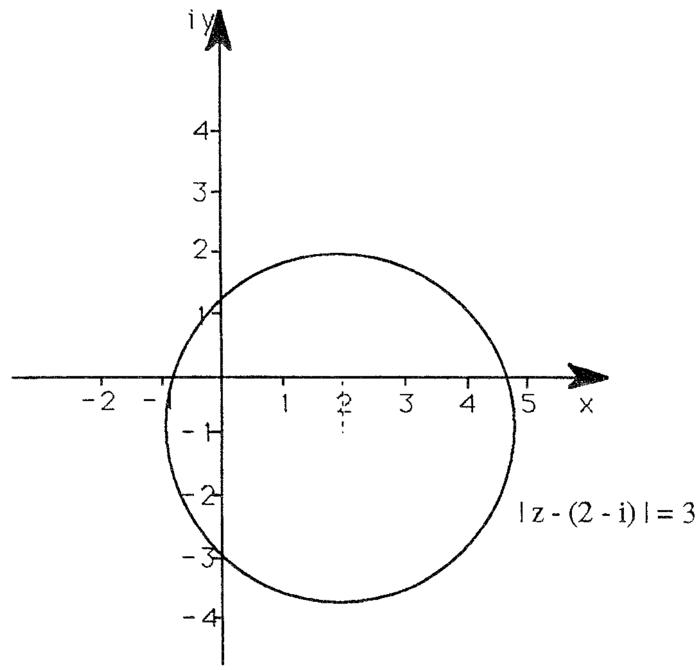
The Locus of /z-3/≥/2i-z/



What is the locus of z given by $|z - (2 - i)| = 3$?

$|z - (2 - i)| = 3$ is the distance from point z to point $(2 - i)$ and the set of points which satifies this is a circle, centre $(2 - i)$, radius 3.

The Locus of /z- (2-i)/=3



6.5 De Moivre's Theorem and Applications

If z and w are two complex numbers such that

$$z_1 = r_1 (\cos \vartheta_1 + i \sin \vartheta_1)$$
$$z_2 = r_2 (\cos \vartheta_2 + i \sin \vartheta_2)$$

Then $$z_1 z_2 = r_1 r_2 (\cos (\vartheta_1 + \vartheta_2) + i \sin (\vartheta_1 + \vartheta_2))$$

and similarly

$$z_1 z_2 \dots z_n = (r_1 r_2 \dots r_n)(\cos (\vartheta_1 + \vartheta_2 + \dots \vartheta_n) + i \sin (\vartheta_1 + \vartheta_2 + \dots + \vartheta_n))$$

If $r_1 = r_2 = \dots = r_n$ and $\vartheta_1 = \vartheta_2 = \dots = \vartheta_n$ we have

$$z^n = r^n(\cos n\vartheta + i \sin n\vartheta) = (\cos \vartheta + i \sin \vartheta)^n$$

This is called **de Moivre's Theorem** and it is valid for n positive, negative or fractional.

We can use de Moivre's theorem as a method of expressing trigonometric functions in terms of powers of cos ϑ and sin ϑ.

eg Express cos 6ϑ and sin 6ϑ in terms of cos ϑ and sin ϑ using de Moivre's theorem.

$$
\begin{aligned}
(\cos 6\vartheta + i \sin 6\vartheta) &= (\cos \vartheta + i \sin \vartheta)^6 \\
&= \cos^6 \vartheta + 6i \sin^5 \vartheta \cos \vartheta - 15\cos^4 \vartheta \sin^2 \vartheta + \\
&\quad 20i^3 \cos^3 \vartheta \sin^3 \vartheta + 15\cos^2 \vartheta \sin^4 \vartheta + 6i\cos \vartheta \sin^5 \vartheta \\
&\quad - \sin^6 \vartheta
\end{aligned}
$$

Now equate the real and imaginary parts of the equation to give the solutions below.

Real Part

$\cos 6\vartheta = \cos^6 \vartheta - 15\cos^4 \vartheta \sin^2 \vartheta + 15\cos^2 \vartheta \sin^4 \vartheta - \sin^6 \vartheta$

Imaginary Part

$\sin 6\vartheta = 6\sin^5 \vartheta \cos \vartheta - 20\cos^3 \vartheta \sin^3 \vartheta \quad 6\cos \vartheta \sin^5 \vartheta$.

Summary

Complex numbers are simple, as long as you remember that $i^2 = -1$ and that if the number is fractional, there is never an imaginary term in the denominator.

Using complex numbers we are able to find a solution to those equations which were previously insoluble. These solutions can be represented on the argand diagram, in cartesian or polar form. The argand diagram is really a vector diagram and all the rules about vector algebra apply.

We have already seen the modulus sign, but not as a specific measure of length. Remember in the inequalities unit it was the distance between two point, here it is the distance from the origin (of the vector) to the point P at the end.

Loci are the paths traced by complex numbers when they are not fixed, ie. if either the modulus or argument are missing, or when two complex numbers are combined. Usually they involve a modulus sign and either one or two numbers z.

Activities

1. Express in the form $z = x + iy$.

(a) $(2 + 3i) + (5 + 7i)$ (b) $(3 + 4i) + (4 - 3i)$
(c) $(5 - 6i) + (2 - 9i)$ (d) $(4 - i) - (3 + 2i)$
(e) $(14 - 3i) - (3 - 7i)$ (f) $(1 - i) - (1 + i)$

2. Write as a single complex number.

(a) $(2 + 3i)(4 + i)$ (b) $(3 - 4i)(4 - 3i)$
(c) $(5 + 2i)(4 - 3i)$ (d) $(1 + i)(1 - i)$
(e) $(2 + 3i)(2 + 3i)$ (f) $(3 - i)(2 + i)(3 + 2i)$
(g) $(4 - 2i)^2 - (3 + i)^2$

3. Express in the form $a + ib$ where a and b are real.

(a) $\dfrac{(2 + i)}{(3 + 2i)}$ (b) $\dfrac{(5 - 3i)}{(3 - 4i)}$ (c) $\dfrac{1}{1 - i}$

(d) $\dfrac{5 + 4i}{5 - 4i}$ (e) $\dfrac{(3 + i)^2}{4 - i^2}$

4. Given $z = -1 + 3i$, express $z + 2/z$ in the form $a + ib$.

5. Simplify $\dfrac{2 + 5i}{5 - 2i}$

6. Find real and complex roots of the equations below.

(a) $x^3 - 11x - 20 = 0$ (b) $x^2 + 4x + 3 = 0$
(c) $2x^2 - 7x + 5 = 0$ (d) $x^2 - 6x + 6 = 0$
(e) $3x^2 + 5x - 1 = 0$ (f) $4x^2 - 7x + 2 = 0$

7. Find complex solutions to the following.

(a) $x^2 + 4x + 3 = 0$ (b) $x^2 - 6x + 10 = 0$
(c) $x^2 - 8x + 17 = 0$ (d) $x^2 + 12x + 40 = 0$

8. Write down the squares of $2i$, $i\sqrt{5}$, $-7i$, $-i\sqrt{3}$ and the square roots of -5, -25, -8, -27.

9. Represent the following on argand diagrams.

   (a) $(2 + 3i) + (5 + i) = 7 + 4i$
   (b) $(3 - 2i) - (4 + 3i) = -1 - 5i$

10. If $z = 4 + 2i$ and $w = 3 - 3i$ illustrate on an argand diagram z, w, $z + w$, $z - w$, $w - z$.

11. If $z = 3 + 4i$ and $w = i$, illustrate z, w, zw, zw^2, zw^3, zw^4. What do you notice happening?

12. Express the following in polar form and illustrate them.

   (a) $1 + i$ (b) $4 + 3i$ (c) $5 - 12i$
   (d) $-12 + 5i$ (e) $4 - 4i$ (f) $-6 - 8i$

13. Rewrite in cartesian form the following complex numbers given in polar form.

   (a) $(2, \pi/4)$ (b) $(4, \pi/6)$ (c) $(1, -\pi/2)$
   (d) $(8, -\pi/3)$ (e) $(3, -5\pi/6)$ (f) $(12, \pi/2)$

14. Write down the product, z_1z_2 and the quotient, z_1/z_2 of the two complex numbers given in polar form.

   (a) $z_1 = (2, \pi/4)$ $z_2 = (3, \pi/4)$ (b) $z_1 = (5, \pi/3)$ $z_2 = (2, \pi/6)$
   (c) $z_1 = (1, \pi/2)$ $z_2 = (5, \pi/12)$ (d) $z_1 = (1, 3\pi/4)$ $z_2 = (7, -\pi/3)$

15. If $\pi/2 < \vartheta < \pi$ find in terms of ϑ the modulus of $\cos^2 \vartheta + i \sin \vartheta \cos \vartheta$.

16. If $z = 4(\cos \pi/3 + i \sin \pi/3)$ and $w = 2(\cos \pi/6 - i \sin \pi/6)$ find the modulus and argument of z, w, z^3 and z^3/w.

17. If $z_1 = 1 - 2i$, $z_2 = 4 + 3i$, $z_3 = 10i$ find z_4 such that $z_1 / z_2 = z_3 / z_4$. Evaluate

   (a) $| z_2 - z_1 |^2$ (b) $| z_4 - z_3 |^2$

   Show that $$\frac{| z_4 - z_3 |}{| z_2 - z_1 |} = \frac{| z_3 |}{| z_1 |}$$

18. Sketch the circle $| z - 3 | = 2$ on an argand diagram. What is the greatest value of z ?

19. Sketch the two numbers $| z - 2 | = 2$ and $| z - i | = | z - 1 |$ on the same diagram.

20. If the following are both true, shade the required region on an argand diagram.

$$|z - 1| \leq 1 \quad 0 \leq \arg z \leq \pi/4.$$

21. Find the real and imaginary parts of,

(a) $\dfrac{2 + 3i}{3 + 2i}$ (b) $1 / i^5$

[Solutions: **1** (a) 7 + 10i (b) 7 - 7i (c) 7 - 15i (d) 1 - 3i (e) 11 + 4i (f) -2i;

2 (a) 5 + 14i (b) -25i (c) 26 - 7i (d) 2 (e) -5 + 12i (f) 19 + 17i (g) 4 - 22i;

3 (a) 8/13 - i/13 (b) 27/25 + 11i/25 (c) 1/2 + i/2 (d) 9/41 + 40i/41 (e) 72/289 + 154i/289;

4 -1.2 + 2.4i;

5 i;

6 (a) 4, -2 ± i, (b) -1, -3, (c) 1, 5/2, (d) 3± √3, (e) (-5 ±√37)/6, (f) (7± √17)/8;

7 (a) 2±i, (b) 3±i, (c) 4±i, (d) -6±2i;

8 -4, -5, -49, -3, ±i√5, ±5i, ±2i√2, ±3i√3;

11 rotation of 90° about the origin;

12 (a) r = √2, ϑ = π/4, (b) r = 5, ϑ = 0.64°, (c) r = 13, ϑ =-1.18°, (d) r = 13, ϑ =2.75°, (e) r = 4√2, ϑ =-π/4, (f) r = 10, ϑ =-2.21°;

13 (a) √2 + i√2, (b) 2√3 + 2i, (c) -1, (d) 4 - 4i√3, (e) (-3/2)√3 - 3i/2, (f) 12i;

14 (a) (6, π/2), (1/3, 0), (b) (10, π/2), (5/2, π/6), (c) (5, 7π/12), (1/5, 5π/12), (d) (7, 5π/12), (1/7, -11π/12);

15 - cos ϑ; **16** (4, π/3), (2, -π/6), (64, π), (32, -5π/6);

17 z_4 = -22 -4i, (a) 34, (b) 680;

18 5;

19 circle, straight line;

21 (a) Re = 12/13, Im = 5i / 13, (b) Re = 0, Im = -1.]