5.11 Analysis of electrical networks

Matrix algebra is very useful for the analysis of certain types of electrical network. For such networks it is possible to produce a mathematical model consisting

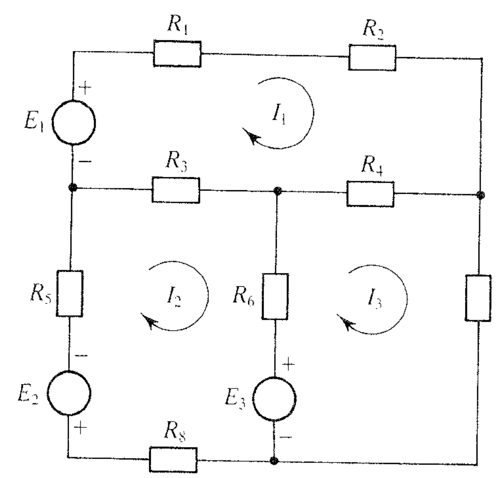


Figure 5.2 An electrical network with mesh currents shown.

of simultaneous equations which can be solved using the techniques just described. We will consider the case when the network consists of resistors and voltage sources. The technique is similar for other types of network.

In order to develop this approach, it is necessary to develop a systematic method for writing the circuit equations. The method adopted depends on what the unknown variables are. A common problem is that the voltage sources and the resistor values are known and it is desired to know the current values in each part of the network. This can be formulated as a matrix equation. Given

$$V = RI'$$

where

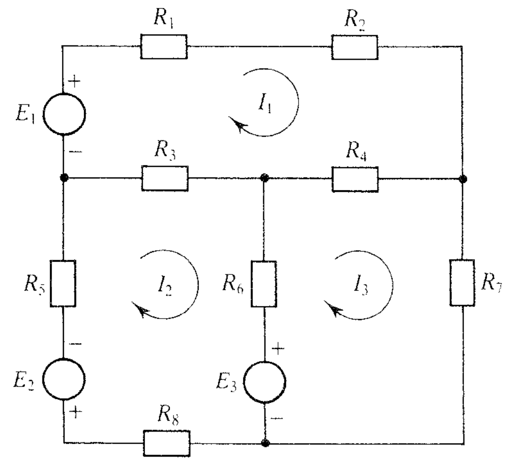
V = voltage vector for the network

I' = current vector for the network

R = matrix of resistor values

the problem is to calculate I' when V and R are known. I' is used to avoid confusion with the identity matrix.

Any size of electrical network can be analysed using this approach. We will limit the discussion to the case where I' has three components, for simplicity. The extension to larger networks is straightforward. Consider the electrical network of Figure 5.2. Mesh currents have been drawn for each of the loops in the circuit. A **mesh** is defined as a loop that cannot contain a smaller closed current path. For convenience, each mesh current is drawn in a clockwise direction even though it may turn out to be in the opposite direction when the calculations have been performed. The net current in each branch of the circuit can be obtained by combining the mesh currents. These are termed the **branch currents**. The concept of a mesh current may appear slightly abstract but it does provide a convenient



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The next stage is to make use of Kirchhoff's voltage law for each of the meshes in the network. This states that the algebraic sum of the voltages around any closed loop in an electrical network is zero. Therefore the sum of the voltage rises must equal the sum of voltage drops. When applying Kirchhoff's voltage law it is important to use the correct sign for a voltage source depending on whether or not it is 'aiding' a mesh current.

For mesh 1

$$E_1 = I_1R_1 + I_1R_2 + (I_1 - I_3)R_4 + (I_1 - I_2)R_3$$

$$E_1 = I_1(R_1 + R_2 + R_4 + R_3) + I_2(-R_3) + I_3(-R_4)$$

For mesh 2

$$-E_2 - E_3 = I_2R_5 + (I_2 - I_1)R_3 + (I_2 - I_3)R_6 + I_2R_8$$

$$-E_2 - E_3 = I_1(-R_3) + I_2(R_5 + R_3 + R_6 + R_8) + I_3(-R_6)$$

For mesh 3

$$E_3 = (I_3 - I_2)R_6 + (I_3 - I_1)R_4 + I_3R_7$$

$$E_3 = I_1(-R_4) + I_2(-R_6) + I_3(R_6 + R_4 + R_7)$$

These equations can be written in matrix form as:

$$\begin{pmatrix} E_1 \\ -E_2 - E_3 \\ E_3 \end{pmatrix} = \begin{pmatrix} R_1 + R_2 + R_4 + R_3 & -R_3 & -R_4 \\ -R_3 & R_5 + R_3 + R_6 + R_8 & -R_6 \\ -R_4 & -R_6 & R_6 + R_4 + R_7 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

Example 5.38

Consider the electrical network of Figure 5.3. It has the same structure as that of Figure 5.2 but with actual values for the voltage sources and resistors. Branch currents as well as mesh currents have been shown. Calculate the mesh currents and hence the branch currents for the network.

Solution

We have already obtained the equations for this network. Substituting actual values for the resistors and voltage sources gives:

$$\begin{pmatrix} 3 \\ -2 - 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 & -3 & -1 \\ -3 & 14 & -2 \\ -1 & -2 & 6 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

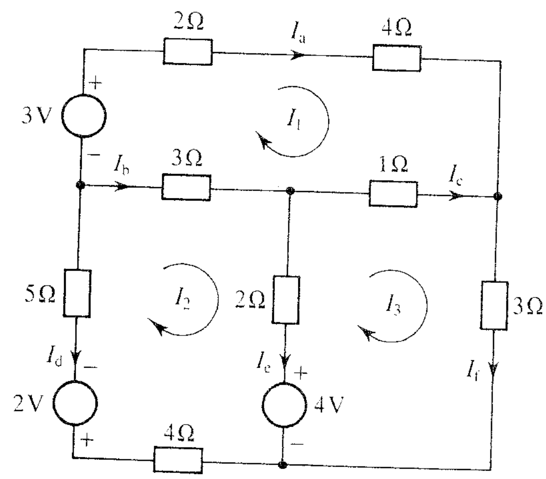

Figure 5.3 The electrical network of Figure 5.2 with values for the source voltages and resistors added.

This is now in the form $V = RI'$. We shall solve these equations by Gaussian elimination. Forming the augmented matrix, we have:

$$\begin{pmatrix} 10 & -3 & -1 & 3 \\ -3 & 14 & -2 & -6 \\ -1 & -2 & 6 & 4 \end{pmatrix}$$

Then

$$\begin{matrix} R_1 \\ R_2 \to 10R_2 + 3R_1 \\ R_3 \to 10R_3 + R_1 \end{matrix} \quad \begin{pmatrix} 10 & -3 & -1 & 3 \\ 0 & 131 & -23 & -51 \\ 0 & -23 & 59 & 43 \end{pmatrix}$$

Then

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \to 131R_3 + 23R_2 \end{matrix} \quad \begin{pmatrix} 10 & -3 & -1 & 3 \\ 0 & 131 & -23 & -51 \\ 0 & 0 & 7200 & 4460 \end{pmatrix}$$

Hence,

$$I_3 = \frac{4460}{7200} = 0.619 \text{ A}$$

Similarly,

$$I_2 = \frac{-51 + 23(0.619)}{131} = -0.281 \text{ A}$$

ıges and

Finally,

$$I_1 = \frac{3 + 0.619 + 3(-0.281)}{10} = 0.278\ \text{A}$$

The branch currents are then

$$I_a = I_1 = 278\ \text{mA}$$

$$I_b = I_2 - I_1 = -281 - 278 = -559\ \text{mA}$$

$$I_c = I_3 - I_1 = 619 - 278 = 341\ \text{mA}$$

$$I_d = -I_2 = 281\ \text{mA}$$

$$I_e = I_2 - I_3 = -281 - 619 = -900\ \text{mA}$$

$$I_f = I_3 = 619\ \text{mA}$$ ▲

An alternative approach to analysing an electrical network is to use the **node voltage method**. For our purposes the nodes of an electrical network can be thought of as the 'islands' of equal potential that lie between electrical components and sources. The procedure is as follows:

(1) Pick a reference node. In order to simplify the equations this is usually chosen to be the node which is common to the largest number of voltage sources and/or the largest number of branches.

(2) Assign a node voltage variable to all of the other nodes. If two nodes are separated solely by a voltage source then only one of the nodes need be assigned a voltage variable. The node voltages are all measured with respect to the reference node.

(3) At each node, write Kirchhoff's current law in terms of the node voltages. Note that once the node voltages have been calculated it is easy to obtain the branch currents.

We will again examine the network of Figure 5.2, but this time use the node voltage method. The network is shown in Figure 5.4 with node voltages assigned and branch currents labelled. The reference node is indicated by using the earth symbol. Writing Kirchhoff's current law for each node, we obtain:

node a

$$I_a = I_a$$

$$\frac{V_b + E_1 - V_a}{R_1} = \frac{V_a - V_d}{R_2}$$

$$V_b R_2 + E_1 R_2 - V_a R_2 = V_a R_1 - V_d R_1$$

$$V_a(R_1 + R_2) - V_b R_2 - V_d R_1 = E_1 R_2$$

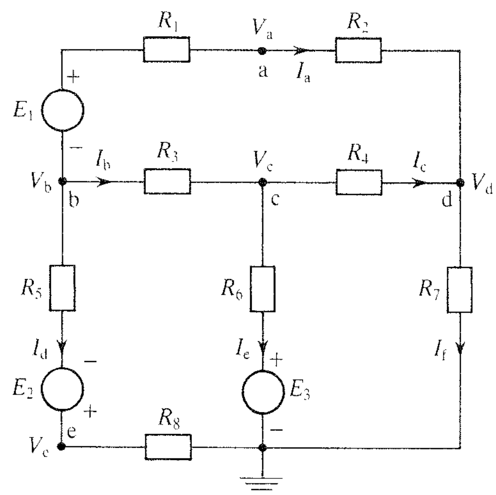

Figure 5.4 The network of Figure 5.2 with node voltages labelled.

node b

$$I_a + I_b + I_d = 0$$

$$\frac{V_b + E_1 - V_a}{R_1} + \frac{V_b - V_c}{R_3} + \frac{V_b + E_2 - V_e}{R_5} = 0$$

Rearrangement yields:

$$V_bR_3R_5 + E_1R_3R_5 - V_aR_3R_5 + V_bR_1R_5 - V_cR_1R_5 + V_bR_1R_3 + E_2R_1R_3 - V_eR_1R_3 = 0$$

that is,

$$V_aR_3R_5 - V_b(R_1R_3 + R_1R_5 + R_3R_5) + V_cR_1R_5 + V_eR_1R_3 = E_1R_3R_5 + E_2R_1R_3$$

node c

$$I_b = I_c + I_e$$

$$\frac{V_b - V_c}{R_3} = \frac{V_c - V_d}{R_4} + \frac{V_c - E_3}{R_6}$$

so that

$$V_bR_4R_6 - V_cR_4R_6 = V_cR_3R_6 - V_dR_3R_6 + V_cR_3R_4 - E_3R_3R_4$$

that is,

$$V_bR_4R_6 - V_c(R_4R_6 + R_3R_6 + R_3R_4) + V_dR_3R_6 = -E_3R_3R_4$$

node d

$$I_a + I_c = I_f$$

$$\frac{V_a - V_d}{R_2} + \frac{V_c - V_d}{R_4} = \frac{V_d}{R_7}$$

$$V_aR_4R_7 - V_dR_4R_7 + V_cR_2R_7 - V_dR_2R_7 = V_dR_2R_4$$

$$V_aR_4R_7 + V_cR_2R_7 - V_d(R_4R_7 + R_2R_7 + R_2R_4) = 0$$

node e

$$I_d = I_d$$

$$\frac{V_b + E_2 - V_e}{R_5} = \frac{V_e}{R_8}$$

$$V_bR_8 - V_e(R_5 + R_8) = -E_2R_8$$

These equations can be written in matrix form $AV = B$, where A is the matrix

$$\begin{pmatrix} R_1 + R_2 & -R_2 & 0 & -R_1 & 0 \\ R_3R_5 & -R_1R_3 - R_1R_5 - R_3R_5 & R_1R_5 & 0 & R_1R_3 \\ 0 & R_4R_6 & -R_4R_6 - R_3R_6 - R_3R_4 & R_3R_6 & 0 \\ R_4R_7 & 0 & R_2R_7 & -R_4R_7 - R_2R_7 - R_2R_4 & 0 \\ 0 & R_8 & 0 & 0 & -R_5 - R_8 \end{pmatrix}$$

and

$$V = \begin{pmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} E_1R_2 \\ E_1R_3R_5 + E_2R_1R_3 \\ -E_3R_3R_4 \\ 0 \\ -E_2R_8 \end{pmatrix}$$

The equations would generally be solved by Gaussian elimination to obtain the node voltages and hence the branch currents.

Using the component values from Example 5.38, these equations become

$$\begin{pmatrix} 6 & -4 & 0 & -2 & 0 \\ 15 & -31 & 10 & 0 & 6 \\ 0 & 2 & -11 & 6 & 0 \\ 3 & 0 & 12 & -19 & 0 \\ 0 & 4 & 0 & 0 & -9 \end{pmatrix} \begin{pmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \end{pmatrix} = \begin{pmatrix} 12 \\ 57 \\ -12 \\ 0 \\ -8 \end{pmatrix}$$

Use of a computer package avoids the tedious arithmetic associated with Gaussian elimination and yields:

$$V_a = 2.969 \qquad V_b = 0.5250 \qquad V_c = 2.200 \qquad V_d = 1.858 \qquad V_e = 1.122$$

It is then straightforward to calculate the branch currents:

$$I_a = \frac{V_a - V_d}{R_2} = 278\,\text{mA} \qquad I_b = \frac{V_b - V_c}{R_3} = -558\,\text{mA} \qquad I_c = \frac{V_c - V_d}{R_4} = 342\,\text{mA}$$

$$I_d = \frac{V_c}{R_8} = 281\ \text{mA} \qquad I_e = \frac{V_c - E_3}{R_6} = -900\ \text{mA} \qquad I_f = \frac{V_d}{R_7} = 619\ \text{mA}$$

Compare these answers with those of Example 5.38.

It is possible to analyse electrical networks containing more complex elements such as capacitors, inductors, active devices, etc., using the same approach. The equations are more complicated but the technique is the same. Often it is necessary to use iterative techniques in view of the size and complexity of the problem. These are examined in the following section.